COLOR LIE RINGS AND PBW DEFORMATIONS OF SKEW GROUP ALGEBRAS

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Abstract. We investigate color Lie rings over finite group algebras and their universal enveloping algebras. We exhibit these universal enveloping algebras as PBW deformations of skew group algebras: Every color Lie ring over a finite group algebra with a particular Yetter-Drinfeld structure has a universal enveloping algebra that is a quantum Drinfeld orbifold algebra. Conversely, every quantum Drinfeld orbifold algebra of a particular type arising from the action of an abelian group is the universal enveloping algebra of some color Lie ring over the group algebra. One consequence is that these quantum Drinfeld orbifold algebras are braided Hopf algebras.

1. Introduction

We examine color Lie rings arising from finite groups acting linearly on finite dimensional vector spaces. We show that color Lie rings with Yetter-Drinfeld structures have universal enveloping algebras that are PBW deformations of skew group rings under a mild condition on their colors. We consider only abelian groups since antisymmetry of the Lie bracket implies that any grading group is abelian. Specifically, we consider a color Lie ring arising from a finite abelian group G acting diagonally on a finite dimensional vector space $V = \mathbb{K}^n$ over a field $\mathbb{K}$, and we exhibit its universal enveloping algebra as a quantum Drinfeld orbifold algebra.

Quantum Drinfeld orbifold algebras [15] generally are filtered PBW deformations of skew group algebras $S_{\mathbf{q}}(V) \rtimes G$ formed by finite groups G acting on quantum symmetric algebras (skew polynomial rings) $S_{\mathbf{q}}(V)$ by graded automorphisms. These types of deformations include as special cases, for example, the

- universal enveloping algebras of Lie algebras (with $\mathbf{q}$ and G both trivial),
- symplectic reflection algebras (with $\mathbf{q}$ trivial and G symplectic) [2],
- graded affine Hecke algebras and generalizations to quantum polynomial rings (with $\mathbf{q}$ and G nontrivial) [8, 9], and
- generalized universal enveloping algebras of color Lie algebras (with G trivial) [16].

When the group G is nontrivial, previous work has concentrated on the case of nonhomogeneous quadratic algebras whose defining relations set commutators of vector space elements equal to elements in the group algebra $\mathbb{K}G$, i.e., $[v, v] \in \mathbb{K}G$ in the algebra. Here we require commutators of vector space elements to equal elements in the space $V \otimes \mathbb{K}G$; our algebras then will be direct generalizations of universal enveloping algebras of Lie algebras (the origin of the term "PBW deformation").

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We examine closely this latter type of PBW deformation of $S_{\mathbf{q}}(V) \rtimes G$ in characteristic 0, searching for Lie structures. In Section 2, starting with an abelian group A, an antisymmetric bicharacter ϵ on A, and a ring R, we define the notion of an (A,ϵ)-*color Lie ring* over R and its universal enveloping algebra. When a finite abelian group G acts *diagonally* on $V = \mathbb{K}^n$ with basis $v_1, \ldots, v_n$, the space

$$L = V \otimes \mathbb{K}G = \bigoplus_{i,g} \mathbb{K}(v_i \otimes g)$$

is naturally graded as a vector space by the abelian group $A = \mathbb{Z}^n \times G$. We define an antisymmetric bicharacter on A and show that $L = V \otimes \mathbb{K}G$ is a Yetter-Drinfeld module over $\mathbb{K}G$, setting the stage for the *Yetter-Drinfeld color Lie rings* defined in Section 2 and featured in later sections.

In Section 3, we define deformations of $S_{\mathbf{q}}(V) \rtimes G$ corresponding to parameter functions $\kappa : V \times V \to V \otimes \mathbb{K}G$:

$$\mathcal{H}_{\mathbf{q},\kappa} := (T(V) \rtimes G) / \big(v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) : 1 \leq i, j \leq n \big).$$

We analyze conditions for $\mathcal{H}_{\mathbf{q},\kappa}$ to satisfy the PBW property, i.e., to be a PBW deformation of $S_{\mathbf{q}}(V) \rtimes G$. Then we present several examples that meet these conditions on κ, the group action, and the parameter set $\mathbf{q} = \{q_{ij}\}$. Under these conditions, we call $\mathcal{H}_{\mathbf{q},\kappa}$ a *quantum Drinfeld orbifold algebra*. In Sections 4 and 5, we determine conditions on the algebra $\mathcal{H}_{\mathbf{q},\kappa}$ under which an underlying subspace $L = V \otimes \mathbb{K}G$ is a color Lie ring over $\mathbb{K}G$, generalizing a result in [16] to nontrivial G. We show that $\mathcal{H}_{\mathbf{q},\kappa}$ is isomorphic to the universal enveloping algebra $\mathcal{U}(L)$ and, in Section 6, exhibit a Hopf algebra structure on $\mathcal{H}_{\mathbf{q},\kappa}$.

Example. Let $V = \mathbb{K}^3$ with basis v_1, v_2, v_3 and suppose $\mathbb{K}$ contains a primitive m-th root of unity q for some $m \geq 2$. Let g be the diagonal matrix $\operatorname{diag}(q,1,1)$ and set $G = \langle g \rangle$. Then $L = V \otimes \mathbb{K}G$ is a *color Lie ring* over $\mathbb{K}G$ with brackets determined by

$$[v_1 \otimes 1, v_2 \otimes 1] = v_1 \otimes g, \quad [v_1 \otimes 1, v_3 \otimes 1] = 0, \quad [v_2 \otimes 1, v_3 \otimes 1] = 0\,.$$

The universal enveloping algebra $\mathcal{U}(L)$ is a PBW deformation of $S_{\mathbf{q}}(V) \rtimes G$ for a choice of parameter set $\mathbf{q}$. Details and more examples are in Section 3.

Color Lie rings carry an additional grading by $\mathbb{Z}/2\mathbb{Z}$, and we may speak of their positive and negative parts with respect to this grading. In Section 7, we determine how color Lie rings with only positive part define deformations of the noncommutative algebra $S_{\mathbf{q}}(V) \rtimes G$. The following result is our main Theorem 7.4.

Theorem. *Let G be a finite group of diagonal matrices acting on $V = \mathbb{K}^n$.*

(i) *The universal enveloping algebra of any Yetter-Drinfeld color Lie ring with purely positive part is isomorphic to a quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$ for some parameters $\mathbf{q}$ and κ.*

(ii) *Any quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$ with $\kappa : V \times V \to V \otimes \mathbb{K}G$ satisfying an additional condition is isomorphic to the universal enveloping algebra $\mathcal{U}(L)$ of some Yetter-Drinfeld color Lie ring L.*

Part (i) shows that certain color Lie rings L over $\mathbb{K}G$ have universal enveloping algebras $\mathcal{U}(L)$ defining PBW deformations of $S_{\mathbf{q}}(V) \rtimes G$. This immediately implies a PBW theorem for these algebras. Part (ii) exhibits many quantum Drinfeld orbifold algebras as the universal enveloping algebras of color Lie rings. In Section 6, we regard color Lie rings L as Lie algebras in certain braided symmetric monoidal categories, and thus their universal enveloping algebras $\mathcal{U}(L)$ inherit the structure of *braided Hopf algebras*. An immediate consequence is that certain quantum Drinfeld orbifold algebras are braided Hopf algebras. We obtain the following corollaries (Corollary 6.7 and 7.3).

Corollary. *A quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$ with G abelian acting diagonally on V and $\kappa : V \times V \to V \otimes \mathbb{K}G$ satisfying an extra condition is a braided Hopf algebra.*

Corollary. *Let L be a Yetter-Drinfeld color Lie ring that is purely positive. Then its universal enveloping algebra $\mathcal{U}(L)$ has the PBW property.*

Throughout, $\mathbb{K}$ will be a field of characteristic 0 unless stated otherwise. An unadorned tensor symbol between modules is understood to be a tensor product over $\mathbb{K}$, that is, $\otimes = \otimes_{\mathbb{K}}$. The symbol $\mathbb{N}$ denotes the set of natural numbers, which we assume includes 0. In any algebra containing a group algebra $\mathbb{K}G$, we identify the unity of the field $\mathbb{K}$ with the identity of G: $1 = 1_{\mathbb{K}} = 1_G$.

2. Color Lie rings and universal enveloping algebras

In this section, we introduce color Lie rings and their universal enveloping algebras. Color Lie rings are a natural generalization of color Lie algebras (see, e.g., [6, 7, 13]).

Gradings. Let A be an abelian group. A $\mathbb{K}$-algebra R is A-graded if $R = \oplus_{a\in A} R_a$ as a vector space and $R_a R_b \subset R_{ab}$ for all $a, b \in A$. An *A-graded R-bimodule* is an R-bimodule L together with a decomposition $L = \oplus_{a\in A} L_a$ as a direct sum of vector spaces L_a such that $R_a L_b R_c \subset L_{abc}$ for all $a, b, c \in A$. If $0 \neq x \in L_a$, we write $|x| = a$ and say x is *A-homogeneous.* We define $|0|$ to be the identity 1_A of A (note $0 \in L_a$ for all $a \in A$).

If an abelian group A is grading a group algebra $R = \mathbb{K}G$, then we make the additional assumption that each g in G is A-homogeneous and we say that *G is graded by A.*

Color Lie rings. A function $\epsilon : A \times A \to \mathbb{K}^*$ is a *bicharacter* on an abelian group A if
$$\epsilon(a, bc) = \epsilon(a, b)\epsilon(a, c) \quad \text{and} \quad \epsilon(ab, c) = \epsilon(a, c)\epsilon(b, c) \qquad \text{for all } a, b, c \in A\ .$$
A bicharacter ϵ is *antisymmetric* if $\epsilon(b, a) = \epsilon(a, b)^{-1}$ for all $a, b \in A$. Note that any antisymmetric bicharacter ϵ satisfies (for all a, b in A)
$$\epsilon(a, a) = \pm 1, \quad \epsilon(a, 1_A) = \epsilon(1_A, a) = 1, \quad \epsilon(a, b^{-1}) = \epsilon(a^{-1}, b) = \epsilon(a, b)^{-1}\ .$$

Definition 2.1. Let A be an abelian group and let ϵ be an antisymmetric bicharacter on A. Let R be an A-graded $\mathbb{K}$-algebra. An *(A, ϵ)-color Lie ring* over R is an A-graded R-bimodule L equipped with an R-bilinear, R-balanced operation $[\ ,\ ]$ for which

(i) $[x, y] \subset L_{|x||y|}$ *(A-grading)*,

(ii) $[x, y] = -\epsilon(|x|, |y|)\ [y, x]$ *(ϵ-antisymmetry)*, and,

(iii) $0 = \epsilon(|z|, |x|)\, [x, [y, z]] + \epsilon(|x|, |y|)\, [y, [z, x]] + \epsilon(|y|, |z|)\, [z, [x, y]]$ *(ϵ-Jacobi identity)*

for all A-homogeneous $x, y, z \in L$; an *ungraded color Lie ring* only satisfies (ii) and (iii) above. (By R-balanced we mean that $[xr, y] = [x, ry]$ for all x, y in L and r in R.)

