GERSTENHABER BRACKETS ON HOCHSCHILD COHOMOLOGY OF QUANTUM SYMMETRIC ALGEBRAS AND THEIR GROUP EXTENSIONS

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Abstract. We construct chain maps between the bar and Koszul resolutions for a quantum symmetric algebra (skew polynomial ring). This construction uses a recursive technique involving explicit formulae for contracting homotopies. We use these chain maps to compute the Gerstenhaber bracket, obtaining a quantum version of the Schouten-Nijenhuis bracket on a symmetric algebra (polynomial ring). We compute brackets also in some cases for skew group algebras arising as group extensions of quantum symmetric algebras.

1. Introduction

Hochschild [8] introduced homology and cohomology for algebras in 1945. Gerstenhaber [5] studied extensively the algebraic structure of Hochschild cohomology—its cup product and graded Lie bracket (or Gerstenhaber bracket)—and consequently algebras with such structure are generally termed Gerstenhaber algebras. Many mathematicians have since investigated Hochschild cohomology for various types of algebras, and it has proven useful in many settings, including algebraic deformation theory [6] and support variety theory [4], [15].

The graded Lie bracket on Hochschild cohomology remains elusive in contrast to the cup product. The latter may be defined via any convenient projective resolution. The former is defined on the bar resolution, which is useful theoretically but not computationally, and one typically computes graded Lie brackets by translating to another more convenient resolution via explicit chain maps. Such chain maps are not always easy to find. One would like to define the graded Lie structure directly on another resolution or to find efficient techniques for producing chain maps.

In this paper, we begin in Section 2 by promoting a recursive technique for constructing chain maps. The technique is not new; for example it appears in the book of Mac Lane [10]. See also Le and the second author [9] for a more general setting. We first use this technique to construct chain maps between the bar and Koszul resolutions for symmetric algebras, reproducing in Theorem 3.5 the chain maps of Shepler and the first author [13] that had been obtained via ad hoc methods. We then construct new chain maps more generally for quantum symmetric algebras (skew polynomial rings) in Theorem 4.6. We generalize an alternative description, due to Carqueville and Murfet [3], of these chain maps for symmetric algebras to quantum symmetric algebras in (4.8). We use these chain maps to compute the Gerstenhaber bracket on quantum symmetric algebras, generalizing the Schouten-Nijenhuis bracket on the Hochschild cohomology of polynomial rings (Theorem 5.1). We then investigate the Hochschild cohomology of a group extension of a quantum symmetric algebra, obtaining results on brackets in the special cases that the action is diagonal (Theorem 7.1) or that the Hochschild cocycles have minimal degree as maps on tensor powers of the algebra (Corollary 7.4). In the latter case, we thereby obtain a new proof that all such Hochschild 2-cocycles are noncommutative Poisson structures (cf. Naidu and the first author [12], in which algebraic deformation theory was used instead). Some results on brackets for group extensions of polynomial rings were previously given by Halbout and Tang [7] and by Shepler and the first author [14].

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Let $\Bbbk$ be a field. All algebras will be associative $\Bbbk$-algebras with unity and tensor products will be taken over $\Bbbk$ unless otherwise indicated.

2. Construction of comparison morphisms

Let A be a ring and let M and N be two left A-modules. Let $P_\bullet$ (respectively, $Q_\bullet$) be a projective resolution of M (respectively, N). It is well known that given a homomorphism of A-modules $f : M \to N$, there exists a chain map $f_\bullet : P_\bullet \to Q_\bullet$ lifting f (and different lifts are equivalent up to homotopy). Sometimes in practice we need an explicit construction of such a chain map, called a comparison morphism, to perform computations. In this section, we recall a method to construct chain maps under the condition that $P_\bullet$ is a free resolution (see Mac Lane [10, Chapter IX, Theorem 6.2]). The second author and Le will present a method for arbitrary projective resolutions in a paper in preparation ([9]).

Let us fix some notation and assumptions. Suppose that

$$\cdots \longrightarrow P_n \xrightarrow{d_n^P} P_{n-1} \xrightarrow{d_{n-1}^P} \cdots \xrightarrow{d_1^P} P_0 \ (\xrightarrow{d_0^P} M \to 0)$$

is a free resolution of M, that is, for each $n \geq 0$, $P_n = A^{(X_n)}$ for some set X_n. (The module $A^{(X_n)}$ is a direct sum of copies of A indexed by X_n. We identify each element of X_n with the identity 1_A in the copy of A indexed by that element.) Suppose that a projective resolution of N,

$$\cdots \longrightarrow Q_n \xrightarrow{d_n^Q} Q_{n-1} \xrightarrow{d_{n-1}^Q} \cdots \xrightarrow{d_1^Q} Q_0 \ (\xrightarrow{d_0^Q} N \to 0),$$

comes equipped with a *chain contraction*: a collection of set maps $t_n : Q_n \to Q_{n+1}$ for each $n \geq 0$ and $t_{-1} : N \to Q_0$ such that for $n \geq 0$, $t_{n-1}d_n^Q + d_{n+1}^Q t_n = \mathrm{Id}_{Q_n}$ and $d_0^Q t_{-1} = \mathrm{Id}_N$. We use these next to construct a chain map, $f_n : P_n \to Q_n$ for $n \geq 0$, lifting $f_{-1} := f$. As P_n is free, we need only specify the values of f_n on elements of X_n, the generating set of P_n.

At first glance, it may appear that f_n defined below will be the zero map, since it is defined recursively by applying the differential more than once. However, the maps t_n are not in general A-module homomorphisms. The formula (2.1) is used only to define f_n on free basis elements, and f_n is then extended to an A-module map. In our examples the maps t_n will be $\Bbbk$-linear, but for the construction, they are only required to be maps of sets, since we apply them only to basis elements. In this weaker setting, such a collection of maps may be called a *weak self-homotopy* as in [1].

