HOPF AUTOMORPHISMS AND TWISTED EXTENSIONS

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ABSTRACT. We give some applications of a Hopf algebra constructed from a group acting on another Hopf algebra A as Hopf automorphisms, namely Molnar's smash coproduct Hopf algebra. We find connections between the exponent and Frobenius-Schur indicators of a smash coproduct and the twisted exponents and twisted Frobenius-Schur indicators of the original Hopf algebra A. We study the category of modules of the smash coproduct.

1. INTRODUCTION

Molnar [Ml] defined smash coproducts of Hopf algebras, putting them on equal footing with the better-known smash products by viewing both as generalizations of semidirect products of groups. Recently smash coproducts have made an appearance as examples of new phenomena in representation theory [BW, DE]. In this paper we propose several applications of smash coproducts. In particular, the smash coproduct construction will allow us to "untwist" some invariants defined via the action of a Hopf algebra automorphism, such as the twisted exponents and the twisted Frobenius-Schur indicators.

We note that considering Hopf automorphisms is a timely topic, since there has been recent progress in determining the automorphism groups of some Hopf algebras [AD, Ke, R3, SV, Y]. There has also been much recent work on indicators; their importance lies in the fact that they are invariants of the category of representations of the Hopf algebra, and may be defined for more abstract categories [NSc]. Moreover the notion of twisted indicators can be extended to pivotal categories [SV3].

We start by defining the smash coproduct $A \natural k^G$, for any Hopf algebra A with an action of a finite group G by Hopf automorphisms, in the next section. In Section 3 we recall the notions of exponent and twisted exponent [SV2] of a Hopf algebra, and find connections between the exponent of $A \natural k^G$ and twisted exponents of A itself. In Section 4 we assume the Hopf algebra A is semisimple. We recall definitions of Frobenius-Schur indicators [KSZ] and twisted Frobenius-Schur indicators [SV] for simple modules over the Hopf algebra, and give relationships between the indicators of the smash coproduct $A \natural k^G$ and twisted indicators of A itself.

In Section 5 we do not assume the Hopf algebra is semisimple. We introduce the twisted Frobenius-Schur indicators of the regular representation of such a Hopf algebra, simultaneously generalizing indicators for not necessarily semisimple Hopf algebras [KMN] and twisted indicators for semisimple Hopf algebras [SV]. Again we find a connection with the Frobenius-Schur indicator of a smash coproduct. We compute an example for which the Hopf algebra

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A is of dimension 8 in Section 6. Finally in Section 7 we study the structure of categories of modules of $A \natural k^G$, showing that they are equivalent to semidirect product tensor categories $\mathcal{C} \rtimes G$, where $\mathcal{C}$ is a category of A-modules.

Throughout, k will be an algebraically closed field of characteristic 0.

2. The smash coproduct

Our Hopf algebra was defined by Molnar [Ml, Theorem 2.14], who called it the smash coproduct, although our definition seems different at first glance. See also [R2, p. 357].

Let A be a Hopf algebra over a field k and let a finite group G act as Hopf algebra automorphisms of A. Let k^G be the algebra of set functions from G to k under pointwise multiplication; that is, if $\{p_x \mid x \in G\}$ denotes the basis of k^G dual to G, then $p_x p_y = \delta_{x,y} p_x$ for all $x, y \in G$. Recall that k^G is a Hopf algebra with comultiplication given by $\Delta(p_x) = \sum_{y \in G} p_y \otimes p_{y^{-1}x}$, counit $\varepsilon(p_x) = \delta_{1,x}$ and antipode $S(p_x) = p_{x^{-1}}$ for all $x \in G$.

Then we may form the *smash coproduct Hopf algebra*

$$K = A \natural k^G$$

with algebra structure the usual tensor product of algebras. Denote by $a \natural p_x$ the element $a \otimes p_x$ in K, for each $a \in A$ and $x \in G$. Comultiplication is given by

$$\Delta(a \natural p_x) = \sum_{y \in G} (a_1 \natural p_y) \otimes ((y^{-1} \cdot a_2) \natural p_{y^{-1}x})$$

for all $x \in G$, $a \in A$. The counit and antipode are determined by

$$\varepsilon(a \natural p_x) = \delta_{1,x} \varepsilon(a) 1 \quad \text{and} \quad S(a \natural p_x) = \ (x^{-1} \cdot S(a)) \natural p_{x^{-1}}.$$

If Λ_A is an integral for A, then $\Lambda_K = \Lambda_A \natural p_1$ is an integral for K.

Note that Molnar defines the smash coproduct for the right coaction of any commutative Hopf algebra H. We show that our construction is actually his smash coproduct with $H = k^G$, by dualizing our G-action to a k^G-coaction.