Color Lie rings generalize other Lie algebras:

- When $A = 1$, a color Lie ring is the usual notion of Lie ring over R.
- When $R = \mathbb{K}$ (a field), a color Lie ring is the usual notion of color Lie algebra or color Lie superalgebra.
- When $R = \mathbb{K}$ and $A = \mathbb{Z}/2\mathbb{Z}$, a color Lie ring is a Lie superalgebra. We give one such example next.

Note that parts (i) and (ii) in the definition of a color Lie ring L force the group A to be abelian since $[L_a, L_b] \subset L_{ab}$ and $[L_b, L_a] \subset L_{ba}$ for all a, b in A.

Example 2.2. Let $L = \mathfrak{gl}(1|1)$, the Lie superalgebra defined as follows (see, e.g., [5, 14]). Let $R = \mathbb{K}$ be a field of characteristic not 2. Let $A = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ and $\epsilon(a, b) = (-1)^{ab}$ for all a, b in A. Let L be the set of 2×2 matrices $M = (m_{ij})$ over $\mathbb{K}$, where for $M \neq 0$ we define $|M| = 0$ if $m_{12} = m_{21} = 0$, and $|M| = 1$ if $m_{11} = m_{22} = 0$. Define $[M, M']$ to be $MM' - \epsilon(|M|, |M'|)M'M$ if M, M' are A-homogeneous; then L is a color Lie ring. In fact, one can define a color Lie ring for any A-graded associative algebra B by replacing the usual commutator with the color bracket $[u, v] = uv - \epsilon(|u|, |v|)vu$ for homogeneous elements $u, v \in B$ ([7, §3, p. 723]). Note that in $L = \mathfrak{gl}(1|1)$ there are matrices M for which $[M, M] \neq 0$, for example, let $M = (m_{ij})$ with $m_{11} = m_{22} = 0$ and both $m_{12} \neq 0$ and $m_{21} \neq 0$. Also note that if $M = (m_{ij})$ with $m_{11} = m_{12} = m_{22} = 0$ and $m_{21} \neq 0$, then $[M, M] = 0$, but $\epsilon(|M|, |M|) = -1$, forcing $M^2 = 0$ in the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(1|1))$ (see Definition 2.3 below), so that $\mathcal{U}(\mathfrak{gl}(1|1))$ has nilpotent elements.

We will see later in Section 6 that color Lie rings are Lie algebras in symmetric monoidal categories.

Color universal enveloping algebras. There is a notion of universal enveloping algebra for color Lie rings:

Definition 2.3. Let L be a color Lie ring (or an ungraded color Lie ring) over R. Consider its tensor algebra $T_R(L) = \bigoplus_{n \geq 0} L^{\otimes_R n}$ and let J be the ideal in $T_R(L)$ defined by generators as

$$J := \left(\{u \otimes_R v - \epsilon(|u|, |v|)\; v \otimes_R u - [u, v] : \text{ A-homogeneous } u, v \text{ in } L\}\right).$$

The *universal enveloping algebra* of L is the quotient

$$\mathcal{U}(L) := T_R(L)/J.$$

Yetter-Drinfeld color Lie rings. A special case of color Lie ring arises from groups acting linearly on finite dimensional vector spaces. Suppose $V \cong \mathbb{K}^n$ and G is a finite group acting linearly on V. We use left superscript to denote the group action in order to distinguish it from multiplication in a corresponding skew group algebra, writing ${}^g v$ for the action of g in G on v in V.

Set $R = \mathbb{K}G$ and view $L = V \otimes \mathbb{K}G$ as an R-bimodule via the natural action

$$g(v \otimes h)g' = {}^g v \otimes ghg' \quad \text{for all } \; g, g', h \in G, \; v \in V. \tag{2.4}$$

When G is abelian, this bimodule structure gives rise to a special kind of color Lie ring over $\mathbb{K}G$, one that is compatible with a Yetter-Drinfeld structure, as we define next.

Definition 2.5. We say that an (A,ϵ)-color Lie ring $L = V \otimes \mathbb{K}G$ over $\mathbb{K}G$ (with $\mathbb{K}G$-bimodule structure as above) is *Yetter-Drinfeld* if

(i) the A-grading on L is induced from A-gradings on G and on V, i.e.,

$$|v \otimes g| = |v||g| \quad \text{for all } g \in G \text{ and } A\text{-homogeneous } v \text{ in } V, \text{ and}$$

(ii) ${}^{g}v = \epsilon(|g|,|v|)\, v$ for all g in G and A-homogeneous v in V.

We make the same definition for ungraded color Lie rings.

When a color Lie ring L is Yetter-Drinfeld, we may choose a basis $v_1,\dots,v_n$ of V which is A-homogeneous since V is A-graded as a $\mathbb{K}G$-bimodule. Condition (ii) in the definition then guarantees that the abelian group G acts diagonally with respect $v_1,\dots,v_n$.

Yetter-Drinfeld modules. We mention briefly the connection with Yetter-Drinfeld modules for context. For G any finite group, a *Yetter-Drinfeld* $\mathbb{K}G$*-module* is a $\mathbb{K}G$-module M that is also G-graded with $M = \oplus_{g\in G} M_g$ in such a way that ${}^{g}(M_h) = M_{ghg^{-1}}$ for all g,h in G. We may view $V\otimes\mathbb{K}G$ as a Yetter-Drinfeld module: It is G-graded with $(V\otimes\mathbb{K}G)_g = V\otimes\mathbb{K}g$ for all g in G, it is a (left) $\mathbb{K}G$-module via

$$ {}^{g}(v\otimes h) = {}^{g}v \otimes ghg^{-1} \quad \text{for all } \ g,h\in G \text{ and } v\in V\,, \tag{2.6}$$

and ${}^{g}((V\otimes\mathbb{K}G)_h) = (V\otimes\mathbb{K}G)_{ghg^{-1}}$ for all g,h in G. Thus one may consider the left action (2.6) to be an adjoint action of G.

3. Quantum Drinfeld orbifold algebras $\mathcal{H}_{\mathbf{q},\kappa}$

In this section, we consider some deformations of skew group algebras, prove a PBW theorem needed later, and give many new examples.

Quantum systems of parameters. We define a *quantum system of parameters* (or *quantum scalars* for short) to be a set of nonzero scalars $\mathbf{q} := \{q_{ij}\}$ with $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i,j $(1\le i,j\le n)$.

Quantum symmetric algebras. Let V be a $\mathbb{K}$-vector space of dimension n, and fix a basis $v_1,\dots,v_n$ of V. Let $\mathbf{q}$ be a quantum system of parameters. Recall that the *quantum symmetric algebra* (or skew polynomial ring) determined by $\mathbf{q}$ and $v_1,\dots,v_n$ is the $\mathbb{K}$-algebra

$$S_{\mathbf{q}}(V) := \mathbb{K}\langle v_1,\dots,v_n\rangle / (v_iv_j = q_{ij}v_jv_i \ \text{ for } 1\le i,j\le n)\,.$$

Note that $S_{\mathbf{q}}(V)$ has the *PBW property*: As a $\mathbb{K}$-vector space, $S_{\mathbf{q}}(V)$ has basis

$$\{v_1^{m_1}v_2^{m_2}\cdots v_n^{m_n} : m_i\in\mathbb{N}\}.$$

Skew group algebras. Let S be a $\mathbb{K}$-algebra on which a finite group G acts by automorphisms. The *skew group algebra* (or semidirect product) $S \rtimes G$ is the $\mathbb{K}$-algebra generated by S and G with multiplication given by

$$(r\ g) \cdot (s\ h) := r({}^g s)\ gh$$

for all $r, s \in S$ and $g, h \in G$. (As a $\mathbb{K}$-vector space, $S \rtimes G$ is isomorphic to $S \otimes \mathbb{K}G$.)

We will use this construction particularly in the following setting. Let G be a finite group, and let V be a $\mathbb{K}G$-module. Assume the linear action of G on V extends to an action by graded automorphisms on $S_{\mathbf{q}}(V)$ (letting $\deg v = 1$ for all $v \in V$). In other words, assume that in $S_{\mathbf{q}}(V)$,

$${}^g v_i\, {}^g v_j = q_{ij}\ {}^g v_j\, {}^g v_i \quad \text{for all } i, j \text{ and } g \in G\ .$$

We can then form the skew group algebra $S_{\mathbf{q}}(V) \rtimes G$. Groups acting diagonally on the choice $v_1, \ldots, v_n$ of basis of V always extend to an action on $S_{\mathbf{q}}(V)$, but many other group actions do *not* extend. For our main results, we will restrict to diagonal actions.

Parameter functions. Let $T(V) = T_{\mathbb{K}}(V)$ be the free algebra over $\mathbb{K}$ generated by V (i.e., the tensor algebra with tensor symbols suppressed). The group G acts on $T(V)$ by automorphisms (extending its action on V). Consider a quotient of the skew group algebra $T(V) \rtimes G$ by relations that lower the degree of the commutators $v_i v_j - q_{ij} v_j v_i$, viewing $\mathbb{K}G$ in degree zero. Specifically, consider a *parameter function*

$$\kappa : V \times V \to V \otimes \mathbb{K}G \tag{3.1}$$

that is bilinear and *quantum antisymmetric*, i.e.,

$$\kappa(v_i, v_j) = -q_{ij}\ \kappa(v_j, v_i)\,,$$

that records such relations. We decompose κ when convenient, writing

$$\kappa(v_i, v_j) = \sum_{g \in G} \kappa_g(v_i, v_j) g$$

with each $\kappa_g : V \times V \to V$. We note that imposing bilinearity is only for convenience; values of κ on the given basis determine the algebra $\mathcal{H}_{\mathbf{q},\kappa}$ defined by (3.2) below.