For $n = 0$, given $x \in X_0$, define $f_0(x) = t_{-1} f d_0^P(x)$. Then $d_0^Q f_0(x) = d_0^Q t_{-1} f d_0^P(x) = f d_0^P(x)$.

Suppose that we have constructed $f_0, \cdots, f_{n-1}$ such that for $0 \leq i \leq n-1$, $d_i^Q f_i = f_{i-1} d_i^P$. For $x \in X_n$, define

$$f_n(x) = t_{n-1} f_{n-1} d_n^P(x). \tag{2.1}$$

Then

$$\begin{aligned} d_n^Q f_n(x) &= d_n^Q t_{n-1} f_{n-1} d_n^P(x) \\ &= f_{n-1} d_n^P(x) - t_{n-2} d_{n-1}^Q f_{n-1} d_n^P(x) \\ &= f_{n-1} d_n^P(x) - t_{n-2} f_{n-2} d_{n-1}^P d_n^P(x) \\ &= f_{n-1} d_n^P(x). \end{aligned}$$

This proves the following.

Proposition 2.2. *The maps f_n defined in equation (2.1) form a chain map from $P_\bullet$ to $Q_\bullet$ lifting $f : M \to N$.*

In the next two sections, we use this formula (2.1) to find explicit chain maps for symmetric and quantum symmetric algebras, and in the rest of this article we use the chain maps thus found in computations of Gerstenhaber brackets for these algebras and their group extensions.

3. Chain contractions and comparison maps for polynomial algebras

Let N be a positive integer. Let V be a vector space over the field $\Bbbk$ with basis $x_1, \cdots, x_N$, and let

$$S(V) := \Bbbk[x_1, \cdots, x_N]$$

be the polynomial algebra in N indeterminates. This is a Koszul algebra, so there is a standard complex $K_{\bullet}(S(V))$ that is a free resolution of $A := S(V)$ as an A-bimodule (equivalently as an A^e-module where $A^e = A \otimes A^{op}$). We recall this complex next: For each p, let $\bigwedge^p(V)$ denote the pth exterior power of V. Then $K_{\bullet}(S(V))$ is the complex

$$\cdots \to A \otimes \textstyle\bigwedge^2(V) \otimes A \xrightarrow{d_2} A \otimes \bigwedge^1(V) \otimes A \xrightarrow{d_1} A \otimes A (\xrightarrow{d_0} A \to 0),$$

that is, for $0 \le p \le N$, the degree p term is $K_p(S(V)) := A \otimes \bigwedge^p(V) \otimes A$. The differential d_p is defined by

$$\begin{aligned} &d_p\big(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1\big) \\ &= \sum_{i=1}^{p} (-1)^{i+1} x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \widehat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes 1 - \sum_{i=1}^{p} (-1)^{i+1} \otimes (x_{j_1} \wedge \cdots \wedge \widehat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes x_{j_i} \end{aligned}$$

whenever $1 \le j_1 < \cdots < j_p \le N$ and $p > 0$; the notation $\widehat{x}_{j_i}$ indicates that the factor x_{j_i} is deleted. The map d_0 is multiplication.

From now on, we denote $\underline{\ell} = (\ell_1, \cdots, \ell_N)$, an N-tuple of nonnegative integers, $\underline{x} = (x_1, \cdots, x_N)$ and $\underline{x}^{\underline{\ell}} = x_1^{\ell_1} \cdots x_N^{\ell_N}$. We shall give a chain contraction of $K_{\bullet}(S(V))$ consisting of maps $t_{-1} : A \to A \otimes A$ and $t_p : A \otimes \bigwedge^p(V) \otimes A \to A \otimes \bigwedge^{p+1}(V) \otimes A$ for $p \ge 0$. These maps will be left A-module homomorphisms, and thus we need only define them on choices of free basis elements of these free left A-modules.

To define t_{-1}, it suffices to specify $t_{-1}(1) = 1 \otimes 1$ and extend it A-linearly. If $p = 0$ and $\underline{\ell} \in \mathbb{N}^N$, define

$$t_0\big(1 \otimes \underline{x}^{\underline{\ell}}\big) = -\sum_{j=1}^{N} \sum_{r=1}^{\ell_j} \big(x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N}\big) \otimes x_j \otimes \big(x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}\big).$$

If $p \ge 1$, it suffices to give $t_p\big(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \underline{x}^{\underline{\ell}}\big)$, for $\underline{\ell} \in \mathbb{N}^N$ and $1 \le j_1 < \cdots < j_p \le N$, and we set

$$\begin{aligned} &t_p\big(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \underline{x}^{\underline{\ell}}\big) \\ &= (-1)^{p+1} \sum_{j_{p+1}=j_p+1}^{N} \sum_{r=1}^{\ell_{j_{p+1}}} \big(x_{j_{p+1}}^{\ell_{j_{p+1}} - r} x_{j_{p+1}+1}^{\ell_{j_{p+1}+1}} \cdots x_N^{\ell_N}\big) \otimes \big(x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}\big) \otimes \big(x_1^{\ell_1} \cdots x_{j_{p+1}-1}^{\ell_{j_{p+1}-1}} x_{j_{p+1}}^{r-1}\big). \end{aligned}$$

We note that in case $j_p = N$, the sum is empty, and so the value of t_p on such an element is 0.

Proposition 3.1. *The above defined maps t_p, $p \ge -1$, form a chain contraction for the resolution $K_{\bullet}(S(V))$.*

Proof It is easy to verify that $d_0 t_{-1} = \mathrm{Id}$. We need to show that for $p \ge 0$, $t_{p-1} d_p + d_{p+1} t_p = \mathrm{Id}$. We first let $p = 0$, and show that $t_{-1} d_0 + d_1 t_0 = \mathrm{Id}$.