Lemma 2.1. *(1) K as above is isomorphic to the smash coproduct as in* [Ml, Theorem 2.14]*, and thus is a Hopf algebra.*

(2) If A is finite-dimensional, then $K^* \cong A^* \# kG$, the smash product Hopf algebra as in [Ml, Theorem 2.13].

Proof. (1) Given the left action of G on A, we define $\rho : A \to A \otimes k^G$ by $a \mapsto \sum_{x \in G} (x \cdot a) \otimes p_x$. Then ρ is a right comodule map, using the fact that the G-action on A satisfies $x \cdot (y \cdot a) = (xy \cdot a)$ and $1 \cdot a = a$ for all $x, y \in G$ and $a \in A$.

Next we note that A is a right comodule algebra under ρ since the G-action is multiplicative, that is $(x \cdot a)(x \cdot b) = x \cdot (ab)$. Also A is a right comodule coalgebra, as the G-action preserves the coalgebra structure of A, that is, $x \cdot (\sum_a a_1 \otimes a_2) = \sum_a (x \cdot a)_1 \otimes (x \cdot a)_2$. Thus A is a right k^G-comodule bialgebra.

Finally the antipode also dualizes to the antipode given by Molnar, and thus Molnar's theorem [Ml, Theorem 2.14] applies.

(2) This is a special case of Molnar's result [Ml, Theorem 5.4].

□

3. Hopf powers and exponents

In any Hopf algebra H, we denote the nth Hopf power of an element $x \in H$ by $x^{[n]} = \sum_x x_1 x_2 x_3 \dots x_n$; that is, first apply Δ_H $n-1$ times to x and then multiply. Note that $x \mapsto x^{[n]}$ is a linear map.

For H semisimple, recall that the exponent of H, $exp(H)$, is the smallest positive integer n, if it exists, such that $x^{[n]} = \varepsilon(x)1$ for all $x \in H$. More generally, this definition makes sense whenever $S^2 = id$. We assume this property of S unless stated otherwise.

Recently [SV2] introduced the *twisted exponent*, where *exp* is twisted by an automorphism of H of finite order. Assume that $\tau \in Aut(H)$ and that n is a multiple of the order of τ. Define the nth *τ-twisted Hopf power* of x to be

$$x^{[n,\tau]} := \sum_x x_1(\tau \cdot x_2)(\tau^2 \cdot x_3) \dots (\tau^{n-1} \cdot x_n).$$

Definition 3.1. $exp_\tau(H)$ is the smallest positive integer n, if it exists, such that n is a multiple of the order of τ and $x^{[n,\tau]} = \varepsilon(x)1$ for all $x \in H$.

Since τ is a Hopf automorphism, $\varepsilon(\tau \cdot x) = \varepsilon(x)$ for any $x \in H$, and thus $\varepsilon(x^{[n,\tau]}) = \varepsilon(x^{[n]}) = \varepsilon(x)$. If H is not semisimple and $S^2 \neq id$ yet S is still bijective, there is a more general definition of the twisted exponent in [SV2].

We will need the following proposition which is a special case of [SV2, Proposition 3.4].

Proposition 3.2. *Suppose that the Hopf automorphism τ of the semisimple Hopf algebra H has order r, $exp_\tau(H)$ is finite, and m is a positive integer. Then $x^{[mr,\tau]} = \varepsilon(x)1$ for all $x \in H$ if and only if $exp_\tau(H)$ divides m.*

Next we give some formulas for our Hopf algebras $K = A \natural k^G$.

Lemma 3.3. *Let $w = a \natural p_x \in A \natural k^G$, the smash coproduct as above. Then*

$$(a \natural p_x)^{[n]} = \sum_{z \in G,\ z^n = x} a^{[n,z^{-1}]} \natural p_z.$$

In particular for $w = \Lambda_K = \Lambda_A \natural p_1$, replace z by z^{-1}. Then

$$\Lambda_K^{[n]} = \sum_{z \in G,\ z^n = 1} \Lambda_A^{[n,z]} \natural p_{z^{-1}}.$$

Proof. A calculation shows that

$$(a \natural p_x)^{[n]} = \sum_{z \in G,\ z^n = x} a_1(z^{-1} \cdot a_2)(z^{-2} \cdot a_3) \cdots (z^{-(n-1)} \cdot a_n) \natural p_z,$$

which gives the first equation in the lemma. The second follows from the first. $\square$

We now find a relation among the (twisted) exponents of A, G, and $K = A \natural k^G$.