Quantum Drinfeld orbifold algebras. Given a quantum system of parameters $\mathbf{q}$ and a parameter function κ as in (3.1), let

$$\mathcal{H}_{\mathbf{q},\kappa} := (T(V) \rtimes G) / \big(v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) : 1 \le i, j \le n\big). \tag{3.2}$$

We say that $\mathcal{H}_{\mathbf{q},\kappa}$ satisfies the *PBW property* if the image of the set

$$\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} g : m_i \in \mathbb{N}, g \in G\}\,,$$

under the quotient map $T(V) \rtimes G \to \mathcal{H}_{\mathbf{q},\kappa}$, is a basis for $\mathcal{H}_{\mathbf{q},\kappa}$ as a $\mathbb{K}$-vector space. In this case, we say $\mathcal{H}_{\mathbf{q},\kappa}$ is a *PBW deformation* of $S_{\mathbf{q}}(V) \rtimes G$, and call $\mathcal{H}_{\mathbf{q},\kappa}$ a *quantum Drinfeld orbifold algebra*. We identify every $v \otimes g$ in $V \otimes \mathbb{K}G$ with its image vg in $\mathcal{H}_{\mathbf{q},\kappa}$.

PBW conditions. We derive necessary and sufficient conditions for $\mathcal{H}_{\mathbf{q},\kappa}$ to satisfy the PBW property in terms of properties of κ. Related PBW theorems appear in Shepler and Levandovskyy [8], Shepler and Witherspoon [12], and Shroff [15]. These previous theorems stated results in a way that allowed direct comparison with homological information, but here we obtain a PBW result in Corollary 3.12 of a different flavor in order to connect with color Lie rings.

In the theorem below, we extend the parameter function κ from the domain $V \times V$ to the domain $(V \otimes \mathbb{K}G) \times (V \otimes \mathbb{K}G)$ by setting

$$\kappa(vg, wh) = \kappa(v, {}^g w)gh \quad \text{for all } g, h \in G \text{ and } \; v, w \in V.$$

We say that κ is *G-invariant* in Condition (1) of the theorem below if it is invariant with respect to the *adjoint action* of the abelian group G on the bimodule $V \otimes \mathbb{K}G$ (as in Equation (2.6)):

$$ {}^g(v \otimes h) = {}^g v \otimes ghg^{-1} = {}^g v \otimes h \quad \text{for all } g, h \in G \text{ and } v \in V. \tag{3.3}$$

We record the action of G acting diagonally on a fixed basis $v_1, \ldots, v_n$ of V with linear characters $\chi_i : G \to \mathbb{K}^*$ giving the i-th diagonal entries:

$$ {}^g v_i = \chi_i(g)\; v_i \quad \text{for } 1 \leq i \leq n.$$

Theorem 3.4. *Let G be an abelian group acting diagonally on V with respect to a $\mathbb{K}$-basis $v_1, \ldots, v_n$. Let $\kappa : V \times V \to V \otimes \mathbb{K}G$ be a parameter function defining the algebra $\mathcal{H}_{\mathbf{q},\kappa}$ as in (3.2), and write $\kappa(v_i, v_j) = \sum_{1 \leq r \leq n,\, g \in G} c_r^{ijg}\; v_r \otimes g$ in $V \otimes \mathbb{K}G$ for some $c_r^{ijg} \in \mathbb{K}$. Then $\mathcal{H}_{\mathbf{q},\kappa}$ satisfies the PBW property if and only if*

(1) *the function κ is G-invariant,*
(2) *for all g in G and all distinct $1 \leq i, j, k \leq n$,*

$$\begin{aligned} 0 &= \big(\chi_k(g) - q_{jk}\, q_{ik}\, q_{kr}\big) c_r^{ijg} \qquad \text{for any } r \neq i, j\,, \\ 0 &= \big(1 - q_{ij}\chi_i(g)\big)\, c_k^{jkg} + \big(q_{jk} - \chi_k(g)\big)\, c_i^{ijg}\,, \qquad \text{and} \end{aligned}$$

(3) *for all g in G and all distinct $1 \leq i, j, k \leq n$,*

$$0 = \sum_{\circlearrowleft} \big(\chi_k(g) - q_{jk}\big)\, c_i^{ijg} \kappa(v_k, v_i) \; + \; \sum_{\circlearrowleft} q_{jk}\; \kappa\big(v_k, \kappa_g(v_i, v_j)\big),$$

where the sums are over all cyclic permutations of i, j, k.

Proof. We use the Diamond-Composition Lemma as outlined in [8], [12], and [15] to write elements of $\mathcal{H}_{\mathbf{q},\kappa}$ in PBW form, up to cyclic permutation. We consider "overlaps" arising from $0 = {}^h(v_i v_j) - ({}^h v_i)({}^h v_j)$ to obtain the condition that $\kappa_g({}^h v_i, {}^h v_j) = {}^h\big(\kappa_g(v_i, v_j)\big)$ for all i, j and all g, h in G, yielding the first condition of the theorem. (See [8, Prop. 9.3] for a substitute for G-invariance when G does not act diagonally.)

We consider overlaps arising from $(v_k v_j)v_i = v_k(v_j v_i)$ to obtain the second and third conditions. We use the relations of the algebra to write $0 = (v_k v_j)v_i - v_k(v_j v_i)$ in the span of the PBW basis $\{v_1^{m_1} \cdots v_n^{m_n} g : m_i \in \mathbb{N}, g \in G\}$. Putting G in degree 0, straightforward calculations show that terms of degree 3 in $v_1, \ldots, v_n$ cancel. Since the group acts diagonally, direct calculations give the coefficient of a fixed g in G (as an element of $T(V)\otimes \mathbb{K}G$) as

$$\sum_{\circlearrowleft} q_{jk}\ v_k\ \kappa_g(v_i,v_j) - q_{ki}\ \chi_k(g)\ \kappa_g(v_i,v_j)\ v_k\,.$$

We assume i,j,k are distinct as this expression trivially vanishes otherwise. We expand each $\kappa_g(v_i,v_j)$ (for fixed g) as $\sum_{1\le r\le n} c_r^{ij} v_r$ for $c_r^{ij} = c_r^{ijg}\in\mathbb{K}$, obtaining

$$0 = \sum_{\circlearrowleft}\ \sum_{1\le r\le n} q_{jk}\ c_r^{ij}\ v_k v_r - q_{ki}\ \chi_k(g)\ c_r^{ij}\ v_r v_k\,.$$

Next, we exchange the order of v_r and v_k so that indices increase, obtaining a sum over cyclic permutations of i,j,k of

$$\sum_{k\le r} q_{jk}\ c_r^{ij}\ v_k v_r - q_{ki}\ \chi_k(g)\ c_r^{ij}\ \big(q_{rk}\ v_k v_r + \kappa(v_r,v_k)\big)$$
$$+\sum_{k>r} q_{jk}\ c_r^{ij}\ \big(q_{kr}\ v_r v_k + \kappa(v_k,v_r)\big) - q_{ki}\ \chi_k(g)\ c_r^{ij}\ v_r v_k\ .$$

The terms of degree 2 vanish exactly when

$$0 = \sum_{\circlearrowleft}\Big(\sum_{k\le r} q_{rk}\,\big(q_{jk}\,q_{kr} - q_{ki}\,\chi_k(g)\big)\,c_r^{ij}\ v_k v_r + \sum_{k>r}\big(q_{jk}\,q_{kr} - q_{ki}\,\chi_k(g)\big)\,c_r^{ij}\ v_r v_k\Big)\ . \tag{3.5}$$

We combine like terms and see that (3.5) holds exactly when six equations hold, obtained by taking cyclic permutations of indices on equations

$$\big(q_{jk} - q_{ki}\,q_{rk}\,\chi_k(g)\big)\,c_r^{ij} = 0 \qquad \text{for } r\ne i,j\,, \tag{3.6}$$

$$\big(1 - q_{ij}\,\chi_i(g)\big)\,c_k^{jk} + \big(q_{jk} - \chi_k(g)\big)\,c_i^{ij} = 0\,, \tag{3.7}$$

whence Condition (2) follows.

We collect terms of degree 1; these vanish exactly when

$$\begin{aligned} 0 &= \sum_{\circlearrowleft}\Big(\sum_{k\le r} -q_{ki}\ \chi_k(g)\ c_r^{ij}\ \kappa(v_r,v_k) + \sum_{k>r} q_{jk}\ c_r^{ij}\ \kappa(v_k,v_r)\Big)\\ &= \sum_{\circlearrowleft}\Big(\sum_{k\le r} q_{ki}\ q_{rk}\ \chi_k(g)\ c_r^{ij}\ \kappa(v_k,v_r) + \sum_{k>r} q_{jk}\ c_r^{ij}\ \kappa(v_k,v_r)\Big).\end{aligned}$$

We add and subtract the second sum over $k>r$ but taken over $k\le r$ instead to obtain

$$0 = \sum_{\circlearrowleft}\sum_{k\le r}\big(q_{ki}\,q_{rk}\,\chi_k(g) - q_{jk}\big)\ c_r^{ij}\ \kappa(v_k,v_r) + \sum_{\circlearrowleft}\sum_{r} q_{jk}\ c_r^{ij}\ \kappa(v_k,v_r)\,.$$

The second sum is just $\sum_{\circlearrowleft} q_{jk}\,\kappa\big(v_k,\kappa_g(v_i,v_j)\big)$. Equation (3.6) implies that terms in the first sum with $r\ne i,j$ vanish; we rewrite the remaining terms using Equation (3.7) and the fact that κ is quantum antisymmetric and obtain $\sum_{\circlearrowleft}\big(\chi_k(g) - q_{jk}\big)c_i^{ij}$. Condition (3) follows. □

Remark 3.8. The conditions in Theorem 3.4 can be written alternatively as

(1) the function κ is G-invariant,

(2) $0 = \sum_{\circlearrowleft} q_{jk}\, v_k\, \kappa_g(v_i, v_j) - q_{ki}\, \chi_k(g)\, \kappa_g(v_i, v_j)\, v_k$ in $S_{\mathbf{q}}(V)$, and

(3) $0 = \sum_{\circlearrowleft} q_{ki}\ \chi_k(g)\ \kappa(\kappa(v_i, v_j), v_k) - q_{jk}\ \kappa(v_k, \kappa(v_i, v_j))$ in $V \otimes \mathbb{K}G$,

for all g in G and all distinct i, j, k.