For $\underline{\ell} \in \mathbb{N}^N$, we have $t_{-1} d_0 (1 \otimes \underline{x}^{\underline{\ell}}) = t_{-1}(\underline{x}^{\underline{\ell}}) = \underline{x}^{\underline{\ell}} \otimes 1$, and

$$
\begin{aligned}
&d_1t_0(1\otimes \underline{x}^{\underline{\ell}})\\
&= d_1\big(-\sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j-r}x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_j\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}x_j^{r-1}\big)\\
&= -\sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j-r+1}x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}x_j^{r-1} + \sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j-r}x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}x_j^{r}\\
&= -\sum_{j=1}^{N}\sum_{r=0}^{\ell_j-1} x_j^{\ell_j-r}x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}x_j^{r} + \sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j-r}x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}x_j^{r}\\
&= -\sum_{j=1}^{N} x_j^{\ell_j}x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}} + \sum_{j=1}^{N} x_{j+1}^{\ell_{j+1}}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}x_j^{\ell_j}\\
&= -\sum_{j=1}^{N} x_j^{\ell_j}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}} + \sum_{j=2}^{N+1} x_j^{\ell_j}\cdots x_N^{\ell_N}\otimes x_1^{\ell_1}\cdots x_{j-1}^{\ell_{j-1}}\\
&= -\underline{x}^{\underline{\ell}}\otimes 1 + 1\otimes \underline{x}^{\underline{\ell}}.
\end{aligned}
$$

We thus obtain $(t_{-1}d_0+d_1t_0)(1\otimes \underline{x}^{\underline{\ell}}) = \underline{x}^{\underline{\ell}}\otimes 1 - \underline{x}^{\underline{\ell}}\otimes 1 + 1\otimes \underline{x}^{\underline{\ell}} = 1\otimes \underline{x}^{\underline{\ell}}$ and therefore confirm the equality. Note that in the above proof, there are many terms which cancel one another.

The proof of the equality $t_{p-1}d_p + d_{p+1}t_p = \mathrm{Id}$ for $p\geq 1$ is similar to the above case $p=0$, but is much more complicated. Note that as in the case $p=0$, in the proof for the cases $p\geq 1$, we must change indices several times in order to cancel many terms.

□

Now we can use the chain contraction of Proposition 3.1 to give formulae for comparison morphisms between the normalized bar resolution and the Koszul resolution. Such comparison morphisms were found by the first author and Shepler [13] by ad hoc methods.

For any $\Bbbk$-algebra A associative with unity, denote by $\overline{A} = A/(\Bbbk\cdot 1)$, a $\Bbbk$-vector space. The *normalized bar resolution* of A has p-th term $B_p(A) = A\otimes \overline{A}^{\otimes p}\otimes A$ and differentials $\delta_p : A\otimes \overline{A}^{\otimes p}\otimes A \to A\otimes \overline{A}^{\otimes (p-1)}\otimes A$ given by

$$\delta_p(a_0\otimes \overline{a_1}\otimes\cdots\otimes\overline{a_p}\otimes a_{p+1}) = \sum_{i=0}^{p}(-1)^i a_0\otimes\cdots\otimes\overline{a_ia_{i+1}}\otimes\cdots\otimes a_{p+1}$$

for $a_0,\ldots,a_{p+1}\in A$, where an overline indicates an image in $\overline{A}$. We shall see that this resolution is suitable for computation using the method from Section 2.

There is a standard chain contraction of the normalized bar resolution, $s_p : A\otimes\overline{A}^{\otimes p}\otimes A\to A\otimes \overline{A}^{\otimes(p+1)}\otimes A$, given by

$$s_p(1\otimes\overline{a_1}\otimes\cdots\otimes\overline{a_p}\otimes a_{p+1}) = (-1)^{p+1}\otimes\overline{a_1}\otimes\cdots\otimes\overline{a_p}\otimes\overline{a_{p+1}}\otimes 1. \tag{3.2}$$

Each s_p is then extended to a left A-module homomorphism. For convenience, we shall from now on abuse notation and write a_i in place of $\overline{a_i}$.

A chain map from the Koszul resolution to the normalized bar resolution is given by the standard embedding: For $p\geq 0$, define $\Phi_p : A\otimes\bigwedge^p(V)\otimes A\to A\otimes\overline{A}^{\otimes p}\otimes A$ by

$$\Phi_p\big(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes 1\big) = \sum_{\pi\in\mathrm{Sym}_p}\mathrm{sgn}\pi\otimes x_{j_{\pi(1)}}\otimes\cdots\otimes x_{j_{\pi(p)}}\otimes 1 \tag{3.3}$$

for $1\leq j_1<\cdots<j_p\leq N$, where Sym_p denotes the symmetric group on p symbols.

The other direction is much more complicated. We shall define $\Psi_p : A\otimes\overline{A}^{\otimes p}\otimes A\to A\otimes\bigwedge^p(V)\otimes A$ for each $p\geq 0$. Let Ψ_0 be the identity map. For $p\geq 1$, define Ψ_p by

$$\begin{aligned}(3.4)\qquad &\Psi_p(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes 1)\\ &= \sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{0\le r_s\le \ell^s_{j_s}-1\\ s=1,\cdots,p}} \underline{x}^{\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\otimes (x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes \underline{x}^{\underline{\widehat{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}},\end{aligned}$$

where

- as in [13], we define the N-tuple $\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$ by
$$\big(\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}\big)_j = \begin{cases} r_j+\ell^1_j+\cdots+\ell^{s-1}_j & \text{if } j=j_s\\ \ell^1_j+\cdots+\ell^s_j & \text{if } j_s<j<j_{s+1}\end{cases};$$
- the N-tuple $\underline{\widehat{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$ is defined to be complementary to $\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$ in the sense that
$$\underline{x}^{\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\ \underline{x}^{\underline{\widehat{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\ x_{j_1}\cdots x_{j_p} = \underline{x}^{\underline{\ell}^1}\cdots \underline{x}^{\underline{\ell}^p}\in \Bbbk[x_1,\cdots,x_N].$$

Theorem 3.5. [13] *Let $\Phi_\bullet$ and $\Psi_\bullet$ be as defined in (3.3) and (3.4). Then*

(i) *the map $\Phi_\bullet$ is a chain map from the Koszul resolution to the normalized bar resolution;*
(ii) *the map $\Psi_\bullet$ is a chain map from the normalized bar resolution to the Koszul resolution;*
(iii) *the composition $\Psi_\bullet\circ\Phi_\bullet$ is the identity map.*

Proof (i). We check that this standard map follows from the method in Section 2, in order to illustrate the method. We proceed by induction, applying (2.1) to the chain contraction $s_\bullet$ of the normalized bar resolution defined in (3.2).