Theorem 3.4. *Assume that $S^2 = id$ in A. Then the exponent of K is the least common multiple of $exp(G)$ and $exp_z(A)$ for all $z \in G$.*

Proof. Let $n = exp(K)$, so that

$$(a \natural p_x)^{[n]} = \varepsilon(a \natural p_x)1 = \varepsilon(a)\delta_{x,1}1 = \varepsilon(a)\delta_{x,1} \sum_z p_z$$

for all $a \in A$ and $x \in G$. When $a = 1$, then $(p_x)^{[n]} = \delta_{x,1}1$ implies that $exp(G) = exp(k^G)$ divides n. Thus $z^n = 1$ for all $z \in G$. By the above calculation, $(a \natural p_1)^{[n]} = \varepsilon(a)1$, and so by Lemma 3.3, $a^{[n,z^{-1}]} = \varepsilon(a)$ for all $z \in G$ and $a \in A$. Therefore by Proposition 3.2, $exp(K)$ is a common multiple of $exp(G)$ and $exp_z(A)$ for all $z \in G$.

Now let m be any common multiple of $exp(G)$ and $exp_z(A)$ for all $z \in G$. By Lemma 3.3 and Proposition 3.2,

$$\begin{aligned}(a \natural p_x)^{[m]} &= \sum_{z\in G,\ z^m = x} a^{[m,z^{-1}]} \natural p_z \\ &= \delta_{1,x} \sum_{z\in G} a^{[m,z^{-1}]} \natural p_z \\ &= \delta_{1,x}\varepsilon(a) \sum_{z\in G} p_z \quad = \quad \varepsilon(a \natural p_x)1_K.\end{aligned}$$

Again by Proposition 3.2, $exp(K)$ divides m. □

We will use the following lemma in calculations.

Lemma 3.5. *Let H be a Hopf algebra and let τ be a Hopf automorphism of H whose order divides n. Then* $S(x^{[n,\tau]}) = \tau^{-1} \cdot (S(x)^{[n,\tau^{-1}]})$ *for all $x \in H$.*

Proof. Since S is an anti-algebra and anti-coalgebra map and $\tau^n = 1$ by hypothesis,

$$\begin{aligned}S\big(x^{[n,\tau]}\big) &= S\left(\sum_x x_1(\tau \cdot x_2)(\tau^2 \cdot x_3)\cdots(\tau^{n-1}\cdot x_n)\right) \\ &= \sum_x (\tau^{n-1}\cdot S(x_n))(\tau^{n-2}\cdot S(x_{n-1}))\cdots(\tau^2\cdot S(x_3))(\tau\cdot S(x_2))S(x_1) \\ &= \sum_x (\tau^{-1}\cdot S(x_n))(\tau^{-2}\cdot S(x_{n-1}))\cdots(\tau^{-1(n-2)}\cdot S(x_3))(\tau^{-1(n-1)}\cdot S(x_2))S(x_1) \\ &= \tau^{-1}\cdot\left(\sum_x S(x_n)(\tau^{-1}\cdot S(x_{n-1}))\cdots(\tau^{-1(n-3)}\cdot S(x_3))(\tau^{-1(n-2)}\cdot S(x_2))(\tau^{-1(n-1)}\cdot S(x_1))\right) \\ &= \tau^{-1}\cdot\big(S(x)^{[n,\tau^{-1}]}\big).\end{aligned}$$

□

Corollary 3.6. *Let H be a Hopf algebra for which $S^2 = id$ and let τ be a Hopf automorphism of H. Then $exp_{\tau^{-1}}(H) = exp_\tau(H)$.*

Proof. It is clear from Lemma 3.5 that $x^{[n,\tau]} = \varepsilon(x)1 \iff S(x)^{[n,\tau^{-1}]} = \varepsilon(x)1$ since τ and S are bijective. Thus the two twisted exponents are the same. □

Question 3.7. We ask if Corollary 3.6 is true more generally. That is, if the order of τ is n and m is relatively prime to n, then is $exp_{\tau^m}(H) = exp_\tau(H)$?

4. Modules and Frobenius-Schur indicators

In this section, we assume A is a semisimple Hopf algebra, and thus we may assume that Λ_A is a normalized integral, that is, $\varepsilon(\Lambda_A) = 1$. Then the integral $\Lambda_K = \Lambda_A \natural p_1$ of $K = A \natural k^G$ is a normalized integral of K.

For any (left) K-module M, we may write

$$M = \bigoplus_{x \in G} M_x$$

where $M_x = p_x \cdot M$ is a K-submodule of M for each $x \in G$. Note that each M_x is also an A-module, by restricting the action to A.

Let ν_m^K denote the mth *Frobenius-Schur indicator* for K-modules as in [KSZ], and let $\nu_{m,x}^A$ denote the mth *twisted Frobenius-Schur indicator* for A-modules, twisted by x, as in [SV]. That is, if V is a K-module with character (or trace function) χ_V, then

$$\nu_m^K(V) = \chi_V(\Lambda_K^{[m]}).$$

If W is an A-module with character χ_W and x is an automorphism of A whose order divides m, then

$$\nu_{m,x}^A(W) = \chi_W(\Lambda_A^{[m,x]}).$$

See [SV] for general results on twisted indicators and for computations of $\nu_{m,x}^A$ when $A = H_8$, the smallest semisimple noncommutative, noncocommutative Hopf algebra.

