Math 415H Homework Assignment 8

1. Find all conjugacy classes in the dihedral group D_4.

2. Let G be a group that has a conjugacy class consisting of exactly two elements. Prove that G has a subgroup of index 2, and conclude that G is not simple.

3. Let G be a group and $H < G$. Let
$$N_G(H) = \{g \in G \mid gHg^{-1} = H\} \quad \text{and } C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\},$$
both subgroups of G, called the *normalizer* and *centralizer* of H in G, respectively. Prove that both H and $C_G(H)$ are normal in $N_G(H)$, and that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

4. Let $G = S_5$ and let H be the subgroup of G generated by (1324) and $(13)(24)$.
   (a) Show that H has order 8.
   (b) To what known group of order 8 is H isomorphic? Justify your answer.

5. Let D_4 be the dihedral group of order 8, that is the symmetry group of a square. Label the vertices of the square as $1, 2, 3, 4$, and consider the resulting action of D_4 on the set $\{1, 2, 3, 4\}$. Find the isotropy subgroup of 1.

6. Recall the *braid group*
$$\begin{aligned} B_n := \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \quad & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (i = 1, 2, \ldots, n-2), \\ & \sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| \geq 2)\rangle. \end{aligned}$$
   (a) Draw a picture to illustrate the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in B_3.
   (b) Draw a picture to illustrate an interesting element in the kernel of the following group homomorphism. (The kernel is called the *pure braid group*.) The homomorphism $\phi : B_n \to S_n$ is defined by $\phi(\sigma_i) = (i,\ i+1)$ (that is, $\phi(\sigma_i)$ is the transposition switching i and $i+1$) for $i = 1, 2, \ldots, n-1$.