The case for $p=0$ is trivial. Now suppose that for $p\ge 0$, $\Phi_p: A\otimes \bigwedge^p(V)\otimes A\to A\otimes \overline{A}^{\otimes p}\otimes A$ is given by (3.3). We compute $\Phi_{p+1}\big(1\otimes (x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\big)$, where Φ_{p+1} is defined by equation (2.1) in terms of Φ_p. We have

$$\begin{aligned}&\Phi_{p+1}\big(1\otimes (x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\big)\\ &\quad = s_p\Phi_p d_{p+1}\big(1\otimes (x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\big)\\ &\quad = s_p\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}x_{j_i}\otimes (x_{j_1}\wedge\cdots\wedge \widehat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\Big)\\ &\qquad - s_p\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}1\otimes (x_{j_1}\wedge\cdots\wedge \widehat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes x_{j_i}\Big).\end{aligned}$$

Notice that the value of s_p on

$$\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}x_{j_i}\otimes (x_{j_1}\wedge\cdots\wedge \widehat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\Big)$$

is 0, since the rightmost tensor factor is 1, and we work with the normalized bar resolution. For a permutation $\pi\in \mathrm{Sym}_p$ that fixes some letter i, $1\le i\le p+1$, consider the permutation $\hat\pi$ of the set $\{1,\cdots,i-1,\hat{i},i+1,\cdots,p+1\}$ corresponding to π via the bijection

$$\{1,\cdots,i-1,i,i+1,\cdots,p\}\simeq\{1,\cdots,i-1,\hat{i},i+1,\cdots,p+1\}$$

sending j to j for $1\le j\le i-1$ and to $j+1$ for $i\le j\le p$.

Define a new permutation $\tilde\pi\in S_{p+1}$ by imposing

$$\tilde\pi(j)=\begin{cases}\hat\pi(j) & \text{for } j<i;\\ \hat\pi(j+1) & \text{for } i\le j<p+1;\\ i & \text{for } j=p+1.\end{cases}$$

Then we have $\mathrm{sgn}(\tilde\pi)=(-1)^{p-i+1}\mathrm{sgn}(\pi)$, and so

$$\begin{aligned}
&\Phi_{p+1}\big(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\big)\\
&=-s_p\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}1\otimes(x_{j_1}\wedge\cdots\wedge\widehat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes x_{j_i}\Big)\\
&=-s_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}\sum_{\tilde\pi\in S_{p+1},\tilde\pi(p+1)=i}(-)^{p-i+1}\mathrm{sgn}(\tilde\pi)\ 1\otimes x_{j_{\tilde\pi(1)}}\otimes\cdots\otimes x_{j_{\tilde\pi(p)}}\otimes x_{j_{\tilde\pi(p+1)}}\Big)\\
&=-(-1)^{p+1}\sum_{i=1}^{p+1}(-1)^{i+1}\sum_{\tilde\pi\in S_{p+1},\tilde\pi(p+1)=i}(-)^{p-i+1}\mathrm{sgn}(\tilde\pi)\ 1\otimes x_{j_{\tilde\pi(1)}}\otimes\cdots\otimes x_{j_{\tilde\pi(p)}}\otimes x_{j_{\tilde\pi(p+1)}}\otimes 1\\
&=\sum_{\tilde\pi\in S_{p+1}}\mathrm{sgn}(\tilde\pi)\ 1\otimes x_{j_{\tilde\pi(1)}}\otimes\cdots\otimes x_{j_{\tilde\pi(p)}}\otimes x_{j_{\tilde\pi(p+1)}}\otimes 1.
\end{aligned}$$

This completes the proof of (i).

(ii). As in (i), we apply the method in Section 2 to the chain contraction $t_\bullet$ of Proposition 3.1 to show that $\Psi_\bullet$ as defined in (3.4) is indeed the resulting chain map. We proceed by induction on p.

Suppose that Ψ_p is given by (3.4). Let us apply (2.1) and show that Ψ_{p+1} results. First notice that we can write

$$t_p\big(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes\underline{x}^{\underline{\ell}}\big)=(-1)^{p+1}\sum_{j_{p+1}=j_p+1}^{N}\sum_{r=1}^{\ell_{j_{p+1}}}\underline{x}^{\underline{Q}_r^{(\underline{\ell};\ j_{p+1})}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\wedge x_{j_{p+1}}\otimes\underline{x}^{\widehat{\underline{Q}}_r^{(\underline{\ell};\ j_{p+1})}}.$$

We have

$$\begin{aligned}
&d_{p+1}(1\otimes\underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^{p+1}}\otimes 1)\\
&\quad=\underline{x}^{\underline{\ell}^1}\otimes\underline{x}^{\underline{\ell}^2}\cdots\otimes\underline{x}^{\underline{\ell}^{p+1}}\otimes 1+\sum_{i=1}^{p}(-1)^p\otimes\underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^i+\underline{\ell}^{i+1}}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^{p+1}}\otimes 1\\
&\qquad+(-1)^{p+1}\otimes\underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^p}\otimes\underline{x}^{\underline{\ell}^{p+1}}.
\end{aligned}$$