Our next theorem gives a relationship between the Frobenius-Schur indicators of K and the twisted Frobenius-Schur indicators of A.

Theorem 4.1. *For every K-module M,*

$$\nu_m^K(M) = \sum_{x \in G,\ x^m = 1} \nu_{m,x^{-1}}^A(M_x).$$

Proof. Write $M = \oplus_{x \in G} M_x$ as before. Then $\nu_m^K(M) = \sum_{x \in G} \nu_m^K(M_x)$, and we will now compute $\nu_m^K(M_x)$ for an element x of G, writing $\Lambda = \Lambda_A$ for ease of notation: By Lemma 3.3,

$$\begin{aligned} \nu_m^K(M_x) &= \chi_{M_x}(\Lambda_K^{[m]}) \\ &= \chi_{M_x}\Big(\sum_{z \in G,\ z^m = 1} \Lambda^{[m,z]} \natural p_{z^{-1}}\Big) \\ &= \delta_{x^m,1} \chi_{M_x}(\Lambda^{[m,x^{-1}]}) \;=\; \delta_{x^m,1} \nu_{m,x^{-1}}^A(M_x). \end{aligned}$$

Summing over all elements of G, we obtain the stated formula. □

As a consequence, for example, if x is an element of G of order n and M is a K-module for which $M = M_x$ (i.e. $M_y = 0$ for all $y \neq x$), then $\nu_m^K(M) = 0$ for all $m < n$.

In our next result, we show that a twisted Frobenius-Schur indicator may always be realized as a Frobenius-Schur indicator for a smash coproduct. Let τ be any Hopf automorphism of A of finite order n, and let $G = \langle \tau \rangle$ be the cyclic subgroup of the automorphism group generated by τ. Set $K = A \natural k^G$.

Theorem 4.2. *For any A-module N, extend N to be a K-module M by letting $M_{\tau^{-1}} = N$ and $M_x = 0$ for all $x \in G$, $x \neq \tau^{-1}$. Then for every positive integer multiple m of n,*

$$\nu_{m,\tau}^A(N) = \nu_m^K(M).$$

Thus every value of a twisted indicator for A is the value of an ordinary indicator for a smash coproduct over A.

Proof. By Theorem 4.1,

$$\nu_m^K(M) = \sum_{x \in G,\ x^m=1} \nu^A_{m,x^{-1}}(M_x) = \nu^A_{m,\tau}(M_{\tau^{-1}}) = \nu^A_{m,\tau}(N).$$

□

Example 4.3. We illustrate the theorem using a non-trivial automorphism of $A = H_8$, the Kac-Palyutkin algebra of dimension 8 which is neither commutative nor cocommutative. The Hopf automorphism group was found in [SV], Section 4.2. Let A be generated by x, y, z with the usual relations $x^2 = y^2 = 1$, $z^2 = \frac{1}{2}(1 + x + y - xy)$, $xy = yx$, $xz = zy$ and $yz = zx$, where x, y are group-like and $\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$.

Let $\tau = \tau_4$ be the automorphism of A of order 2 that interchanges x and y and sends z to $\frac{1}{2}(-z + xz + yz + xyz)$, and let χ be the character of the unique two-dimensional simple module N of A. Then from [SV], $\nu^A_{2,\tau}(N) = -1$.

Letting $G = \langle \tau \rangle$ and $K = A \natural k^G$, N becomes a K-module M by setting $M_\tau = N$ and $M_1 = 0$. Then $\nu_2^K(M) = -1$.

5. Frobenius-Schur indicators for non-semisimple Hopf algebras

Let A be a finite-dimensional Hopf algebra that is not necessarily semisimple and for which S^2 is not necessarily the identity map. When A is not semisimple, there does not exist a normalized integral, and so we cannot use the definition of indicator from the previous section. Instead we extend the work in [KMN] and define twisted Frobenius-Schur indicators for A itself and obtain connections to Frobenius-Schur indicators of smash coproducts. Fix τ, a Hopf automorphism of A whose order divides the positive integer m. We define a variant of the mth twisted Hopf power map of A to be $P_{m-1,\tau} : A \to A$, given by

$$P_{m-1,\tau}(a) = \sum_a (\tau^{m-1} \cdot a_1)(\tau^{m-2} \cdot a_2) \cdots (\tau^2 \cdot a_{m-2})(\tau \cdot a_{m-1})$$

for all $a \in A$. We will use this map to define twisted Frobenius-Schur indicators, and then we will show how it relates to the twisted Hopf power maps defined in Section 3, by giving equivalent definitions of twisted Frobenius-Schur indicators in Theorem 5.1 and Corollary 5.2.

