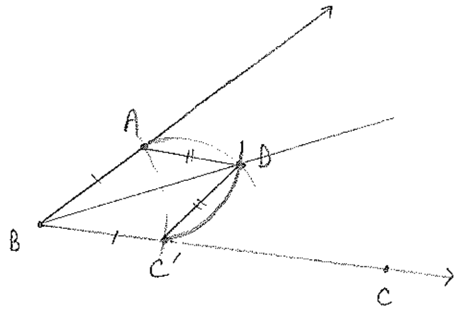
Thm 32 Every angle has a bisector.



Proof: Let $\angle ABC$ be an angle. Consider a circle of radius $\mathcal{L}(AB)$ and center B. Let C' be the intersection point of the circle with ray $\overrightarrow{BC}$. So $AB \cong C'B$ by construction.

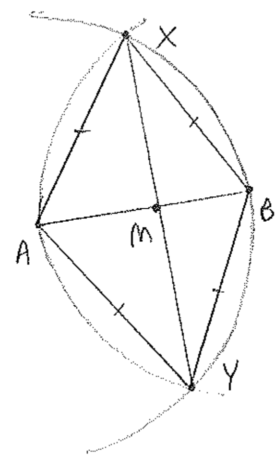
Let D be the intersection point of two circles:

(1) the circle with center A and radius $\mathcal{L}(AC')$ (other choices of radius work as well)

(2) the circle with center C' and radius $\mathcal{L}(AC')$.

By construction, then, $AD \cong C'D$. Since $BD \cong BD$, the SSS Axiom now implies that $\triangle ABD \cong \triangle C'BD$. By CPCFC, $\angle ABD \cong \angle C'BD$. Therefore $\overrightarrow{BD}$ is a bisector of $\angle ABC$. $\square$

Thm 35 Every segment has a midpoint.



Proof: Let AB be a segment. Consider two circles:

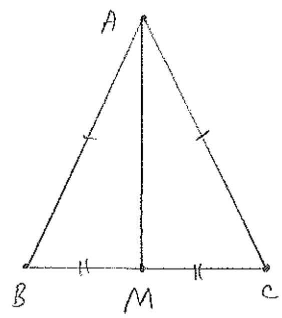
(1) the circle with center A and radius $\mathcal{L}(AB)$
(other choices of radius work, as long as they are not too short)

(2) the circle with center B and radius $\mathcal{L}(AB)$.

Let X and Y be their two intersection points.
Let M be the intersection of AB and XY.

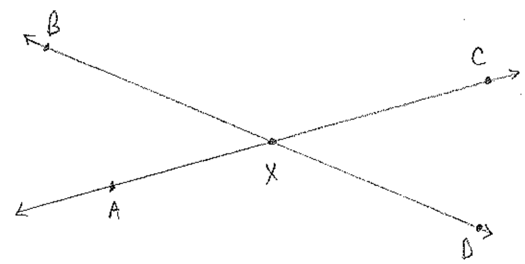
By construction, $AX \cong BX$, $AY \cong BY$, and certainly $XY \cong XY$. By the SSS Axiom, $\triangle AXY \cong \triangle BXY$. By CPCFC, $\angle AXY \cong \angle BXY$, so that $\angle AXM \cong \angle BXM$ (these angles are the same as the others). Now apply the SAS Axiom to $\triangle AXM$ and $\triangle BXM$: By construction, $AX \cong BX$, and certainly $XM \cong XM$. We have also just shown that $\angle AXM \cong \angle BXM$. So by the SAS Axiom, $\triangle AXM \cong \triangle BXM$. Now by CPCFC, $AM \cong MB$. Therefore M is the midpoint of AB. $\square$

Thm 36 The base angles of an isosceles triangle are congruent angles.



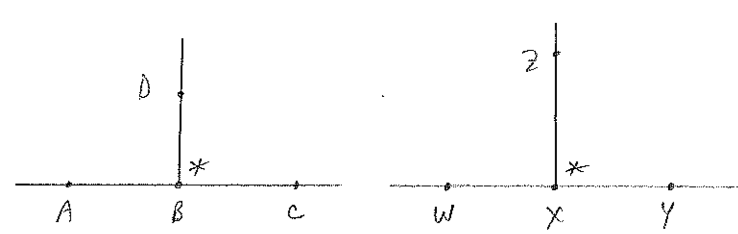
Proof Let $\triangle ABC$ be an isosceles triangle in which $AB \cong AC$. By Theorem 35, BC has a midpoint; call it M. So $BM \cong MC$. Since $AM \cong AM$, by the SSS Axiom, $\triangle ABM \cong \triangle ACM$. By CPCFC, $\angle ABM \cong \angle ACM$. Thus the base angles of $\triangle ABC$ are congruent. $\square$

Cor 38 Vertical angles are congruent.



Proof Let $\angle AXB$ and $\angle CXD$ be a pair of vertical angles. Note that $\angle AXB$ is supplementary to $\angle BXC$, and that $\angle CXD$ is also supplementary to $\angle BXC$. Since $\angle BXC$ is congruent to itself, by Thm 37, $\angle AXB \cong \angle CXD$. $\square$

Thm 39 (Weak Right Angle Thm) An angle that is congruent to a right angle is also a right angle.



Proof Suppose $\angle CBD \cong \angle YXZ$, as indicated in the picture by an asterisk, and that $\angle CBD$ is a right angle. By definition then, $\angle CBD \cong \angle ABD$ as shown since $\angle ABD$ is supplementary to $\angle CBD$. By Thm 37, $\angle ABD \cong \angle WXZ$, since these are supplements to angles that are congruent. So, collecting all these congruences together, we have:

$$\angle WXY \cong \angle ABD \cong \angle CBD \cong \angle YXZ.$$

So all of these angles are congruent to each other, and in particular, $\angle WXY \cong \angle YXZ$. Now $\angle WXY$ and $\angle YXZ$ are supplementary as well, and so by definition of right angle, $\angle YXZ$ is a right angle. $\square$