Now consider

$$\begin{aligned}
&\Psi_p(\underline{x}^{\underline{\ell}^1}\otimes\underline{x}^{\underline{\ell}^2}\cdots\otimes\underline{x}^{\underline{\ell}^{p+1}}\otimes 1)\\
&\quad=\sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{1\le r_s\le\ell^{s+1}_{j_s}\\ 1\le s\le p}}\underline{x}^{\underline{\ell}^1}\underline{x}^{\underline{Q}^{(\underline{\ell}^2,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes\underline{x}^{\widehat{\underline{Q}}^{(\underline{\ell}^2,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}.
\end{aligned}$$

However, by definition, $\widehat{\underline{Q}}^{(\underline{\ell}^2,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$ has no terms of the form x_u^v with $u>j_p$. Therefore, $t_p\Psi_p(\underline{x}^{\underline{\ell}^1}\otimes\underline{x}^{\underline{\ell}^2}\cdots\otimes\underline{x}^{\underline{\ell}^{p+1}}\otimes 1)=0$.

Similarly we can prove that for $1\le i\le p$,

$$t_p\Psi_p(1\otimes\underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^i+\underline{\ell}^{i+1}}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^{p+1}}\otimes 1)=0.$$

The only term left is $t_p\Psi_p((-1)^{p+1}\otimes\underline{x}^{\underline{\ell}^1}\otimes\underline{x}^{\underline{\ell}^2}\cdots\otimes\underline{x}^{\underline{\ell}^p}\otimes\underline{x}^{\underline{\ell}^{p+1}})$. We obtain

$$
\begin{aligned}
&t_p\Psi_p((-1)^{p+1}\otimes \underline{x}^{\underline{\ell}^1}\otimes \underline{x}^{\underline{\ell}^2}\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes \underline{x}^{\underline{\ell}^{p+1}})\\
&=(-1)^{p+1}\sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{1\le r_s\le \ell^s_{j_s}\\ 1\le s\le p}} t_p\big(\underline{x}^{\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes \underline{x}^{\widehat{\underline{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\underline{x}^{\underline{\ell}^{p+1}}\big)\\
&=\sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{1\le r_s\le \ell^s_{j_s}\\ 1\le s\le p}}\ \sum_{j_{p+1}=j_p+1}^{N}\ \sum_{r=1}^{\ell^{p+1}_{j_{p+1}}} \underline{x}^{\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\underline{x}^{\underline{Q}^{(\underline{\ell};\ j_{p+1})}_{r_{p+1}}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}}\otimes \underline{x}^{\widehat{\underline{Q}}^{(\underline{\ell};\ j_{p+1})}_{r_{p+1}}},
\end{aligned}
$$

where

$$\underline{\ell}=\widehat{\underline{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}+\underline{\ell}^{p+1}.$$

Now notice that

$$\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}+\underline{Q}^{(\underline{\ell};\ j_{p+1})}_{r_{p+1}}=\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_{p+1})}_{(r_1,\cdots,r_{p+1})}$$

and

$$\widehat{\underline{Q}}^{(\underline{\ell};\ j_{p+1})}_{r_{p+1}}=\widehat{\underline{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_{p+1})}_{(r_1,\cdots,r_{p+1})}.$$

We have the desired result:

$$
\begin{aligned}
&t_p\Psi_p d_{p+1}(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1)\\
&=t_p\Psi_p((-1)^{p+1}\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes \underline{x}^{\underline{\ell}^{p+1}})\\
&=\sum_{1\le j_1<\cdots<j_{p+1}\le N}\ \sum_{\substack{1\le r_s\le \ell^s_{j_s}\\ 1\le s\le p+1}} \underline{x}^{\underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_{p+1})}_{(r_1,\cdots,r_{p+1})}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}}\otimes \underline{x}^{\widehat{\underline{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^{p+1};\ j_1,\cdots,j_{p+1})}_{(r_1,\cdots,r_{p+1})}}\\
&=\Psi_{p+1}(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1).
\end{aligned}
$$

(iii). For $1\le i_1<\cdots<i_p\le N$, we have

$$
\begin{aligned}
&\Psi_p\Phi_p\big(1\otimes (x_{i_1}\wedge\cdots\wedge x_{i_p})\otimes 1\big)\\
&=\Psi_p\Big(\sum_{\pi\in\mathrm{Sym}_p}\mathrm{sgn}\pi\otimes x_{i_{\pi(1)}}\otimes\cdots\otimes x_{i_{\pi(p)}}\otimes 1\Big)\\
&=\sum_{\pi\in\mathrm{Sym}_p}\mathrm{sgn}\pi\sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{0\le r_s\le (e_{i_{\pi(s)}})_{j_s}-1\\ s=1,\cdots,p}} \underline{x}^{\underline{Q}^{(e_{i_{\pi(1)}},\cdots,e_{i_{\pi(p)}};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes \underline{x}^{\widehat{\underline{Q}}},
\end{aligned}
$$

where e_u is the uth canonical basis vector $(0,\ldots,0,1,0,\ldots,0)$, the 1 in the uth position, and $\widehat{\underline{Q}}=\widehat{\underline{Q}}^{(e_{i_{\pi(1)}},\cdots,e_{i_{\pi(p)}};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$.

