Math 415 Homework Assignment 9

Assume that each ring has unity (i.e. a multiplicative identity).

1. For each of the following sets R with indicated binary operations, decide whether R is a ring. Justify your answers.
   (a) $R = \{ai \mid a \in \mathbb{R}\}$, under addition and multiplication of complex numbers. (Here i is a square root of -1.)
   (b) $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$, under addition and multiplication of real numbers.

2. Show that $\begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$ is a zero divisor in $M_2(\mathbb{R})$.

3. For each of the following functions f, decide whether f is a ring homomorphism. Justify your answers.
   (a) $f : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $f(A) = \det(A)$ (the determinant of A) for each $A \in M_2(\mathbb{R})$.
   (b) $f : \mathbb{R} \to M_2(\mathbb{R})$ defined by $f(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for all $a \in \mathbb{R}$.

4. The *center* of a ring R is $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$. Prove that $Z(R)$ is a subring of R.

5. Let R be a ring. An element $r \in R$ is *nilpotent* if $r^n = 0$ for some positive integer n.
   (a) Prove that if R is commutative and $r, s \in R$ are nilpotent, then $r + s$ is nilpotent.
   (b) The conclusion in (a) can fail if R is noncommutative: Let $R = M_2(\mathbb{R})$, $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Verify that r and s are both nilpotent, but that $r + s$ is not nilpotent.

6. Let R be a commutative ring of characteristic p, a prime.
   (a) Prove that $(a + b)^p = a^p + b^p$ for all $a, b \in R$.
   (b) Let $\phi : R \to R$ be the function defined by $\phi(a) = a^p$ for all $a \in R$. Prove that ϕ is a ring homomorphism (called the *Frobenius homomorphism*).

7. Let R be any ring and let $f, g : \mathbb{Q} \to R$ be two ring homomorphisms. Suppose that $f(z) = g(z)$ for all $z \in \mathbb{Z}$. Prove that $f = g$.

Bonus. (Optional) Let R be a ring such that $\mathbb{C} \subseteq Z(R)$. Let $q \in \mathbb{C}^\times$. Suppose $a, b \in R$ such that $ba = qab$. Prove the *q-binomial theorem*: For all positive integers n,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \qquad \text{where} \qquad \binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!},$$

$(j)_q! = (j)_q (j-1)_q \cdots (3)_q (2)_q (1)_q$, $(j)_q = 1 + q + q^2 + \cdots + q^{j-1}$, $(0)_q! = 1$. (Note that if $q = 1$, then $(j)_q = j$. *Hint: First prove the identity for all positive integers n, $1 \leq k \leq n$:*

$$\binom{n}{k-1}_q + q^k \binom{n}{k}_q = \binom{n+1}{k}_q.)$$