Math 300 Final Exam Practice Problems
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The following are just a few representative problems. They are not meant to include examples of all possible problems that may be on the exam. You should also be prepared to work any problems similar to homework, examples from class, and the first two exams.

1. Prove or find a counterexample:

(a) For all real numbers x, x is irrational if, and only if, $10x$ is irrational.

True.

Proof: Let x be a real number. We wish to prove that if x is irrational, then $10x$ is irrational, and that if $10x$ is irrational, then x is irrational. We will prove the contrapositive of each statement, that is we will prove that
(1) if $10x$ is rational, then x is rational, and that
(2) if x is rational, then $10x$ is rational. Since the contrapositive of any statement is logically equivalent to the statement, this will suffice.

Proof of (1): Assume $10x$ is rational, i.e. $10x = \frac{p}{q}$ for $p, q \in \mathbb{Z}$, $q \neq 0$.
Then $x = \frac{p}{10q}$, and so x is rational since $10q$ is an integer.

Proof of (2): Assume x is rational, i.e. $x = \frac{a}{b}$ for $a, b \in \mathbb{Z}$, $b \neq 0$.
Then $10x = \frac{10a}{b}$, which is rational, since $10a$ is an integer.

(b) For all real numbers x, x is irrational if, and only if, $\sqrt{2}x$ is irrational.

False.

Counterexample: Let $x = \sqrt{2}$. Then x is irrational and
$\sqrt{2}x = \sqrt{2} \cdot \sqrt{2} = 2$, which is rational.

(This suffices to show that the statement is false, since a biconditional statement is the conjunction of two conditional statements, both of which must be true in order for the biconditional statement to be true.
In this case, the converse statement is also false:
Take $x = 1$. Then $\sqrt{2}x = \sqrt{2}$, which is irrational, and x is rational.)

2. Prove by induction that for each positive integer n,

$$P(n): \quad 1+5+9+\cdots+(4n-3)=n(2n-1).$$

$P(1)$ is true: $\quad 4(1)-3=1 \quad$ and $\quad 1(2\cdot 1-1)=1.$

Assume that $1+5+9+\cdots+(4k-3)=k(2k-1)$ for some positive integer k, i.e. that $P(k)$ is true for some positive integer k.

Then $1+5+9+\cdots+(4k-3)+(4(k+1)-3)$

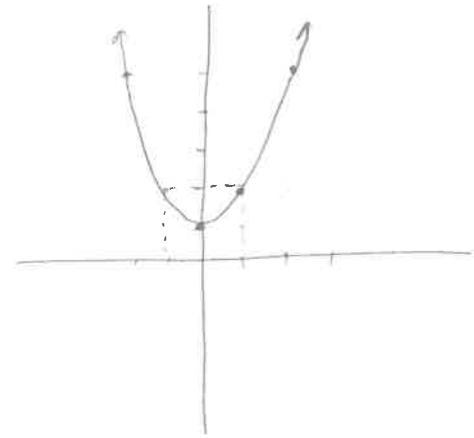
$$\begin{aligned}
&= k(2k-1) + (4(k+1)-3) \\
&= 2k^2-k+4k+4-3 \\
&= 2k^2+3k+1 \\
&= (2k+1)(k+1) \\
&= (k+1)(2(k+1)-1),
\end{aligned}$$

that is, $P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$.
(a) Find $f^{-1}((0, 2))$ (where $(0, 2)$ denotes an open interval).

$$f^{-1}((0,2)) = \{x \in \mathbb{R} \mid 0 < f(x) < 2\}$$
$$= \{x \in \mathbb{R} \mid 0 < x^2+1 < 2\}$$
$$= \{x \in \mathbb{R} \mid -1 < x^2 < 1\}$$
$$= (-1, 1) \quad \text{(open interval)}$$



(b) Find the range of f.

The range is $\{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in \mathbb{R}\} = [1, \infty)$

(c) Is f injective? Justify your answer.

No.

$f(1) = 2 = f(-1)$

4. Let X and Y be sets, let A and B be subsets of X, and let $f : X \to Y$ be a function. Prove that if f is injective and $f(A) \subseteq f(B)$, then $A \subseteq B$.

Assume f is injective and $f(A) \subseteq f(B)$.
let $a \in A$. By definition, $f(a) \in f(A)$. Since $f(A) \subseteq f(B)$,
it follows that $f(a) \in f(B)$. This implies that there is
an element $b \in B$ such that $f(a) = f(b)$. By assumption,
f is injective, which implies $a = b$. Since $b \in B$, it follows
that $a \in B$. Since this is true for all $a \in A$, we have shown
that $A \subseteq B$.

5. Let A and B be sets, and let $f : A \to B$ be a function. Let X be a subset of A, and let Y be a subset of B for which $f(X) \subseteq Y$.
(a) Prove that $X \subseteq f^{-1}(Y)$.

Let $x \in X$. By assumption, $f(X) \subseteq Y$, so $f(x) \in Y$.
By definition, $f^{-1}(Y) = \{z \in X \mid f(z) \in Y\}$, and so $x \in f^{-1}(Y)$.
Since this is true for all $x \in X$, we have shown that $X \subseteq f^{-1}(Y)$.