Notice that $\underline{Q}^{(e_{i_{\pi(1)}},\cdots,e_{i_{\pi(p)}};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$ occurs in the sum only if $(i_{\pi(1)},\cdots,i_{\pi(p)})=(j_1,\cdots,j_p)$. In this case, π is the identity, $r_1=\cdots=r_p=0$ and $\underline{Q}^{(e_{i_{\pi(1)}},\cdots,e_{i_{\pi(p)}};\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}$ is the zero vector. Therefore,

$$\Psi_p\Phi_p\big(1\otimes x_{i_1}\wedge\cdots\wedge x_{i_p}\otimes 1\big)=1\otimes x_{i_1}\wedge\cdots\wedge x_{i_p}\otimes 1.$$

□

For comparison, we give an alternative description of the maps Ψ_p due to Carqueville and Murfet [3]: For each i, let $\tau_i: S(V)^e\to S(V)^e$ be the $\Bbbk$-linear map that is defined on monomials as follows. (We denote application of the map τ_i by a left superscript.)

$${}^{\tau_i}(x_1^{j_1}\cdots x_N^{j_N}\otimes x_1^{l_1}\cdots x_N^{l_N})=x_1^{j_1}\cdots x_{i-1}^{j_{i-1}}x_{i+1}^{j_{i+1}}\cdots x_N^{j_N}\otimes x_1^{l_1}\cdots x_{i-1}^{l_{i-1}}x_i^{j_i+l_i}x_{i+1}^{l_{i+1}}\cdots x_N^{l_N}.$$

Define difference quotient operators $\partial_{[i]} : S(V) \to S(V)^e$ for each i, $1 \le i \le N$, as in [3, (2.12)] by

$$\partial_{[i]}(f) := \frac{{}^{\tau_1\cdots\tau_{i-1}}(f\otimes 1) - {}^{\tau_1\cdots\tau_i}(f\otimes 1)}{x_i\otimes 1 - 1\otimes x_i}.$$

For example, ${}^{\tau_1}(x_1^2x_2\otimes 1) = x_2\otimes x_1^2$, so that

$$\partial_{[1]}(x_1^2x_2) = \frac{x_1^2x_2\otimes 1 - x_2\otimes x_1^2}{x_1\otimes 1 - 1\otimes x_1} = x_1x_2\otimes 1 + x_2\otimes x_1.$$

Similarly, $\partial_{[2]}(x_1^2x_2) = 1\otimes x_1^2$.

Identify elements in $S(V)^e\otimes\bigwedge^p(V)$ with elements in $S(V)\otimes\bigwedge^p(V)\otimes S(V)$ via the canonical isomorphism between these two spaces. Then Ψ_p may be expressed as in [3, (2.22)]:

$$\Psi_p(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes\underline{x}^{\underline{\ell}^p}\otimes 1) = \sum_{1\le j_1<\cdots<j_p\le N}\Big(\prod_{s=1}^p \partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\Big)\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}.$$

For example, if $N=2$, then $\Psi_1(1\otimes x_1^2x_2\otimes 1) = x_1x_2\otimes 1\otimes x_1 + x_2\otimes x_1\otimes x_1 + 1\otimes x_1^2\otimes x_2$. We may similarly express the chain contraction t_p as

$$t_p(1\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes\underline{x}^{\underline{\ell}}) = (-1)^{p+1}\sum_{j_{p+1}=j_p+1}^{N}\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})\otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}}.$$

4. Chain contractions and comparison maps for quantum symmetric algebras

Let N be a positive integer, and for each pair $i,j\in\{1,2,\cdots,N\}$, let $q_{i,j}$ be a nonzero scalar in the field $\Bbbk$ such that $q_{i,i}=1$ and $q_{j,i}=q_{i,j}^{-1}$ for all i,j. Denote by $\mathbf{q}$ the corresponding tuple of scalars, $\mathbf{q} := (q_{i,j})_{1\le i,j\le N}$. Let V be a vector space with basis $x_1,\cdots,x_N$, and let

$$S_{\mathbf{q}}(V) := k\langle x_1,\cdots,x_N \mid x_ix_j = q_{i,j}x_jx_i, \text{ for all } 1\le i,j\le N\rangle, \tag{4.1}$$

the quantum symmetric algebra determined by $\mathbf{q}$. This is a Koszul algebra, and there is a standard complex $K_\bullet(S_{\mathbf{q}}(V))$ that is a free resolution of $S_{\mathbf{q}}(V)$ as an $S_{\mathbf{q}}(V)$-bimodule (see, e.g., Wambst [16, Proposition 4.1(c)]). Setting $A = S_{\mathbf{q}}(V)$, the complex is

$$\cdots\to A\otimes\textstyle\bigwedge^2(V)\otimes A\xrightarrow{d_2} A\otimes\bigwedge^1(V)\otimes A\xrightarrow{d_1} A\otimes A(\xrightarrow{d_0} A\to 0),$$

with differential d_p defined by

$$\begin{aligned}
&d_p\big(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes 1\big)\\
&\quad=\sum_{i=1}^p(-1)^{i+1}\Big(\prod_{s=1}^i q_{j_s,j_i}\Big)\, x_{j_i}\otimes(x_{j_1}\wedge\cdots\wedge\widehat{x}_{j_i}\wedge\cdots\wedge x_{j_p})\otimes 1\\
&\qquad-\sum_{i=1}^p(-1)^{i+1}\Big(\prod_{s=i}^p q_{j_i,j_s}\Big)\, 1\otimes(x_{j_1}\wedge\cdots\wedge\widehat{x}_{j_i}\wedge\cdots\wedge x_{j_p})\otimes x_{j_i}
\end{aligned}$$

whenever $1\le j_1<\cdots<j_p\le N$ and $p>0$; the map d_0 is multiplication.

As in the previous section, we denote $\underline{\ell} = (\ell_1,\cdots,\ell_N)$, $\underline{x} = (x_1,\cdots,x_N)$ and $\underline{x}^{\underline{\ell}} = x_1^{\ell_1}\cdots x_N^{\ell_N}$. We shall give a chain contraction of $K_\bullet(S_{\mathbf{q}}(V))$, $t_p : A\otimes\bigwedge^p(V)\otimes A\to A\otimes\bigwedge^{p+1}(V)\otimes A$ for $p\ge 0$ and $t_{-1}: A\to A\otimes A$, which are moreover left A-module homomorphisms (cf. Wambst [16]).

