Math 415 Homework Assignment 10

Assume that each ring has unity (i.e. a multiplicative identity) not equal to 0.

1. Find the remainder of 9^{380} when it is divided by 19.

2. Find the remainder of 11^{37} when it is divided by 27.

3. Let R be a ring, and let $R^\times$ denote the subset of R consisting of all units in R. Prove that $R^\times$ is a group under multiplication.

4. Consider the following groups of units of finite fields. For each, to what known group (in the notation of the Fundamental Theorem of Finitely Generated Abelian Groups) is it isomorphic? Justify your answers.
   (a) $\mathbb{Z}_{11}^\times$ (b) $\mathbb{Z}_{13}^\times$

5. Find all solutions to each of the following congruences.
   (a) $2x \equiv 10 (\text{mod} 11)$ (b) $9x \equiv 12 (\text{mod} 24)$

6. Let p be a prime and n a positive integer. Find and prove a formula for $\phi(p^n)$, where ϕ is the Euler phi function.

7. Let F be a field.
   (a) Prove that the characteristic of F must be either 0 or prime.
   (b) Prove that F contains a unique smallest subfield F_0 that is isomorphic either to $\mathbb{Q}$ or to $\mathbb{Z}_p$ for some prime p. (The field F_0 is called the *prime subfield* of F.)

8. Let $D = \{a + bi \mid a, b \in \mathbb{Z}\}$, the ring of *Gaussian integers*, an integral subdomain of $\mathbb{C}$. Describe its field of quotients F (that is, F consists of which elements of $\mathbb{C}$?).

9. Let R be a commutative ring, and let t be a nonzero element of R that is not a zero divisor. Let $T = \{t^n \mid n \in \mathbb{Z}\}$.
   (a) Consider the following relation on $R \times T$: $(a, b) \sim (c, d)$ if and only if $ad = bc$. Prove that this is an equivalence relation.
   (b) Let $Q(R, T)$ be the set of equivalence classes. By analogy with the construction in Section 21, define addition and multiplication on $Q(R, T)$. Outline a proof that your addition and multiplication are well-defined, and that $Q(R, T)$ is a ring under these operations. (You need not give all details, just enough to be convincing, referring to pages in the text where the proof would be very similar.)
   (c) Let $R = \mathbb{Z}$, $t = 2$, and $T = \{2^n \mid n \in \mathbb{Z}\}$. Describe $Q(\mathbb{Z}, T)$. (That is, since it will be a subring of $\mathbb{Q}$, describe which elements of $\mathbb{Q}$ it consists of).