Math 300 Final Exam Practice Problems
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The following are just a few representative problems. They are not meant to include examples of all possible problems that may be on the exam. You should also be prepared to work any problems similar to homework, examples from class, and the first two exams.

1. Prove or find a counterexample:

(a) For all real numbers x, x is irrational if, and only if, $10x$ is irrational.

(b) For all real numbers x, x is irrational if, and only if, $\sqrt{2}x$ is irrational.

2. Prove by induction that for each positive integer n,

$$1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1).$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$.
(a) Find $f^{-1}((0, 2))$ (where $(0, 2)$ denotes an open interval).

(b) Find the range of f.

(c) Is f injective? Justify your answer.

4. Let X and Y be sets, let A and B be subsets of X, and let $f : X \to Y$ be a function. Prove that if f is injective and $f(A) \subseteq f(B)$, then $A \subseteq B$.

5. Let A and B be sets, and let $f : A \to B$ be a function. Let X be a subset of A, and let Y be a subset of B for which $f(X) \subseteq Y$.
(a) Prove that $X \subseteq f^{-1}(Y)$.

(b) If $f(X) = Y$, is it necessarily true that $X = f^{-1}(Y)$? Justify your answer.

6. Find a solution to the equation $14x + 18y = 114$ in which x and y are integers.

7. Consider the following two sets:

$$
\begin{aligned}
A &= \{n \in \mathbb{Z} \mid n = 4x + 3y \text{ for some } x, y \in \mathbb{Z}\} \\
B &= \{n \in \mathbb{Z} \mid n = 4x + 15y \text{ for some } x, y \in \mathbb{Z}\}
\end{aligned}
$$

(a) List at least 5 elements of A and at least 5 elements of B.

(b) Is $A = B$? Prove or disprove.

8. Let $A = \{1, 2, 3\}$ and let X be the set of all bijective functions $f : A \to A$. Define a relation R on X by fRg provided that $f(1) = g(1)$.
(a) Prove that R is an equivalence relation.

(b) Find all elements in the equivalence class of the function f defined by $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$.

9. Prove that if n is an integer for which $5 \nmid n$, then $n^2 \equiv 1 \mod 5$ or $n^2 \equiv 4 \mod 5$.

10. Let $\mathbb{Z}_9$ be the set of congruence classes of integers modulo 9. Find the subset of $\mathbb{Z}_9$ consisting of all elements $[a]$ for which there exists $[x] \in \mathbb{Z}_9$ such that $[a] \odot [x] = [0]$.