We will see in the proof of Theorem 4.1 that the color Lie Jacobi identity is equivalent to the condition $0 = \sum_{\circlearrowleft} q_{jk}\ \kappa\big(v_k, \kappa_g(v_i, v_j)\big)$ for distinct i, j, k. Hence, we are interested in a condition that recovers this expression from Theorem 3.4(3), i.e., a condition that forces half of that third PBW condition to vanish. This condition, defined next, is a precursor to a stronger condition that we will need later to guarantee a Hopf structure on the quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$.

Definition 3.9. A quotient algebra $\mathcal{H}_{\mathbf{q},\kappa}$ as in (3.2) satisfies the *vanishing condition* if for all g in G, distinct $1 \le i, j, k \le n$, and $1 \le r \le n$,

$$^{g}v_k = q_{ik}\, q_{jk}\, q_{kr}\ v_k$$

whenever the coefficient of $v_r\ g$ is nonzero in $\kappa(v_i, v_j) \in V \otimes \mathbb{K}G$.

This vanishing condition indeed implies a simplified PBW theorem as a corollary of Theorem 3.4:

Corollary 3.10. *Let G be an abelian group acting diagonally on V with respect to a $\mathbb{K}$-basis $v_1, \dots, v_n$. Let $\kappa : V \times V \to V \otimes \mathbb{K}G$ be a parameter function defining the algebra $\mathcal{H}_{\mathbf{q},\kappa}$ as in (3.2) and satisfying the vanishing condition (3.9). Then $\mathcal{H}_{\mathbf{q},\kappa}$ satisfies the PBW property if and only if*

- *κ is G-invariant, and*
- *$0 = \sum_{\circlearrowleft} q_{jk}\ \kappa\big(v_k, \kappa(v_i, v_j)\big)$ for distinct $1 \le i, j, k \le n$.*

Groups acting on V without fixed points will provide us with a wide class of examples where the vanishing condition holds and thus a color Jacobi identity holds. To establish this connection, we first need a lemma.

Lemma 3.11. *Let G and $\mathcal{H}_{\mathbf{q},\kappa}$ be as in Theorem 3.4.*

(i) If κ is G-invariant, then $\chi_i\chi_j = \chi_r$ for $1 \le i, j, r \le n$ whenever $v_r\, g$ has a nonzero coefficient in $\kappa(v_i, v_j)$.

(ii) If the action of G is fixed-point-free, then Condition (2) in Theorem 3.4 is equivalent to the vanishing condition (3.9).

Proof. If κ is G-invariant, then for all g, h in G,

$$\sum_r \chi_r(h)\, c_r^{ijg}\, v_r \;=\; {}^{h}\Big(\sum_r c_r^{ijg}\, v_r\Big) \;=\; {}^{h}\big(\kappa_g(v_i, v_j)\big)$$
$$=\; \kappa_g\big({}^{h}v_i, {}^{h}v_j\big) \;=\; \chi_i(h)\, \chi_j(h)\ \kappa_g(v_i, v_j) \;=\; \sum_r \chi_i(h)\chi_j(h)\, c_r^{ijg}\, v_r\,.$$

This implies that $\chi_i(h)\chi_j(h) = \chi_r(h)$ for all h in G whenever some $c_r^{ijg} \neq 0$. In particular, in the case $r = j$, we have $\chi_i(h) = 1$ for all $h \in G$ whenever $c_r^{ijg} \neq 0$, i.e., G fixes

v_i. Thus, if the action of G is fixed-point-free, then $c_i^{ijg} = 0 = c_j^{ijg}$ for all distinct i, j and all g in G, and the second part of Condition (2) is trivial. □

Corollary 3.10 and Lemma 3.11 together imply the following PBW result.

Corollary 3.12. *The three PBW conditions of Theorem 3.4 are implied by conditions*

($1'$) *the function κ is G-invariant,*

($2'$) *the vanishing condition (3.9) holds, and*

($3'$) $0 = \sum_{\circlearrowleft} q_{jk}\ \kappa\big(v_k, \kappa(v_i, v_j)\big)$ *for all distinct* $1 \le i, j, k \le n$.

In addition, if the action of G on V is fixed-point-free, then the three conditions in Theorem 3.4 may be replaced by the three above conditions.

There are many examples in the literature of quotient algebras defined by (3.2) but with κ taking values in $\mathbb{K}G$ instead of in $V \otimes \mathbb{K}G$, particularly in the case where $q_{ij} = 1$ for all i, j. This includes the noncommutative deformations of Kleinian singularities [1] in which $n = 2$ and G is a subgroup of $\mathrm{SL}_2(\mathbb{K})$. There are also examples for more general quantum systems of parameters $\mathbf{q}$, however for those in the literature, again κ typically takes values in $\mathbb{K}G$. We give here some examples of a different nature, focusing in this paper on the less studied case where κ takes values in $V \otimes \mathbb{K}G$. For all these examples, it will follow from results in the next section that $L = V \otimes \mathbb{K}G$ is an ungraded Yetter-Drinfeld color Lie ring. One may check that they all satisfy the vanishing condition (3.9). Examples 3.14 and 3.15 in fact satisfy a stronger condition, the strong vanishing condition (5.1) that will be defined in Section 5.

Example 3.13. Let $V = \mathbb{K}^3$ with basis v_1, v_2, v_3. Let q in $\mathbb{K}$ be a primitive m-th root of unity for some fixed $m \ge 2$. Let $G \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ be generated by diagonal matrices $g_1 = \mathrm{diag}(q, 1, q)$ and $g_2 = \mathrm{diag}(1, q, q)$ acting on V. The action of G on V induces an action on $S_{\mathbf{q}}(V)$ where $q_{12} = q_{13} = q$ and $q_{23} = 1$. Then the algebra

$$\mathcal{H} = (T(V) \rtimes G)/(v_1v_2 - qv_2v_1 - qv_3\, g_1,\ v_1v_3 - qv_3v_1,\ v_2v_3 - v_3v_2)$$

satisfies the PBW property by Theorem 3.4. Corollary 4.2 will imply that $\mathcal{H}$ is the universal enveloping algebra $\mathcal{U}(L)$ of an ungraded color Lie ring $L = V \otimes \mathbb{K}G$ with

$$[v_1, v_2] = q\, v_3\, g_1, \quad [v_1, v_3] = 0, \quad [v_2, v_3] = 0.$$

See [11, Proposition 3.5, Remark 3.6, and Remark 3.11], where this example appears.

Example 3.14. Let $V = \mathbb{K}^3$ with basis v_1, v_2, v_3. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on V by the diagonal matrix $g = \mathrm{diag}(-1, -1, 1)$. Fix some nonzero q in $\mathbb{K}$. The action of G on V induces an action on $S_{\mathbf{q}}(V)$ for $q_{12} = q^{-1}$, $q_{13} = -q^{-1}$, and $q_{23} = -q$. Fix some λ in $\mathbb{K}$. Then

$$\mathcal{H}_\lambda = (T(V) \rtimes G)/(v_1v_2 - q^{-1}v_2v_1 - \lambda v_3\, g,\ v_1v_3 + q^{-1}v_3v_1,\ v_2v_3 + qv_3v_2)$$

satisfies the PBW property by Theorem 3.4. Theorem 4.1 will imply that $\mathcal{H}_\lambda \cong \mathcal{U}(L_\lambda)$ for a 1-parameter family of graded color Lie rings $L_\lambda = V \otimes \mathbb{K}G$ with brackets

$$[v_1, v_2] = \lambda v_3\, g, \quad [v_1, v_3] = 0, \quad [v_2, v_3] = 0.$$

Example 3.15. Let $V = \mathbb{K}^3$ with basis v_1, v_2, v_3. Let q in $\mathbb{K}$ be a primitive m-th root of unity for some fixed $m \geq 2$ and let $G = \mathbb{Z}/m\mathbb{Z}$ act on V by generator $g = \mathrm{diag}(q, 1, 1)$. Let λ, p in $\mathbb{K}$ be nonzero. The action of G on V induces an action on $S_{\mathbf{q}}(V)$ where $q_{12} = q^{-1}$, $q_{13} = p$, and $q_{23} = 1$. Then

$$\mathcal{H}_{\lambda,p} = (T(V) \rtimes G)/(v_1v_2 - q^{-1}v_2v_1 - \lambda v_1\, g,\ v_1v_3 - pv_3v_1,\ v_2v_3 - v_3v_2)$$

satisfies the PBW property by Theorem 3.4. Theorem 4.1 will imply that $\mathcal{H}_{\lambda,p} \cong \mathcal{U}(L_{\lambda,p})$ for a 2-parameter family of graded color Lie rings $L_{\lambda,p} = V \otimes \mathbb{K}G$ with

$$[v_1, v_2] = \lambda\, v_1\, g, \quad [v_1, v_3] = 0, \quad [v_2, v_3] = 0.$$

Example 3.16. Fix some $n \geq 1$. Let $V = \mathbb{K}^{2n}$ with basis $v_1, \ldots, v_{2n}$. Consider $G = (\mathbb{Z}/2\mathbb{Z})^n$ acting on V by generators $g_1, \ldots, g_n$ with ${}^{g_i}v_j = (-1)^{\delta_{i,j}}v_j$ for $1 \leq i \leq n$, $1 \leq j \leq 2n$. Set $q_{ij} = (-1)^{\delta_{j-i,n}}$ for $1 \leq i < j \leq 2n$. The action of G on V induces an action on $S_{\mathbf{q}}(V)$. Let λ_i be a scalar in $\mathbb{K}$ $(1 \leq i \leq n)$. Then

$$\begin{aligned}\mathcal{H}_{\boldsymbol{\lambda}} = (T(V) \rtimes G)/(&v_iv_{n+i} + v_{n+i}v_i - \lambda_i v_i\, g_i \quad \text{for } 1 \leq i \leq n,\\ &v_iv_j - v_jv_i \quad \text{for } 1 \leq i < j \leq 2n \text{ with } j \neq n+i)\end{aligned}$$

satisfies the PBW property (by Theorem 3.4) for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $L_{\boldsymbol{\lambda}} = V \otimes \mathbb{K}G$ is an ungraded color Lie ring with

$$\begin{aligned}[v_i, v_{n+i}] &= \lambda_i\, v_i\, g_i \quad &&\text{for } 1 \leq i \leq n,\\ [v_i, v_j] &= 0 &&\text{for } 1 \leq i < j \leq 2n \text{ with } j \neq n+i,\end{aligned}$$

and Theorem 4.1 will imply that $\mathcal{H}_{\boldsymbol{\lambda}} \cong \mathcal{U}(L_{\boldsymbol{\lambda}})$.

