Math 415H Homework Assignment 7

1. Describe the factor group $(\mathbb{Z}_3 \times \mathbb{Z}_9)/\langle(1,3)\rangle$ in the notation of the fundamental theorem of finitely generated abelian groups. Is it cyclic? Justify your answer.

2. Let $\mathbb{C}^\times$ be the group of nonzero complex numbers, under multiplication. Let $f : \mathbb{C}^\times \to \mathrm{GL}(2,\mathbb{R})$ be defined by $f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for all $a, b \in \mathbb{R}$.
   (a) Show that f is an injective group homomorphism.
   (b) Let U denote the unit circle in the complex plane. Show that $U = f^{-1}(\mathrm{SL}(2,\mathbb{R}))$.

3. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Let $G = \langle A, B\rangle$, the subgroup of $\mathrm{GL}(2,\mathbb{C})$ generated by A and B.
   (a) Show that G is a group of order 8, isomorphic to the *quaternion group* Q_8.
   (b) Find all subgroups of Q_8.
   (c) Show that all subgroups of Q_8 are normal.

4. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R},\ ac \neq 0 \right\}$ and let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$, subgroups of $\mathrm{GL}(2,\mathbb{R})$. Prove that N is normal in G, and that $G/N \cong \mathbb{R}^\times \times \mathbb{R}^\times$.

5. Let $\mathrm{GL}(2,\mathbb{Z}_2)$ denote the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2$, a group under multiplication of matrices (which involves both addition and multiplication modulo 2).
   (a) Find all elements of $\mathrm{GL}(2,\mathbb{Z}_2)$.
   (b) Show that $\mathrm{GL}(2,\mathbb{Z}_2) \cong S_3$.

6. Let A and B be normal subgroups of a group G such that G/A and G/B are both abelian. Prove that $G/A \cap B$ is abelian.

7. The *center* of a group G is the set $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \in G\}$. For a fixed $g \in G$, the *centralizer* of g is the set $Z_G(g) = \{a \in G \mid ag = ga\}$.
   (a) Prove that $Z_G(g)$ is a subgroup of G. (You already proved on Exam 1 that $Z(G)$ is a subgroup.)
   (b) Prove that $Z(G)$ is normal in G.

8. If G is a simple abelian group, what is $Z(G)$? If G is a simple nonabelian group, what is $Z(G)$? Justify your answers.

9. Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

10. Let G be a group, and let $\mathrm{Aut}(G)$ denote the set of all *automorphisms* of G (that is all bijective group homomorphisms from G to G).
    (a) Prove that $\mathrm{Aut}(G)$ is a group under composition of functions.
    (b) Find $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.