Math 367 Notes: Functions

In Euclidean geometry we will be looking closely at some types of functions on the plane, as well as their compositions and inverses. In these notes, we will consider functions more generally, and some of their properties that will be important later in Euclidean geometry.

Functions and notation.

Definition 1. A *function* from a set A to a set B is a correspondence f from A to B in which each element of A is paired with one, and only one, element of B. We call A the *domain* of f, and B the *codomain* of f.

Common notation for such a function is $f : A \to B$.

Examples 1. Draw a diagram or graph to help visualize each function:

- To each state, associate its capital city.

- $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$.

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f((x, y)) = (x + 1, y + 1)$ for all $x, y \in \mathbb{R}$.

Composition and inverse.

Definition 2. Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of f and g is the function

$$g \circ f : A \to C$$

given by $(g \circ f)(a) = g(f(a))$ for all a in A.

Examples 2. Find the following compositions.

- Let A be the set of children in a fourth grade classroom. Let $B = \mathbb{N}$, and let $C = \{ \text{ yes }, \text{ no } \}$. Let $f : A \to B$ assign to each child their height in inches. Let $g : B \to C$ be given by

$$g(n) = \begin{cases} \text{yes}, & \text{if } n \geq 52 \\ \text{no}, & \text{if } n < 52. \end{cases}$$

- Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 1$, $g(x) = x^2$, and $h(x) = \frac{1}{2}(x - 1)$. Find the compositions $g \circ f$, $f \circ g$, $h \circ f$.

Definition 3. Let $f : A \to B$ be a function. We say that f is *invertible* if there is a function, denoted $f^{-1} : B \to A$, such that $f^{-1} \circ f(a) = a$ for all a in A and $f \circ f^{-1}(b) = b$ for all b in B. In this case, we call f^{-1} the *inverse* function to f.

Examples 3. For each function, find its inverse if it exists. If the function is not invertible, explain why not.

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 1$.

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sqrt[3]{x}$.

- Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. *(The notation* $[0, \infty)$ *refers to the set of all real numbers* x *for which* $x \geq 0$.*)*

- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f((x, y)) = (2x, 2y)$.

Question. Let $f : A \to B$ and $g : B \to C$ be two invertible functions. Is $g \circ f$ invertible? If so, what is its inverse?

Functions that are onto.

Definition 4. Let $f : A \to B$ be a function. We say f is *onto* (or *surjective*) if the following statement is true: For every element b in B, there is an element a in A for which $f(a) = b$.

Examples 4.

Functions that are one-to-one.

Definition 5. Let $f : A \to B$ be a function. We say f is *one-to-one* (or *injective*) if the following statement is true: For all elements a_1 and a_2 in A, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Examples 5.

Functions that are one-to-one correspondences.

Definition 6. Let $f : A \to B$ be a function. We say f is a *one-to-one correspondence* (or *bijective*) if f is one-to-one and onto.

Examples 6.

Proposition 1. *Let $f : A \to B$ be a function. Then f is invertible if, and only if, f is a one-to-one correspondence.*

Proposition 2. *Let $f : A \to B$ and $g : B \to C$ be functions. If f and g are both one-to-one correspondences, then $g \circ f$ is a one-to-one correspondence.*