4. Drinfeld orbifold algebras defining ungraded color Lie rings

We now highlight the distinction between the color Lie rings and the ungraded color Lie rings. We show in this section that every quantum Drinfeld orbifold algebra with G acting fixed-point-free on V is the universal enveloping algebra of some ungraded Yetter-Drinfeld color Lie ring.

As before, let G be a finite abelian group acting diagonally on $V = \mathbb{K}^n$ with basis $v_1, \ldots, v_n$ and use linear characters $\chi_i : G \to \mathbb{K}^*$ to record the i-th diagonal entries of each g in G by setting ${}^{g}v_i = \chi_i(g)v_i$ for $1 \leq i \leq n$.

Natural grading used to construct Yetter-Drinfeld color Lie rings. We work with a choice of grading group A throughout this section: Set $A = \mathbb{Z}^n \times G$. The space

$$L = V \otimes \mathbb{K}G = \bigoplus_{i,g} \mathbb{K}(v_i \otimes g)$$

is naturally graded as a vector space by the abelian group A after setting

$$|v_i| = (a_i, 1_G), \quad |g| = (0, g), \quad \text{and} \quad |v_i \otimes g| = |v_i||g| \quad \text{for all } g \in G$$

where $a_1, \ldots, a_n$ is the standard basis of $\mathbb{Z}^n$. We will use this specific grading throughout this section to relate quantum Drinfeld orbifold algebras to color Lie rings.

Quantum Drinfeld orbifold algebras as universal enveloping algebras. We show that quantum Drinfeld orbifold algebras define ungraded color Lie rings when the underlying action of G is fixed-point-free. We begin with a more general theorem that merely requires the vanishing condition (3.9) on quotient algebras $\mathcal{H}_{\mathbf{q},\kappa}$ defined as in (3.2).

Theorem 4.1. *Let G be a finite abelian group acting diagonally on $V = \mathbb{K}^n$ with respect to a basis $v_1, \dots, v_n$. Suppose that $\mathcal{H}_{\mathbf{q},\kappa}$ is a quantum Drinfeld orbifold algebra for some parameter function $\kappa : V \times V \to V \otimes \mathbb{K}G$ satisfying the vanishing condition (3.9). Let $A = \mathbb{Z}^n \times G$ and consider the A-graded vector space $L = V \otimes \mathbb{K}G$ as above. Then*

(a) *There is an antisymmetric bicharacter $\epsilon : A \times A \to \mathbb{K}^*$ with*
$$\epsilon\big(|v_i \otimes g|, |v_j \otimes h|\big) = q_{ij}\ \chi_i^{-1}(h)\ \chi_j(g) \quad \text{for } g, h \in G,\ 1 \le i, j \le n\,.$$

(b) *The space L is an ungraded Yetter-Drinfeld (A, ϵ)-color Lie ring for a color Lie bracket $[\ ,\ ] : L \times L \to L$ defined by*
$$[v_i \otimes g, v_j \otimes h] = \kappa(v_i, {}^g v_j)\, gh \quad \text{for } g, h \in G\,.$$

(c) *The algebra $\mathcal{H}_{\mathbf{q},\kappa}$ is isomorphic to the universal enveloping algebra of L.*

Proof. We use the natural $\mathbb{K}G$-bimodule structure on $L = V \otimes \mathbb{K}G$ given by Equation (2.4). The function ϵ given in the statement of the proposition extends to a bicharacter on A by setting its values on $A \times A$ to be those determined by the bicharacter condition and its values on generators. Note that this forces
$$\epsilon(|g|, |v_j|) = \epsilon(|v_i \otimes g|\,|v_i \otimes 1_G|^{-1}, |v_j|) = \epsilon(|v_i \otimes g|, |v_j|)\ \epsilon\big(|v_i \otimes 1_G|, |v_j|\big)^{-1} = \chi_j(g)$$
and $\epsilon(|g|, |h|) = 1$ for all $g, h \in G$ and $1 \le i, j \le n$. Since $q_{ji} = q_{ij}^{-1}$, the function ϵ is antisymmetric. Then ${}^g v_i = \chi_i(g) v_i = \epsilon(|g|, |v_i|)\, v_i$ and L satisfies the conditions to be Yetter-Drinfeld. Thus we need only show that L is an ungraded (A, ϵ)-color Lie ring.

Bilinear and balanced bracket. The bracket $[\ ,\ ]$ is $\mathbb{K}G$-balanced by construction. We argue that it is also $\mathbb{K}G$-bilinear with respect to the bimodule structure given by Equation (2.4). We use the G-invariance of κ, guaranteed by Corollary 3.10, under the adjoint action of G given in Equation (3.3): For all a, b, g, h in G and $1 \le i, j \le n$,
$$\begin{aligned} [\, a(v_i \otimes g),\ (v_j \otimes h)b\,] &= [\, {}^a v_i \otimes ag,\ v_j \otimes hb\,] &&= \kappa({}^a v_i,\ {}^{ag} v_j)\ aghb \\ &= {}^a\big(\kappa(v_i,\ {}^g v_j)\big)\ aghb &&= a\ \big(\kappa(v_i,\ {}^g v_j) gh\big)\ b \\ &= a\ [v_i \otimes g,\ v_j \otimes h]\ b. \end{aligned}$$

We suppress tensor symbols on elements of L in the rest of the proof for clarity of notation.

Color antisymmetry of bracket. We check directly that for all i, j and $g, h \in G$,
$$\begin{aligned} [v_i g, v_j h] &= \kappa(v_i, {}^g v_j)\ gh \ = \ \chi_j(g)\ \kappa(v_i, v_j)\ gh \\ &= -q_{ij}\ \chi_j(g)\ \kappa(v_j, v_i)\ gh \ = \ -q_{ij}\ \chi_j(g)\chi_i^{-1}(h)\ \kappa(v_j h, v_i g) \\ &= -\epsilon(|v_i g|, |v_j h|)\ [v_j h, v_i g]\,. \end{aligned}$$

Color Jacobi identity. For basis vectors v_i, v_j, v_k and g_i, g_j, g_k in G,

$$\sum_{\circlearrowleft} \epsilon(|v_k g_k|, |v_i g_i|)\big[v_i g_i, [v_j g_j, v_k g_k]\big]$$

$$= \sum_{\circlearrowleft} q_{ki}\ \chi_k^{-1}(g_i)\chi_i(g_k)\ \kappa\big(v_i, {}^{g_i}\kappa(v_j, {}^{g_j}v_k)\big)\ g_i g_j g_k\,.$$

By Corollary 3.10, κ is G-invariant, and so this is

$$\sum_{\circlearrowleft} q_{ki}\ \chi_k^{-1}(g_i)\chi_i(g_k)\ \kappa\big(v_i, \kappa({}^{g_i}v_j, {}^{g_i g_j}v_k)\big)\ g_i g_j g_k$$

$$= \sum_{\circlearrowleft} q_{ki}\ \chi_k^{-1}(g_i)\chi_i(g_k)\ \chi_j(g_i)\chi_k(g_i)\chi_k(g_j)\ \kappa\big(v_i, \kappa(v_j, v_k)\big)\ g_i g_j g_k$$

$$= \chi_i(g_k)\chi_j(g_i)\chi_k(g_j)\Big(\sum_{\circlearrowleft} q_{ki}\ \ \kappa\big(v_i, \kappa(v_j, v_k)\big)\Big)\ g_i g_j g_k\,,$$

which vanishes by Corollary 3.10 for distinct i, j, k. When i, j, k are not distinct, then the ϵ-Jacobi identity holds automatically since $[v, v] = \kappa(v, v) = 0$ for all v in V as κ is quantum antisymmetric with $q_{mm} = 1$ for all m.

Color universal enveloping algebra. We now check that $\mathcal{H}_{\mathbf{q},\kappa}$ is isomorphic to $\mathcal{U}(L)$, the universal enveloping algebra of L. Recall Definition 2.3: $\mathcal{U}(L) = T_{\mathbb{K}G}(V \otimes \mathbb{K}G)/J$, where J is the ideal generated by

$$\{v_i g \otimes_{\mathbb{K}G} v_j h - \epsilon(|v_i g|, |v_j h|)(v_j h \otimes_{\mathbb{K}G} v_i g) - [v_i g, v_j h] : 1 \le i, j \le n, g, h \in G\}.$$

The $\mathbb{K}$-vector space isomorphism

$$\begin{aligned}(V \otimes \mathbb{K}G) \otimes_{\mathbb{K}G} (V \otimes \mathbb{K}G)\ &\xrightarrow{\cong}\ V \otimes V \otimes \mathbb{K}G,\\ v_i g \otimes_{\mathbb{K}G} v_j h\ &\mapsto\ v_i \otimes {}^{g}v_j \otimes gh,\end{aligned}$$

induces an isomorphism of $\mathbb{K}$-algebras

$$\begin{aligned}T_{\mathbb{K}G}(V \otimes \mathbb{K}G)\ &\xrightarrow{\cong}\ T(V) \rtimes G\,,\\ v_i g \otimes_{\mathbb{K}G} v_j h\ &\mapsto\ v_i\ {}^{g}v_j\ gh.\end{aligned}$$

The images of generators of the ideal J under this isomorphism vanish in $\mathcal{H}_{\mathbf{q},\kappa}$:

$$\begin{aligned}&v_i g \otimes_{\mathbb{K}G} v_j h\ -\ \epsilon(|v_i g|, |v_j h|)(v_j h \otimes_{\mathbb{K}G} v_i g) - [v_i g, v_j h]\\ &\quad\mapsto\ v_i\, {}^{g}v_j\ gh - \epsilon(|v_i g|, |v_j h|)\ v_j\, {}^{h}v_i\ hg - \kappa(v_i, {}^{g}v_j)\ gh\\ &\quad\quad= v_i\, {}^{g}v_j\ gh - q_{ij}\ \chi_i^{-1}(h)\,\chi_j(g)\ v_j\, {}^{h}v_i\ hg - \kappa(v_i, {}^{g}v_j)\ gh\\ &\quad\quad= \big(v_i\, {}^{g}v_j - q_{ij}\ {}^{g}v_j\, v_i - \kappa\ (v_i, {}^{g}v_j)\big)\ gh\\ &\quad\quad= \chi_j(g)\ \big(v_i\, v_j - q_{ij}\ v_j\, v_i - \kappa(v_i, v_j)\big)\ gh.\end{aligned}$$

One may check that this isomorphism extends to an algebra isomorphism $\mathcal{U}(L) \to \mathcal{H}_{\mathbf{q},\kappa}$:

$$\mathcal{U}(L) = T_{\mathbb{K}G}(V \otimes \mathbb{K}G)/J\ \xrightarrow{\cong}\ T(V) \rtimes G/I = \mathcal{H}_{\mathbf{q},\kappa}\,,$$

where I is the ideal $(v_i\, v_j - q_{ij}\, v_j\, v_i - \kappa(v_i, v_j) : 1 \le i, j \le n)$. Indeed, one may verify that the generators of I correspond to elements of J under the inverse of the isomorphism $T_{\mathbb{K}G}(V \otimes \mathbb{K}G) \xrightarrow{\cong} T(V) \rtimes G$. □

Together, Theorem 4.1 and Corollary 3.12 give the following corollary.