Let $t_{-1}(1) = 1\otimes 1$ and extend t_{-1} to be left A-linear. For $p\ge 0$, $1\le j_1<\cdots<j_p\le N$, and $\underline{\ell}\in\mathbb{N}^N$, let

$$\begin{aligned}
&t_p\big(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes\underline{x}^{\underline{\ell}}\big)\\
&= (-1)^{p+1}\sum_{j_{p+1}=j_p+1}^{N}\sum_{r=1}^{\ell_{j_{p+1}}}\lambda^{(\underline{\ell};\, j_1,\cdots,j_p)}_{j_{p+1},r}\, x_{j_{p+1}}^{\ell_{j_{p+1}}-r}\, x_{j_{p+1}+1}^{\ell_{j_{p+1}+1}}\cdots x_N^{\ell_N}\otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}}\otimes x_1^{\ell_1}\cdots x_{j_{p+1}-1}^{\ell_{j_{p+1}-1}}\, x_{j_{p+1}}^{r-1}
\end{aligned}$$

where

$$\lambda_{j_{p+1},r}^{(\underline{\ell};\, j_1,\cdots,j_p)} = \Big(\prod_{s=1}^{j_{p+1}-1}\prod_{t=j_{p+1}}^{N} q_{s,t}^{\ell_s\ell_t}\Big)\Big(\prod_{t=1}^{N} q_{j_{p+1},t}^{\ell_t}\Big)^{r-1}\Big(\prod_{t=1}^{p} q_{j_t,j_{p+1}}^{\ell_{j_{p+1}}-r}\Big)\Big(\prod_{s=1}^{p+1}\prod_{t=j_{p+1}+1}^{N} q_{j_s,t}^{\ell_t}\Big).$$

We remark that compared with the maps in the previous section for polynomial algebras, the only difference is that now there is a new coefficient. This (rather complicated) coefficient $\lambda_{j_{p+1},r}^{(\underline{\ell};\, j_1,\cdots,j_p)}$ can be obtained as follows: In the right-hand side of the formula for t_p, in comparison to its argument $1\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes \underline{x}^{\underline{\ell}}$ on the left-hand side, whenever a factor x_i of $\underline{x}^{\underline{\ell}}$ has changed positions so that it is now to the left of a factor x_j with $i>j$ (including factors of the exterior product), one should include one factor of $q_{j,i}$. One verifies easily that $\lambda_{j_{p+1},r}^{(\underline{\ell};\, j_1,\cdots,j_p)}$ has the given form. We shall call this rule the *twisting principle* and shall use it several times later.

Proposition 4.2. *The above defined maps t_p, $p\geq -1$, form a chain contraction over the resolution $K_\bullet(S_{\mathbf{q}}(V))$.*

Proof One needs to verify that for $n\geq 0$, $t_{n-1}d_n + d_{n+1}t_n = \mathrm{Id}$, and $d_0t_{-1}=\mathrm{Id}$. Notice that the computation used in the above equalities is the same as for polynomial algebras, except that now for quantum symmetric algebras, we have some extra coefficents. One needs to show that these extra coefficients do not cause any problem.

Recall that in the proof of Proposition 3.1, the concrete computation is simplified by many terms which cancel one another. For example, this occurs in the verification of the equation $t_{-1}d_0+d_1t_0=\mathrm{Id}$ in the proof of Proposition 3.1. For polynomial algebras, the proof works due to these cancelling terms.

For quantum symmetric algebras, things are not so easy. However, the twisting principle always holds, that is, when we apply a differential or chain contraction, once we produce a monomial (always in lexographical order) or tensor of monomials, we need to include a coefficient before this monomial according to the twisting principle. Thus, if two terms cancel each other for polynomial algebras, as we have included the same coefficient, they still cancel each other for quantum symmetric algebras. This completes the proof of the result.

□

Now we can use (2.1) and the chain contraction of Proposition 4.2 to give formulae for comparison morphisms between the normalized bar resolution and the Koszul resolution.

A chain map from the Koszul resolution to the normalized bar resolution is induced from the standard embedding of the Koszul resolution into the (unnormalized) bar resolution. See also Wambst [16, Lemma 5.3 and Theorem 5.4] for a more general setting. We give the formula as it appears in [11, §2.2(3)]. For $p\geq 0$, we define $\Phi_p : A\otimes \bigwedge^p(V)\times A\to A\otimes \overline{A}^{\otimes p}\otimes A$ by

$$\Phi_p\big(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes 1\big) = \sum_{\pi\in\mathrm{Sym}_p}(\mathrm{sgn}\pi)q_\pi^{j_1,\cdots,j_p}\otimes x_{j_{\pi(1)}}\otimes\cdots\otimes x_{j_{\pi(p)}}\otimes 1 \tag{4.3}$$

for $1\leq j_1<\cdots<j_p\leq N$. In the above formula, the coefficients $q_\pi^{j_1,\cdots,j_p}$ are the scalars obtained from the twisting principle, that is,

$$q_\pi^{j_1,\ldots,j_p}x_{j_{\pi(1)}}\cdots x_{j_{\pi(p)}} = x_{j_1}\cdots x_{j_p}. \tag{4.4}$$

The other direction is much more complicated. We shall see that for quantum symmetric algebras, the comparison morphism is a twisted version of that for a polynomial ring given in the previous section, with certain coefficients included according to the twisting principle.