The mth *twisted Frobenius-Schur indicator* of A is

$$\nu_{m,\tau}(A) := \mathrm{Tr}(S \circ P_{m-1,\tau}),$$

the trace of the map $S \circ P_{m-1,\tau}$ from A to A, where S is the antipode of A.

We choose this definition as it specializes to the definition of the Frobenius-Schur indicator of the regular representation A for an arbitrary finite-dimensional Hopf algebra in [KMN] when τ is the identity, and also to the definition of twisted Frobenius-Schur indicators in the semisimple case given in [SV, Theorem 5.1]. The indicator of the regular representation has also been considered in [Sh].

The following theorem generalizes part of [KMN, Theorem 2.2].

Theorem 5.1. *Let Λ be a left integral of A and λ a right integral of A^* for which $\lambda(\Lambda) = 1$. Then*

$$\nu_{m,\tau}(A) = \lambda(S(\Lambda)^{[m,\tau]}).$$

Proof. By [R, Theorem 1],

$$\begin{aligned}
\mathrm{Tr}(S \circ P_{m-1,\tau}) &= \sum \lambda(S(\Lambda_2) S \circ P_{m-1,\tau}(\Lambda_1)) \\
&= \sum \lambda(S(\Lambda_m) S((\tau^{m-1} \cdot \Lambda_1)(\tau^{m-2} \cdot \Lambda_2) \cdots (\tau \cdot \Lambda_{m-1})) \\
&= \sum \lambda(S(\Lambda_m)(\tau \cdot S(\Lambda_{m-1})) \cdots (\tau^{m-1} \cdot S(\Lambda_1))) \\
&= \sum \lambda(S(\Lambda)_1 (\tau \cdot S(\Lambda)_2) \cdots (\tau^{m-1} \cdot S(\Lambda)_m)) \quad = \quad \lambda(S(\Lambda)^{[m,\tau]}).
\end{aligned}$$

□

A similar proof to that of [KMN, Corollary 2.6] yields the following result that will be useful for computations.

Corollary 5.2. *Let Λ_r be a right integral of A and λ_r be a right integral of A^* for which $\lambda_r(\Lambda_r) = 1$. Then*

$$\nu_{m,\tau}(A) = \lambda_r(\Lambda_r^{[m,\tau]}).$$

Similarly let Λ_l be a left integral of A and λ_l be a left integral of A^* for which $\lambda_l(\Lambda_l) = 1$. Then

$$\nu_{m,\tau}(A) = \lambda_l(\tau^{-1} \cdot \Lambda_l^{[m,\tau^{-1}]}).$$

Proof. The first statement follows immediately from Theorem 5.1 and the fact that if Λ_l is a left integral, then $\Lambda_r := S(\Lambda_l)$ is a right integral, and the value of λ_r on each is the same.

For the second statement, if λ_r is a right integral, let $\lambda_l := \lambda_r \circ S$, a left integral of A^*. Then again by Theorem 5.1 and also Lemma 3.5,

$$\begin{aligned}
\lambda_l(\tau^{-1} \cdot \Lambda_l^{[m,\tau^{-1}]}) &= \lambda_r(S(\tau^{-1} \cdot \Lambda_l^{[m,\tau^{-1}]})) \\
&= \lambda_r(\tau^{-1} \cdot (S(\Lambda_l^{[m,\tau^{-1}]}))) \\
&= \lambda_r(S(\Lambda_l)^{[m,\tau]}) \quad = \quad \lambda_r(\Lambda_r^{[m,\tau]}).
\end{aligned}$$

□

Now let G be a group of Hopf algebra automorphisms of A, as in Section 2. The next result is a connection between twisted indicators of A and indicators of the smash coproduct $K = A \natural k^G$.

Theorem 5.3. $\nu_m(K) = \sum_{g \in G,\ g^m = 1} \nu_{m,g}(A).$

Proof. Note that $\Lambda_K = \Lambda \natural p_1$ and $\lambda_{K^*} = \lambda \otimes (\sum_{z \in G} z)$ (since e.g. $\varepsilon(z \cdot a) = \varepsilon(a)$). By [KMN, Theorem 2.2] and our Lemmas 3.3 and 3.5,

$$\begin{aligned}
\nu_m(K) &= \lambda_{K^*}(S_K(\Lambda_K^{[m]})) \\
&= \left(\lambda \otimes \big(\sum_{z \in G} z\big)\right) \left(S_K\big(\sum_{g \in G,\ g^m = 1} \Lambda^{[m,g]} \otimes p_{g^{-1}}\big)\right) \\
&= \left(\lambda \otimes \big(\sum_{z \in G} z\big)\right) \left(S_K\big(\sum_{g \in G,\ g^m = 1} \Lambda_1 (g \cdot \Lambda_2) \cdots (g^{m-1} \cdot \Lambda_m)\big) \otimes p_{g^{-1}}\right) \\
&= \sum_{g \in G,\ g^m = 1} \lambda(g \cdot S(\Lambda_1 (g \cdot \Lambda_2) \cdots (g^{m-1} \cdot \Lambda_m))) \\
&= \sum_{g \in G,\ g^m = 1} \lambda(S(\Lambda)^{[m, g^{-1}]}) \\
&= \sum_{g \in G,\ g^m = 1} \nu_{m, g^{-1}}(A) \quad = \sum_{g \in G,\ g^m = 1} \nu_{m,g}(A).
\end{aligned}$$

□

In the next section, we compute an example, a non-semisimple Hopf algebra of dimension 8 and its Hopf automorphism group.