(b) If $f(X) = Y$, is it necessarily true that $X = f^{-1}(Y)$? Justify your answer.

No.
For example, let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$.
Let $X = [0, 1]$. Then $f([0,1]) = [0,1]$. Set $Y = [0,1]$.
By definition, $f^{-1}(Y) = \{x \in \mathbb{R} \mid f(x) \in [0,1]\} = [-1,1]$.
So $X \neq f^{-1}(Y)$.

6. Find a solution to the equation $14x + 18y = 114$ in which x and y are integers.

First use the Euclidean Algorithm to find gcd (14, 18):

$18 = 14 \cdot 1 + 4$
$14 = 4 \cdot 3 + 2$
$4 = 2 \cdot 2$

so $2 = \gcd(14, 18)$ and

$$\begin{aligned} 2 &= 14 - 4 \cdot 3 \\ &= 14 - (18 - 14) \cdot 3 \\ &= 14 - 18 \cdot 3 + 14 \cdot 3 \\ &= 14(4) + 18(-3) \end{aligned}$$

Now multiply both sides of this equation,

$$14 \cdot 4 + 18 \cdot (-3) = 2,$$

by 57, to obtain

$$14 \cdot 4 \cdot 57 + 18 \cdot (-3) \cdot 57 = 2 \cdot 57, \text{ or}$$

$$14 \cdot (228) + 18 \cdot (-171) = 114$$

so a solution is $x = 228$, $y = -171$

(What are all solutions? See §8.3. This is not asked here, but you should know how to find them.)

7. Consider the following two sets:

$$A = \{n \in \mathbb{Z} \mid n = 4x + 3y \text{ for some } x, y \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 4x + 15y \text{ for some } x, y \in \mathbb{Z}\}$$

(a) List at least 5 elements of A and at least 5 elements of B.

A: 0, 3, 4, 7, -3

B: 0, 4, 15, 19, -4

(b) Is $A = B$? Prove or disprove.

Yes.

Since gcd $(4,3)=1$, there are $x, y \in \mathbb{Z}$ such that $1 = 4x + 3y$.
Let $n \in \mathbb{Z}$. Then $n = 4(xn) + 3(yn)$, so $n \in A$.

Similarly, since gcd $(4, 15) = 1$, $n \in B$ for all $n \in \mathbb{Z}$.
So in fact $A = \mathbb{Z}$ and $B = \mathbb{Z}$, and it follows that $A = B$.

8. Let $A = \{1, 2, 3\}$ and let X be the set of all bijective functions $f : A \to A$. Define a relation R on X by fRg provided that $f(1) = g(1)$.
(a) Prove that R is an equivalence relation.

R is reflexive : let $f \in X$. Then $f(1) = f(1)$ so fRf.

R is symmetric : let $f, g \in X$ with fRg. That is, $f(1) = g(1)$. Then $g(1) = f(1)$, so gRf.

R is transitive : Let $f, g, h \in X$ with fRg and gRh, that is $f(1) = g(1)$ and $g(1) = h(1)$. Then $f(1) = h(1)$, i.e. fRh.

(b) Find all elements in the equivalence class of the function f defined by $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$.

let $g \in [f] = \{ g \in X \mid f(1) = g(1) \}$.
Then $g(1) = f(1) = 2$. There is no restriction on $g(2)$ or $g(3)$, other than that g is bijective. The possibilities are :
(i) $g = f$, i.e. $g(2) = 3$, $g(3) = 1$

or (ii) $g(1) = 2$, $g(2) = 1$, $g(3) = 3$.
These are the two functions in the equivalence class of f.

9. Prove that if n is an integer for which $5 \nmid n$, then $n^2 \equiv 1 \mod 5$ or $n^2 \equiv 4 \mod 5$.

Case 1 $\quad n \equiv 1 \pmod 5$

Then $n^2 \equiv 1^2 \equiv 1 \pmod 5$

Case 2 $\quad n \equiv 2 \pmod 5$

Then $n^2 \equiv 2^2 \equiv 4 \pmod 5$

Case 3 $\quad n \equiv 3 \pmod 5$

Then $n^2 \equiv 3^2 \equiv 9 \equiv 4 \pmod 5$

Case 4 $\quad n \equiv 4 \pmod 5$

Then $n^2 \equiv 4^2 \equiv 16 \equiv 1 \pmod 5$

10. Let $\mathbb{Z}_9$ be the set of congruence classes of integers modulo 9. Find the subset of $\mathbb{Z}_9$ consisting of all elements $[a]$ for which there exists $[x] \in \mathbb{Z}_9$ such that $[a] \odot [x] = [0]$.

$[a] \odot [x] = [ax]$

So we want all $[a]$ for which $[ax] = [0]$, i.e. $9 \mid ax$, for some $[x] \in \mathbb{Z}_9$

We could look at the multiplication table for $\mathbb{Z}_9$,

or note that since

$\mathbb{Z}_9 = \{[0], [1], [2], [3], [4], [5], [6], [7], [8]\}$,

the only possible $[a], [x]$ with $9 \mid ax$ will be when each of a, x is a multiple of 3:

$[0], [3], [6]$