Corollary 4.2. *Let G be a finite abelian group acting fixed-point-free and diagonally on $V = \mathbb{K}^n$ with respect to a $\mathbb{K}$-basis $v_1, \ldots, v_n$. Suppose that $\mathcal{H}_{\mathbf{q},\kappa}$ defined by parameter function $\kappa : V \times V \to V \otimes \mathbb{K}G$ is a quantum Drinfeld orbifold algebra. Then the conclusion of Theorem 4.1 holds.*

5. Color Lie rings

We now consider the case of color Lie rings (compared to ungraded color Lie rings). We saw in the last section that quantum Drinfeld orbifold algebras define ungraded color Lie rings when the vanishing condition (3.9) holds. We show in this section that we obtain color Lie rings when the vanishing condition holds for *all* indices i, j, k, not just distinct indices i, j, k.

Again, we consider a finite abelian group G acting diagonally on $V = \mathbb{K}^n$ with basis $v_1, \ldots, v_n$ and linear characters $\chi_i : G \to \mathbb{K}^*$ with ${}^g v_i = \chi_i(g) v_i$ for $1 \le i \le n$.

Definition 5.1. We say a quotient algebra $\mathcal{H}_{\mathbf{q},\kappa}$ as in (3.2) satisfies the *strong vanishing condition* if for all g in G, $1 \le i, j, k \le n$, and $1 \le r \le n$,

$$ {}^g v_k = q_{ik}\, q_{jk}\, q_{kr}\ v_k $$

whenever the coefficient of $v_r\ g$ is nonzero in $\kappa(v_i, v_j) \in V \otimes \mathbb{K}G$.

One may check that Examples 3.14 and 3.15 satisfy this strong vanishing condition, while Examples 3.13 and 3.16 do not.

We now modify the grading by the abelian group A given in the last section to show that quantum Drinfeld orbifold algebras satisfying the strong vanishing condition define color Lie rings. In Section 4, we graded $L = V \otimes \mathbb{K}G$ by $A = \mathbb{Z}^n \times G$ by setting

$$ |v_i| = (a_i, 1_G), \quad |g| = (0, g), \quad \text{and} \quad |v_i \otimes g| = |v_i||g| \quad \text{for all } g \in G\ . $$

Using this grading, we defined ungraded color Lie rings from quantum Drinfeld orbifold algebras. In order to obtain color Lie rings, we replace A by its quotient by a normal subgroup N in order to recapture grading Condition (i) in Definition 2.1 of a color Lie ring.

Consider the dual basis $\{(v_k \otimes g)^*\}$ of $(V \otimes \mathbb{K}G)^* = \mathrm{Hom}_{\mathbb{K}}(V \otimes \mathbb{K}G, \mathbb{K})$. Define a subgroup N of A by a set of generators as follows:

$$ N := \langle\, |v_i||v_j||v_r|^{-1}|g|^{-1} : (v_r \otimes g)^* \kappa(v_i, v_j) \neq 0 \,\rangle. $$

The condition $(v_r \otimes g)^* \kappa(v_i, v_j) \neq 0$ is precisely the condition $c_r^{ijg} \neq 0$ in the expansion $\kappa(v_i, v_j) = \sum_{r,g} c_r^{ijg} v_r g$. We then obtain

Theorem 5.2. *Fix a finite abelian group G acting diagonally on $V = \mathbb{K}^n$. Let $\mathcal{H}_{\mathbf{q},\kappa}$ be a quantum Drinfeld orbifold algebra defined by a parameter function $\kappa : V \times V \to V \otimes \mathbb{K}G$ satisfying the strong vanishing condition (5.1). Let $A = \mathbb{Z}^n \times G$ with subgroup N defined above. Then*

(a) *There exists an antisymmetric bicharacter $\epsilon : A/N \times A/N \to \mathbb{K}^*$ satisfying*
$$\epsilon(|v_i \otimes g|, |v_j \otimes h|) = q_{ij}\ \chi_j(g)\ \chi_i^{-1}(h) \quad \textit{for } g, h \textit{ in } G, 1 \le i, j \le n.$$

(b) *The space $L = V \otimes \mathbb{K}G$ is an A/N-graded algebra under the bilinear operation $[\ ,\ ] : L \times L \to L$ defined by*
$$[v_i \otimes g, v_j \otimes h] = \kappa(v_i, {}^g v_j)\ gh \quad \textit{for } g, h \textit{ in } G, 1 \le i, j \le n\,.$$

(c) *The algebra L is a Yetter-Drinfeld $(A/N, \epsilon)$-color Lie ring with universal enveloping algebra $\mathcal{U}(L)$ isomorphic to $\mathcal{H}_{\mathbf{q},\kappa}$.*

Proof. The function ϵ given in the statement extends to an antisymmetric bicharacter on A as in the proof of Theorem 4.1, with
$$\epsilon(|g|, |v_j|) = \chi_j(g)$$
for all $g \in G$ and $1 \le j \le n$. We first check that it is well-defined on $A/N \times A/N$ and thus defines an antisymmetric bicharacter on A/N. Suppose the coefficient of $v_r \otimes g$ in $\kappa(v_i, v_j)$ is nonzero and consider any $v_k \otimes h$ in L with h in G. Note that
$$\epsilon(|v_r|^{-1}|g|^{-1}, |v_k \otimes h|) = \epsilon(|v_r||g|, |v_k \otimes h|)^{-1}\,.$$
Then
$$\begin{aligned}
&\epsilon(|v_i||v_j||v_r|^{-1}|g|^{-1}, |v_k \otimes h|)\\
&\quad = \epsilon(|v_i|, |v_k \otimes h|) \cdot \epsilon(|v_j|, |v_k \otimes h|) \cdot \epsilon(|v_r \otimes g|, |v_k \otimes h|)^{-1}\\
&\quad = q_{ik}\ \chi_i^{-1}(h)\ q_{jk}\ \chi_j^{-1}(h)\ q_{rk}^{-1}\ \chi_r(h)\ \chi_k(g)^{-1}\\
&\quad = \chi_r(h)\,\chi_i^{-1}(h)\chi_j^{-1}(h)\ q_{ik}\,q_{jk}\ q_{rk}^{-1}\ \chi_k(g)^{-1}\,.
\end{aligned}$$
But Theorem 3.4(1) implies that κ is G-invariant and hence $\chi_r(h) = \chi_i(h)\chi_j(h)$ by Lemma 3.11. By the strong vanishing condition (5.1), the above expression is just 1 and ϵ is well-defined on $A/N \times A/N$.

By Theorem 4.1, L is an ungraded color Lie ring. We check now that the first condition in the definition of a color Lie ring holds as well. Note that in A,
$$|[v_i \otimes h_i, v_j \otimes h_j]| = |\kappa(v_i, {}^{h_i}v_j) \otimes h_ih_j| = |\kappa(v_i, v_j)|\ |h_ih_j|\,.$$
If the coefficient of $v_r \otimes g$ in $\kappa(v_i, v_j)$ is nonzero, then
$$|v_i||v_j| = |v_r||g| \quad \text{in } A/N. \tag{5.3}$$
Hence $|\kappa(v_i, v_j)|$ is A/N-homogeneous with $|\kappa(v_i, v_j)| = |v_i||v_j|$, and the color Lie bracket $[\ ,\ ]$ is A/N-graded. □

From now on, we may replace $A = \mathbb{Z}^n \times G$ with A/N where convenient, in order to work with color Lie rings instead of ungraded color Lie rings.

The following corollary gives some alternate conditions under which one obtains the same conclusion as in the theorem.

Corollary 5.4. *Fix a finite abelian group G acting diagonally on $V = \mathbb{K}^n$ and fixed-point-free. Let $\mathcal{H}_{\mathbf{q},\kappa}$ be a quantum Drinfeld orbifold algebra defined by a parameter function $\kappa : V \times V \to V \otimes \mathbb{K}G$ such that for all $1 \leq i, j, r \leq n$,*

$$^{g}v_i = q_{ji}\, q_{ir}\ v_i \quad \textit{whenever} \quad (v_r \otimes g)^* \kappa(v_i, v_j) \neq 0\,.$$

Then the conclusion of Theorem 5.2 holds.

Proof. We argue that the given hypothesis implies the strong vanishing condition (5.1) needed for Theorem 5.2. By Theorem 3.4 and Lemma 3.11, the algebra $\mathcal{H}_{\mathbf{q},\kappa}$ satisfies vanishing condition (3.9). Hence we need only check the vanishing condition when indices coincide. If $i = j$, then $\kappa(v_i, v_i) = 0$ so there is nothing to check. The condition for $k = i$ is the assumption stated. Since $\kappa(v_i, v_j) = -q_{ij}\, \kappa(v_j, v_i)$, the condition for $k = i$ implies the condition for $k = j$. □

6. Braided Lie algebras and Hopf algebras

In this section, we view color Lie rings as Lie algebras in a particular category and derive a braided Hopf algebra structure for certain quantum Drinfeld orbifold algebras.