6. A non-semisimple example

Let A be the Hopf algebra defined as

$$A = k\langle g, x, y \mid gx = -xg,\ gy = -yg,\ xy = -yx,\ g^2 = 1,\ x^2 = y^2 = 0 \rangle$$

with coalgebra structure given by:

$$\Delta(g) = g \otimes g,\ \varepsilon(g) = 1,\ S(g) = g,$$
$$\Delta(x) = x \otimes g + 1 \otimes x,\ \varepsilon(x) = 0,\ S(x) = gx,$$
$$\Delta(y) = y \otimes g + 1 \otimes y,\ \varepsilon(y) = 0,\ S(y) = gy.$$

The element $\Lambda = xy + xyg$ is both a right and left integral for A, and $\lambda = (xy)^*$ is both a right and left integral for A^* such that $\lambda(\Lambda) = 1$.

Lemma 6.1. *Let V be the k-span of x and y. Then $Aut(A) \cong Gl_2(V)$.*

Proof. This is close to the examples considered in [AD], as A is pointed and generated by its group-like and skew-primitive elements. However we provide an elementary proof for completeness.

The coradical of A is given by $A_0 = k\langle g \rangle$. Any automorphism τ of A stabilizes A_0 and so fixes g. The next term of the coradical filtration is

$$A_1 = A_0 \oplus V \oplus gV,$$

since V is the set of $(g,1)$-primitives and gV is the set of $(1,g)$-primitives. Consequently V and gV are each stable under the action of τ. But the τ-action on V determines the τ-action on gV, and also on $A = A_1 \oplus W$, where W is the span of xy and gxy.

Conversely it is easy to check that any invertible linear action on V preserves all of the relations of A, and thus gives an automorphism. □

For an automorphism τ of order 2 or 3, we are able to compute some values of the indicators, using Corollary 5.2. We identify τ with a matrix

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\tau}{\longmapsto} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $a, b, c, d \in k$, such that $\mathrm{Det}(\tau) = ad - bc \neq 0$.

Proposition 6.2. *Case (1). If $\tau^2 = 1$ and m is even, then $\nu_{m,\tau}(A) = \frac{m^2}{2}\big(1 + \mathrm{Det}(\tau)\big)$. Consequently,*

$$\nu_{m,\tau}(A) = \begin{cases} m^2, & \textit{if } \mathrm{Det}(\tau) = 1 \\ 0, & \textit{if } \mathrm{Det}(\tau) = -1. \end{cases}$$

Case (2). If $\tau^3 = 1$, then $\nu_{3,\tau}(A) = (\mathrm{Tr}(\tau) + \mathrm{Det}(\tau))^2 + (\mathrm{Tr}(\tau) + 1)(1 - \mathrm{Det}(\tau))$. Consequently,

$$\nu_{3,\tau}(A) = \begin{cases} 9, & \textit{if } \tau = id \\ 0, & \textit{if } \tau \neq id. \end{cases}$$

Proof. We verify the formulas by using the first part of Corollary 5.2.

Case (1): Recall that $\Lambda = xy + xyg$ and $\lambda = (xy)^*$ are right integrals. We must find $\lambda(\Lambda^{[m,\tau]})$. First we will show that $\lambda((xy)^{[m,\tau]}) = \frac{m^2}{4}(1 + \mathrm{Det}(\tau))$, and then we will argue that $\lambda((xyg)^{[m,\tau]}) = \lambda((xy)^{[m,\tau]})$. In order to find $(xy)^{[m,\tau]}$, first note that

$$\begin{aligned} \Delta^{m-1}(x) &= x \otimes g^{\otimes^{m-1}} + 1 \otimes x \otimes g^{\otimes^{m-2}} + \cdots + 1^{\otimes^{i-1}} \otimes x \otimes g^{\otimes^{m-i}} + \cdots + 1^{\otimes^{m-1}} \otimes x, \\ \Delta^{m-1}(y) &= y \otimes g^{\otimes^{m-1}} + 1 \otimes y \otimes g^{\otimes^{m-2}} + \cdots + 1^{\otimes^{i-1}} \otimes y \otimes g^{\otimes^{m-i}} + \cdots + 1^{\otimes^{m-1}} \otimes y, \end{aligned}$$