Lie algebras in a symmetric monoidal category. For any commutative unital ring $\mathbb{K}$ and $\mathbb{K}$-linear symmetric monoidal category $\mathcal{C}$, one can define a Lie algebra as follows (see, e.g., [6, 7, 10]). Denote the monoidal product in $\mathcal{C}$ by $\otimes$ and the braiding by τ. A *Lie algebra* in $\mathcal{C}$ is an object L in $\mathcal{C}$ together with a morphism $[\ ,\ ] : L \otimes L \to L$ satisfying antisymmetry and the Jacobi identity, i.e.,

$$[\ ,\ ] + [\ ,\ ] \circ \tau = 0, \tag{6.1}$$

$$[\ ,[\ ,\ ]] + [\ ,[\ ,\ ]] \circ (\mathrm{id} \otimes \tau) \circ (\tau \otimes \mathrm{id}\,) + [\ ,[\ ,\ ]] \circ (\tau \otimes \mathrm{id}\,) \circ (\mathrm{id} \otimes \tau) = 0. \tag{6.2}$$

The category for color Lie rings. Some of our results in this paper can be phrased alternatively in the language of category theory. We again fix an abelian group A and ring R and take $\mathcal{C}$ to be the category of A-graded R-bimodules with monoidal product $\otimes_R$ and graded morphisms. Any bicharacter ϵ on A gives rise to a braiding τ on $\mathcal{C}$ in the following way: For objects V, W of $\mathcal{C}$, define $\tau = \tau_{V,W} : V \otimes_R W \to W \otimes_R V$ by

$$\tau(v \otimes_R w) = \epsilon(|v|, |w|)\ w \otimes_R v \tag{6.3}$$

for all A-homogeneous $v \in V$, $w \in W$. Then the bicharacter condition on ϵ is equivalent to the braiding condition on τ, that is, τ satisfies the hexagon identities

$$\tau_{U,V \otimes_R W} = (1_V \otimes_R \tau_{U,W})(\tau_{U,V} \otimes_R 1_W) \quad \text{and} \quad \tau_{U \otimes_R V, W} = (\tau_{U,W} \otimes_R 1_V)(1_U \otimes_R \tau_{V,W})$$

for all U, V, W in $\mathcal{C}$. Since ϵ is antisymmetric, $\tau^2 = 1$, and $\mathcal{C}$ is a symmetric monoidal category.

Color Lie rings as Lie algebras. An (A, ϵ)-color Lie ring over R is a Lie algebra in this category $\mathcal{C}$. Indeed, Definition 2.1(i) states that $[\ ,\ ]$ is a graded map and hence a graded morphism in $\mathcal{C}$ (consider the product $L \times L$ as an object in $\mathcal{C}$ with grading $(L \times L)_c = \oplus_{a,b:\, ab=c}\, L_a \times L_b$). Definition 2.1(ii) and (iii) are equivalent to (6.1) and (6.2), respectively. A calculation shows that Definition 2.1(i) also implies that the bracket $[\ ,\ ]$ is compatible with the braiding τ in the sense that

$$\tau(1 \otimes_R [\ ,\ ]) = ([\ ,\ ] \otimes_R 1)(1 \otimes_R \tau)(\tau \otimes_R 1). \tag{6.4}$$

A second compatibility condition

$$\tau([\ ,\ ] \otimes_R 1) = (1 \otimes_R [\ ,\ ])(\tau \otimes_R 1)(1 \otimes_R \tau)$$

follows from the first since $\tau^2 = 1$; just multiply both sides of the first condition by τ on the left and by $(\tau \otimes_R 1)(1 \otimes_R \tau)$ on the right. (More generally, these two compatibility conditions are in fact necessary conditions to have a Lie algebra in a symmetric monoidal category: The braiding τ by definition consists of functorial isomorphisms and thus must satisfy commutative diagrams corresponding to morphisms in the category. For the particular morphism given by the bracket operation, the compatibility conditions as given above are equivalent to commuting diagrams arising from morphisms from three copies of the Lie algebra to two.)

Universal enveloping algebras in the category. We next observe that color universal enveloping algebras are universal enveloping algebras in the specific category above. One can define the notion of *universal enveloping algebra* of a Lie algebra in any symmetric monoidal category $\mathcal{C}$ via a universal property as follows (see, e.g., [6, 7, 10]). We first define an associative algebra in $\mathcal{C}$ to be an object B together with a morphism $m : B \otimes B \to B$ satisfying $m \circ (m \otimes 1) = m \circ (1 \otimes m)$. A calculation shows that B defines a Lie algebra in $\mathcal{C}$, denoted $\mathrm{Lie}(B)$, by setting $[\ ,\ ] := m - m \circ \tau$. For a Lie algebra L in $\mathcal{C}$, the universal enveloping algebra $\mathcal{U}(L)$ is an associative algebra together with an injective map $i : L \hookrightarrow \mathcal{U}(L)$ satisfying the universal property that for any associative algebra B in $\mathcal{C}$ and Lie algebra map $\phi : L \to \mathrm{Lie}(B)$, there exists a unique map of associative algebras $\hat{\phi} : \mathcal{U}(L) \to B$ such that $\phi = \hat{\phi} \circ i$. Note that such an associative algebra $\mathcal{U}(L)$ may not exist in general. If it does exist, then it is unique. In the case where $\mathcal{C}$ is the category of A-graded R-bimodules described above, the universal enveloping algebra exists for all Lie algebras in $\mathcal{C}$, and it is given explicitly in Definition 2.3.

Hopf algebra structures. We now point out a Hopf algebra structure on color universal enveloping algebras defined via the category setting. Generally, a *Hopf algebra* in a $\mathbb{K}$-linear symmetric monoidal category $\mathcal{C}$ is an object in $\mathcal{C}$ with defining maps (unit, multiplication, counit, comultiplication, antipode) all morphisms in the category satisfying the standard Hopf algebra properties. We sometimes simply speak of a *braided Hopf algebra* when it is clear from context which braiding and category are intended.

In certain categories, universal enveloping algebras always exhibit the structure of a braided Hopf algebra. Let $\mathcal{C}$ be a $\mathbb{K}$-linear symmetric monoidal category for which the universal enveloping algebra exists for all Lie algebras in $\mathcal{C}$ and the assignment $L \mapsto \mathcal{U}(L)$ is functorial. Then $\mathcal{U}(L)$ has a braided Hopf algebra structure, i.e., it has a counit map $\epsilon : \mathcal{U}(L) \to \mathcal{U}(0) = \mathbf{1}$ induced by the Lie algebra map $L \to 0$, a comultiplication $\Delta : \mathcal{U}(L) \to \mathcal{U}(L \times L) \cong \mathcal{U}(L) \otimes \mathcal{U}(L)$ induced by the diagonal map $L \to L \times L$, and an antipode $S : \mathcal{U}(L) \to \mathcal{U}(L)$ induced by $-1 : L \to L$. These maps also satisfy some properties with respect to the braiding; e.g., see [6, 7].

The morphisms giving color universal enveloping algebras the structure of braided Hopf algebras are given explicitly in the next proposition, a consequence of [7, Section 4]. See also [3, Proposition 2.7] for a special case.

Proposition 6.5. *Let L be an (A,ϵ)-color Lie ring over R, $\mathcal{U}(L)$ its universal enveloping algebra, and $\mathcal{C}$ the category of A-graded R-bimodules with monoidal product $\otimes_R$. Then $\mathcal{U}(L)$ is a Hopf algebra in $\mathcal{C}$:*

(1) The tensor algebra $T_R(L)$ of L is a Hopf algebra in $\mathcal{C}$ with coproduct, counit, and antipode defined by

$$\Delta(l) = l\otimes_R 1 + 1\otimes_R l, \qquad \epsilon(l)=0, \qquad S(l)=-l, \qquad \text{for all } l\in L.$$

(2) The ideal J is a Hopf ideal in $\mathcal{C}$, and consequently $\mathcal{U}(L)$ is a Hopf algebra in $\mathcal{C}$.

We wish to conclude that there is a braided Hopf structure on the quantum Drinfeld orbifold algebras $\mathcal{H}_{\mathbf{q},\kappa}$ of Theorem 5.2. We first make an observation interesting in its own right: The strong vanishing condition assumed in Theorem 5.2 is equivalent to compatibility of the bracket operation $[\ ,\ ]$ with the braiding τ. We give a direct proof for interest, although we use similar arguments in other sections.

Proposition 6.6. *Let G be a finite group acting diagonally on $V=\mathbb{K}^n$. Let $\mathcal{H}_{\mathbf{q},\kappa}$ be a quantum Drinfeld orbifold algebra satisfying the vanishing condition (3.9). The strong vanishing condition (5.1) is equivalent to the compatibility (6.4) of the braiding τ defined by (6.3) with the bracket $[\ ,\ ]$ on $L=V\otimes\mathbb{K}G$ given in Theorem 4.1(b).*

Proof. The compatibility condition (6.4), applied to $v_k\otimes_{\mathbb{K}G} v_i\otimes_{\mathbb{K}G} v_j$, may be written

$$\tau(v_k\otimes_{\mathbb{K}G}\kappa(v_i,v_j)) = ([\ ,\ ]\otimes_{\mathbb{K}G} 1)(1\otimes_{\mathbb{K}G}\tau)(\tau(v_k,v_i)\otimes_{\mathbb{K}G} v_j),$$

and applying the definition (6.3) of τ, this is equivalent to

$$\epsilon(|v_k|,|\kappa(v_i,v_j)|) = \epsilon(|v_k|,|v_i|)\,\epsilon(|v_k|,|v_j|).$$

In turn, this equation may be rewritten as

$$q_{kr}\,\chi_k^{-1}(g) = q_{ki}\,q_{kj}$$

for all r,g for which $c_r^{ijg}\neq 0$. This is required to hold for all i,j,k, and that is precisely the strong vanishing condition (5.1). We note that the compatibility condition (6.4) applied more generally to elements $v_kg\otimes_{\mathbb{K}G} v_ih\otimes_{\mathbb{K}G} v_jl$ follows from this case $g=h=l=1$ by the definitions of τ, κ, and $[\ ,\ ]$. □

The following corollary is an immediate consequence of Theorem 5.2 and Proposition 6.5.

Corollary 6.7. *Let G be a finite abelian group acting diagonally on the vector space V. Let $\mathcal{H}_{\mathbf{q},\kappa}$ be a corresponding quantum Drinfeld orbifold algebra for which the strong vanishing condition (5.1) holds. Then $\mathcal{H}_{\mathbf{q},\kappa}$ is a braided Hopf algebra.*

Proof. By Theorem 5.2(c), $L=V\otimes\mathbb{K}G$ is an $(A/N,\epsilon)$-color Lie ring with universal enveloping algebra $\mathcal{U}(L)\cong\mathcal{H}_{\mathbf{q},\kappa}$. Let $\mathcal{C}$ be the category of A/N-graded $\mathbb{K}G$-bimodules with monoidal product $\otimes_{\mathbb{K}G}$, graded morphisms, and braiding τ given by (6.3). By Proposition 6.5, $\mathcal{U}(L)\cong\mathcal{H}_{\mathbf{q},\kappa}$ is a Hopf algebra in $\mathcal{C}$, that is, a braided Hopf algebra. □

7. Color universal enveloping algebras as quantum Drinfeld orbifold algebras

In this section, we determine those Yetter-Drinfeld color Lie rings that arise from quantum Drinfeld orbifold algebras and establish a converse to Theorem 4.1 and Theorem 5.2. But first we discuss positive and negative parts of color Lie rings.