each sum consisting of m terms. Set

$$x_1 = x \otimes g^{\otimes^{m-1}}, \ldots,\ x_i = 1^{\otimes^{i-1}} \otimes x \otimes g^{\otimes^{m-i}}, \ldots,\ x_m = 1^{\otimes^{m-1}} \otimes x,$$

the index indicating the position of x in the tensor product, and similarly define $y_1, y_2, \ldots, y_m$. Letting μ denote the multiplication map, by definition we have

$$(xy)^{[m,\tau]} = \mu((1 \otimes \tau)^{\otimes^{\frac{m}{2}}}) \left(\sum_{i,j=1}^{m} x_i y_j \right).$$

Since $\tau \cdot g = g$ and $\lambda = (xy)^*$, in computing $\lambda((xy)^{[m,\tau]})$, the only terms in the above expansion of $(xy)^{[m,\tau]}$ yielding a nonzero value of λ are those with an even number of factors of g. These are precisely the terms $x_i y_j$ for which i, j have the same parity, of which there are $\frac{m^2}{2}$ terms. If i, j are both odd (of which there are $\frac{m^2}{4}$ pairs), then in $(xy)^{[m,\tau]}$, the (i, j) term is simply xy by the following observations: (1) τ is applied only to factors of g or 1, which are fixed by τ, (2) if $i \leq j$, there are an even number of factors of g between x and y after applying μ, and (3) if $i > j$, there are an odd number of factors of g between x and y after applying μ (since x_i is to the left of y_j), so moving factors of g to the right, past x, results in a factor of (-1), and then applying the relation $yx = -xy$ results in another factor of (-1), so that the end result is a term xy. If i, j are both even (of which there are $\frac{m^2}{4}$ pairs), then in $(xy)^{[m,\tau]}$, the (i, j) term is $\tau \cdot xy = \mathrm{Det}(\tau) xy$, by similar reasoning. Therefore

$$\lambda((xy)^{[m,\tau]}) = \lambda \left(\frac{m^2}{4} xy + \frac{m^2}{4} \mathrm{Det}(\tau) xy \right) = \frac{m^2}{4} (1 + \mathrm{Det}(\tau)).$$

Finally, in order to compute $\lambda((xyg)^{[m,\tau]})$, note that we need only include an extra factor of $g^{\otimes m}$ on the right:

$$(xyg)^{[m,\tau]} = \mu((1 \otimes \tau)^{\otimes \frac{m}{2}}) \left(\sum_{i,j=1}^{m} x_i y_j \right) (g^{\otimes m}).$$

Since m is even, the number of new factors of g to be included, in comparison to our previous calculation, is even, and so a similar analysis applies. One checks that the extra factors of g do not affect the result, and so

$$\lambda((xyg)^{[m,\tau]}) = \lambda((xy)^{[m,\tau]}) = \frac{m^2}{4} (1 + \mathrm{Det}(\tau)).$$

Consequently, $\nu_{m,\tau}(A) = \lambda\big(\Lambda^{[m,\tau]}\big) = \dfrac{m^2}{2} (1 + \mathrm{Det}(\tau))$.

To see the conclusion of Case (1), note that since $\tau^2 = 1$, the determinant of τ is either 1 or -1.

Case (2): A similar analysis applies. Note that $\lambda((xy)^{[3,\tau]}) = \mu(1 \otimes \tau \otimes \tau^2)(\sum_{i,j=1}^{3} x_i y_j)$ and that $\tau^2(x) = (a^2 + bc)x + b(a + d)y$, $\tau^2(y) = c(a + d)x + (d^2 + bc)y$. In evaluating $\lambda((xy)^{[3,\tau]})$, we again need only consider (i, j) terms for which i, j have the same parity. By contrast, in evaluating $\lambda((xyg)^{[3,\tau]})$, we need only consider (i, j) terms for which i, j have different parity. Thus we find

$$\begin{aligned}
\lambda((xy)^{[3,\tau]}) &= \lambda\left(xy + x(\tau^2 \cdot y) + yg(\tau^2 \cdot x)g + (\tau^2 \cdot xy) + (\tau \cdot xy)\right) \\
&= 1 + (d^2 + bc) + (a^2 + bc) + (a^2 + bc)(d^2 + bc) - bc(a + d)^2 + (ad - bc), \\
\lambda((xyg)^{[3,\tau]}) &= \lambda\left(xg^2(\tau \cdot y)g^4 + yg(\tau \cdot x)g^5 + g(\tau \cdot x)g^2(\tau^2 \cdot y)g + g(\tau \cdot y)g(\tau^2 \cdot x)g^2\right) \\
&= d + a + a(d^2 + bc) - bc(a + d) - bc(a + d) + d(a^2 + bc).
\end{aligned}$$