Positive and negative parts of color Lie rings. Recall that in any (A,ϵ)-color Lie ring, $\epsilon(|x|,|x|)=\pm 1$ for all A-homogeneous x, introducing a $\mathbb{Z}/2\mathbb{Z}$-grading. In fact,

$$[x,x]=0 \quad \text{or} \quad \epsilon(|x|,|x|)=-1 \quad \text{for all } A\text{-homogeneous } x.$$

For a color Lie ring $L=V\otimes\mathbb{K}G$ with G a finite group acting on V, we define the *positive* and *negative parts* of V (just as for color Lie algebras),

$$\begin{aligned} V_- &= \{A\text{-homogeneous } v\in V : \epsilon(|v|,|v|)=-1\} \quad \text{and}\\ V_+ &= \{A\text{-homogeneous } v\in V : \epsilon(|v|,|v|)=+1\}, \end{aligned}$$

and define

$$L_+ = V_+\otimes\mathbb{K}G \quad \text{and} \quad L_- = V_-\otimes\mathbb{K}G$$

so that $L=L_-\oplus L_+$. We say a color Lie ring $L=V\otimes\mathbb{K}G$ has *purely positive part* when $L=L_+$. We will see that Yetter-Drinfeld color Lie rings with purely positive part arise from quantum Drinfeld orbifold algebras. (See Example 2.2 for a color Lie ring with x satisfying $[x,x]\neq 0$.)

Note that for a Yetter-Drinfeld color Lie ring $L=V\otimes\mathbb{K}G$, the set

$$\{q_{ij}=\epsilon(|v_i|,|v_j|)\}$$

may fail to be a quantum system of parameters. Indeed, if $[v_i\otimes 1_G, v_i\otimes 1_G]\neq 0$ for some i, then $q_{ii}=\epsilon(|v_i|,|v_i|)=-1$. In fact, if $[v_i\otimes 1_G, v_i\otimes 1_G]=0$ for some i but $q_{ii}=\epsilon(|v_i|,|v_i|)=-1$, the element v_i^2 is nilpotent in the universal enveloping algebra $\mathcal{U}(L)$; see Example 2.2. In such cases, the set $\{q_{ij}\}$ could be used to define truncated quantum Drinfeld Hecke algebras as in Grimley and Uhl [4].

The next proposition shows how the last condition in the definition of a Yetter-Drinfeld color Lie ring arises from the positive and negative parts.

Proposition 7.1. *Suppose that $L=V\otimes\mathbb{K}G$ is an (A,ϵ)-color Lie ring over $\mathbb{K}G$ with A-grading on L induced from gradings on V and G. Assume G acts diagonally on $V\cong\mathbb{K}^n$ with respect to a basis $v_1,\dots,v_n$. Then for all $1\le i\le n$ and g in G,*

$$[v_i\otimes 1_G, v_i\otimes g]=0 \quad \text{or} \quad {}^g v_i = -\epsilon(|v_i|,|v_i|)\ \epsilon(|g|,|v_i|)\ v_i\,.$$

If, in addition, L is Yetter-Drinfeld with purely positive part, then $[v_i\otimes 1_G, v_i\otimes g]=0$ for all g in G.

Proof. Suppose ${}^g v_i=\chi_i(g)\ v_i$. As the bracket is $\mathbb{K}G$-balanced and $|v_i\otimes g|=|v_i||g|$,

$$\begin{aligned} [v_i\otimes 1_G, v_i\otimes g] &= [v_i\otimes 1_G,\ g({}^{g^{-1}}v_i\otimes 1_G)] = [v_i\otimes g, {}^{g^{-1}}v_i\otimes 1_G]\\ &= \chi_i(g)^{-1}\ [v_i\otimes g, v_i\otimes 1_G]\\ &= \chi_i(g)^{-1}\ (-1)\ \epsilon(|v_i\otimes g|,|v_i\otimes 1_G|)\ [v_i\otimes 1_G, v_i\otimes g]\,. \end{aligned}$$

Then for nonzero $[v_i \otimes 1_G, v_i \otimes g]$, since ϵ is a bicharacter and $|1_G| = 1_A$,

$$\begin{aligned} -\chi_i(g) &= \epsilon(|v_i \otimes g|, |v_i \otimes 1_G|) = \epsilon\big(|v_i||g|, |v_i||1_G|\big) \\ &= \epsilon\big(|v_i||g|, |v_i|\big) = \epsilon(|v_i|, |v_i|)\ \epsilon(|g|, |v_i|)\,, \end{aligned}$$

establishing the first claim. If L is Yetter-Drinfeld with purely positive part, then

$$\epsilon(|g|, |v_i|)\ v_i = {}^g v_i = -\epsilon(|v_i|, |v_i|)\, \epsilon(|g|, |v_i|)\ v_i = -\epsilon(|g|, |v_i|)\ v_i\,,$$

from which the second claim follows. □

Yetter-Drinfeld color Lie rings with purely positive part. We show now that every Yetter-Drinfeld color Lie ring over $\mathbb{K}G$ with purely positive part corresponds to a quantum Drinfeld orbifold algebra. We consider an arbitrary Yetter-Drinfeld color Lie ring defined by some abelian grading group A and bicharacter ϵ.

Theorem 7.2. *Let $L = V \otimes \mathbb{K}G$ be a Yetter-Drinfeld (A, ϵ)-color Lie ring over $\mathbb{K}G$ for some group G with purely positive part. Then there exist parameters $\kappa : V \times V \to V \otimes \mathbb{K}G$ and $\mathbf{q}$ so that the universal enveloping algebra $\mathcal{U}(L)$ is isomorphic to the quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$.*

Proof. Let $v_1, \ldots, v_n$ be an A-homogeneous basis of V and recall that G must act diagonally with respect to this basis since L is Yetter-Drinfeld. Define a parameter function $\kappa : V \otimes V \to V \otimes \mathbb{K}G$ by

$$\kappa(v_i, v_j) = [v_i \otimes 1_G, v_j \otimes 1_G] \quad \text{for all } 1 \le i, j \le n$$

and a quantum system of parameters $\mathbf{q} = \{q_{ij}\}$ (using that L is purely positive) by

$$q_{ij} = \epsilon(|v_i|, |v_j|) \quad \text{for all } 1 \le i, j \le n\,.$$

Then κ is quantum symmetric as ϵ is antisymmetric. We show that $\mathcal{H}_{\mathbf{q},\kappa}$ is a quantum Drinfeld orbifold algebra by checking Conditions $(1')$, $(2')$, and $(3')$ of Corollary 3.12.

Condition $(3')$ follows from the ϵ-Jacobi identity as L is purely positive.

Condition $(1')$, the G-invariance of κ, follows from the fact that $[\ ,\ ]$ is $\mathbb{K}G$-bilinear and $\mathbb{K}G$-balanced:

$$\begin{aligned} \kappa({}^g v_i,\ {}^g v_j) = &\ \ [{}^g v_i \otimes g g^{-1},\ {}^g v_j \otimes 1_G] &= &\ \ [g(v_i \otimes 1_G) g^{-1},\ {}^g v_j \otimes 1_G] \\ = &\ \ g\,[v_i \otimes 1_G, g^{-1}({}^g v_j \otimes 1_G)] &= &\ \ g\,[v_i \otimes 1_G, v_j \otimes g^{-1}] \\ = &\ \ g\,[v_i \otimes 1_G, (v_j \otimes 1_G)\, g^{-1}] &= &\ \ g\,[v_i \otimes 1_G, v_j \otimes 1_G]\, g^{-1}. \end{aligned}$$

But this last expression is ${}^g(\kappa(v_i, v_j))$ (with G-action on κ induced from the adjoint action (3.3)), and thus κ is G-invariant.

We argue that Condition $(2')$ follows from the fact that the color Lie bracket is A-graded. Suppose the coefficient of $v_r \otimes g$ in $\kappa(v_i, v_j)$ is nonzero (so $i \neq j$) for some $1 \le r \le n$ and $g \in G$. Then as $|v_i||v_j| = |\kappa(v_i, v_j)|$ in A, we have $|v_i||v_j| = |v_r \otimes g|$. Condition $(2')$ then follows since ϵ is a bicharacter, $|1_G| = 1_A$, and L is Yetter-Drinfeld:

$$\begin{aligned} q_{rk}\ \chi_k(g) &= \epsilon(|v_r|, |v_k|)\ \epsilon(|g|, |v_k|) = \epsilon(|v_r||g|, |v_k||1_G|) = \epsilon(|v_r \otimes g|, |v_k \otimes 1|) \\ &= \epsilon(|v_i||v_j|, |v_k|) = \epsilon(|v_i|, |v_k|)\ \epsilon(|v_j|, |v_k|) = q_{ik}\ q_{jk}\,. \end{aligned}$$

□

By the definition of the quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$, we immediately conclude that the universal enveloping algebra $\mathcal{U}(L)$ of a Yetter-Drinfeld color Lie ring L is a PBW deformation of $S_{\mathbf{q}}(V) \rtimes G$:

Corollary 7.3. *Let $L = V \otimes \mathbb{K}G$ be a Yetter-Drinfeld (A, ϵ)-color Lie ring with purely positive part. Then $\mathcal{U}(L)$ has the PBW property.*

We collect our main results from this section and the last, making precise the connection between the universal enveloping algebras of Yetter-Drinfeld color Lie rings and quantum Drinfeld orbifold algebras. Compare with [16, Theorem 3.9] in the special case $G = 1$. Recall that in any color Lie ring L, $\epsilon(|x|, |x|) = 1$ forces $[x, x] = 0$ for x in L. The following statement is now a consequence of Theorem 5.2 and Theorem 7.2.

Theorem 7.4. *Let G be a finite abelian group acting diagonally on $V = \mathbb{K}^n$.*

(i) *The universal enveloping algebra of any Yetter-Drinfeld color Lie ring with purely positive part is isomorphic to a quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$.*

(ii) *Any quantum Drinfeld orbifold algebra $\mathcal{H}_{\mathbf{q},\kappa}$ with $\kappa : V \times V \to V \otimes \mathbb{K}G$ satisfying the strong vanishing condition (5.1) is isomorphic to the universal enveloping algebra of some Yetter-Drinfeld color Lie ring with purely positive part.*

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