Adding these together, we have

$$\begin{aligned}
\lambda(\Lambda^{[3,\tau]}) &= 1 + a + d + a^2 + ad + d^2 + a^2d + ad^2 + a^2d^2 - abc - 2abcd - bcd + bc + b^2c^2 \\
&= (\mathrm{Tr}(\tau) + \mathrm{Det}(\tau))^2 + (1 + \mathrm{Tr}(\tau))(1 - \mathrm{Det}(\tau)).
\end{aligned}$$

To see the conclusion in Case (2), one can check the possible Jordan forms of the matrix for τ. □

7. Tensor products and category of modules

The following theorem generalizes [BW, Theorem 2.1] from the case that A is a group algebra, to the case that A is a Hopf algebra. Let $K = A \natural k^G$ as before, and recall that for M a K-module and $x \in G$, M_x denotes $p_x \cdot M$, a K-submodule of M, and $M = \oplus_{x \in G} M_x$. If $y \in G$, define ${}^yM_x$ to be M_x as a vector space, with A-module structure given by $a \cdot_y m = (y^{-1} \cdot a) \cdot m$ for all $a \in A$, $m \in M$.

Theorem 7.1. *Let M, N be K-modules. Then*

(i) $(M \otimes N)_x \cong \bigoplus_{\substack{y,z \in G \\ yz = x}} M_y \otimes {}^yN_z$, *and*

(ii) $(M^*)_x = {}^x(M_{x^{-1}})^*$.

Proof. The proof is a straightforward generalization of that of [BW, Theorem 2.1]. We include details for completeness. We will prove the statement for modules of the form $M = M_y$, $N = N_z$. Let $\phi : M_y \otimes N_z \to M_y \otimes {}^y N_z$, where the target module is a K-module on which p_{yz} acts as the identity and p_w acts as 0 for $w \neq yz$, be defined by $\phi(m \otimes n) = m \otimes n$ for all $m \in M_y$, $n \in N_z$. We check that ϕ is a K-module homomorphism: Let $x \in G$, $a \in A$. Apply Δ to $a \natural p_x$ to obtain

$$\phi((a \natural p_x)(m \otimes n)) = \sum \delta_{x,yz} \phi(a_1 m \otimes (y^{-1} \cdot a_2) n).$$

On the other hand,

$$(a \natural p_x)\phi(m \otimes n) = \sum \delta_{x,yz} a_1 m \otimes (y^{-1} \cdot a_2) n.$$

As ϕ is a bijection by its definition, it is an isomorphism of K-modules.

We will prove that since $M = M_y$, its dual satisfies $M^* = (M^*)_{y^{-1}}$, and that the corresponding underlying A-module structure on the vector space $(M^*)_{y^{-1}}$ is isomorphic to ${}^{y^{-1}}(M_y)^*$. To see this, first let $x \in G$, $f \in M^*$, and $m \in M$. Then

$$((1 \natural p_x)(f))(m) = f((1 \natural p_{x^{-1}})m) = \delta_{x^{-1},y} f(m).$$

It follows that $(M_y)^* = M^* = (M^*)_{y^{-1}}$, as claimed. The A-module structure on $(M^*)_{y^{-1}}$ may be determined by considering the action on M^* of all elements of K of the form $a \natural p_{y^{-1}}$ where $a \in A$. Let $f \in M^*$ and $m \in M$. Then

$$((a \natural p_{y^{-1}})(f))(m) = f(S(a \natural p_{y^{-1}})m) = f((y \cdot S(a))m).$$

Considering the restriction of $M^* = (M^*)_{y^{-1}}$ to an A-module in this way, we see that the action of a on the vector space $(M_y)^*$ is that of a on the A-module ${}^{y^{-1}}(M_y)^*$:

$$(a \cdot_{y^{-1}} f)(m) = ((y \cdot a)f)(m) = f(S(y \cdot a)m) = f((y \cdot S(a))m).$$

Therefore the A-module structure on the vector space $(M^*)_{y^{-1}}$ is that of the A-module ${}^{y^{-1}}(M_y)^*$. □

Remark 7.2. As a consequence of the theorem, the category of K-modules is equivalent to the *semidirect product tensor category* $\mathcal{C} \rtimes G$ where $\mathcal{C}$ is the category of A-modules. By definition, $\mathcal{C} \rtimes G$ is the category $\oplus_{g \in G} \mathcal{C}$, with objects $\oplus_{g \in G}(M_g, g)$ where each M_g is an object of $\mathcal{C}$, and tensor product $(M, g) \otimes (N, h) = (M \otimes {}^g N, gh)$. See [T], where the notation $\mathcal{C}[G]$ is used instead for this semidirect product category. For other occurrences of $\mathcal{C} \rtimes G$ in the literature, see, for example, [GNaNi, Ni].

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