We define the maps $\Psi_p : A\otimes\overline{A}^{\otimes p}\otimes A\to A\otimes\bigwedge^p(V)\otimes A$ as follows. Let Ψ_0 be the identity map. For $p\geq 1$, define Ψ_p by

(4.5)

$$\Psi_p(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes 1)$$
$$=\sum_{1\leq j_1<\cdots<j_p\leq N}\ \sum_{\substack{0\leq r_s\leq \ell^s_{j_s}-1\\ s=1,\cdots,p}} \mu_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)} \underline{x}^{\underline{Q}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes \underline{x}^{\widehat{\underline{Q}}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}},$$

where

- as before, we define the N-tuple $\underline{Q}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}$ by
$$\big(\underline{Q}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}\big)_j=\begin{cases} r_j+\ell^1_j+\cdots+\ell^{s-1}_j & \text{if } j=j_s\\ \ell^1_j+\cdots+\ell^s_j & \text{if } j_s<j<j_{s+1}\end{cases};$$
- the N-tuple $\widehat{\underline{Q}}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}$ and scalar $\mu_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}$ are (uniquely) defined by the equation
$$\mu_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)} \underline{x}^{\underline{Q}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}} x_{j_1}\cdots x_{j_p}\underline{x}^{\widehat{\underline{Q}}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}}=\underline{x}^{\underline{\ell}^1}\cdots\underline{x}^{\underline{\ell}^p}\in S_{\mathbf{q}}(V).$$

Note that the coefficient $\mu_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}$ is obtained using the twisting principle in the right-hand side of the formula for Ψ_p, and that $\underline{Q}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}$ and $\widehat{\underline{Q}}_{(r_1,\cdots,r_p)}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}$ are the same as in the case of the polynomial algebra $\Bbbk[x_1,\dots,x_n]$. For comparison, we note that Wambst gave such a chain map in degree 1 [16, Lemma 6.7].

Theorem 4.6. *Let $\Phi_\bullet$ and $\Psi_\bullet$ be as defined in (4.3) and (4.5). Then*

(i) *the map $\Phi_\bullet$ is a chain map from the Koszul resolution to the normalized bar resolution;*
(ii) *the map $\Psi_\bullet$ is a chain map from the normalized bar resolution to the Koszul resolution;*
(iii) *the composition $\Psi_\bullet\circ\Phi_\bullet$ is the identity map.*

Proof (i). One direct proof was given in [11, Lemma 2.3]. (The characteristic of $\Bbbk$ was assumed to be 0 in [11], however this assumption is not needed in that proof.) Another proof can be given by applying (2.1) to a chain contraction $s_\bullet$ over the normalized bar resolution as in the proof of Theorem 3.5 (i). The twisting principle gives the coefficients.

(ii). One direct computational proof can be given by applying (2.1) to the chain contraction $t_\bullet$ of Proposition 4.2, as in the proof of Theorem 3.5 (ii). Thus the same proof as that of Theorem 3.5 (ii) works, taking care with the coefficients, by the twisting principle.

(iii). The same proof as in the proof of Theorem 3.5 (iii) works; by the twisting principle, the coefficients on both sides of the equation coincide.

□

We now give alternative descriptions of the maps t_p and Ψ_p in this case of a quantum symmetric algebra. The description of Ψ_p will generalize that of Carqueville and Murfet [3] from $S(V)$ to $S_{\mathbf{q}}(V)$. To this end, it is convenient to replace each term $S_{\mathbf{q}}(V)\otimes\bigwedge^p(V)\otimes S_{\mathbf{q}}(V)$ of the Koszul resolution by $S_{\mathbf{q}}(V)\otimes S_{\mathbf{q}}(V)\otimes\bigwedge^p(V)$, using the canonical isomorphism
$$\sigma_p: S_{\mathbf{q}}(V)\otimes S_{\mathbf{q}}(V)\otimes\textstyle\bigwedge^p(V)\to S_{\mathbf{q}}(V)\otimes\bigwedge^p(V)\otimes S_{\mathbf{q}}(V)$$
in which coefficients are inserted according to the twisting principle. For example, for $\underline{x}^{\underline{\ell}}\in S_{\mathbf{q}}(V)$ and $1\leq j_1<\cdots<j_p\leq N$,
$$\sigma_p(1\otimes\underline{x}^{\underline{\ell}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p})=\Big(\prod_{s=1}^N\prod_{t=1}^p q_{s,j_t}^{\ell_s}\Big)\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes\underline{x}^{\underline{\ell}}.$$
Via this isomorphism between the two spaces, consider t_p as a map from $S_{\mathbf{q}}(V)\otimes S_{\mathbf{q}}(V)\otimes\bigwedge^p(V)$ to $S_{\mathbf{q}}(V)\otimes S_{\mathbf{q}}(V)\otimes\bigwedge^{p+1}(V)$. By abuse of notation, we still denote by t_p this new map; the same rule applies to Ψ_p.

For $1 \le j \le N$, define $\tau_j : S_{\mathbf{q}}(V)^e \to S_{\mathbf{q}}(V)^e$ to be the operator that replaces all factors of the form $x_j \otimes 1$ with $1 \otimes x_j$, but with coefficient inserted according to the twisting principle. For example, if $\underline{x}^{\underline{\ell}} \in S_{\mathbf{q}}(V)$, then

$$^{\tau_j}(\underline{x}^{\underline{\ell}} \otimes 1) = (\prod_{s=j+1}^{N} q_{j,s}^{\ell_j \ell_s}) x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_j^{\ell_j}.$$

It is not difficult to see that for $1 \le i \ne j \le N$, $\tau_i \tau_j = \tau_j \tau_i$. Define quantum difference quotient operators $\partial_{[i]} : S_{\mathbf{q}}(V) \to S_{\mathbf{q}}(V) \otimes S_{\mathbf{q}}(V)$ for each i, $1 \le i \le N$ by

$$\partial_{[i]}(f) := (x_i \otimes 1 - 1 \otimes x_i)^{-1} ({}^{\tau_1 \cdots \tau_{i-1}}(f \otimes 1) - {}^{\tau_1 \cdots \tau_i}(f \otimes 1)). \tag{4.7}$$

This definition should be understood as follows: By writing f as a linear combination of monomials, it suffices to define $\partial_{[i]}$ on each monomial $\underline{x}^{\underline{\ell}}$. The difference ${}^{\tau_1 \cdots \tau_{i-1}}(\underline{x}^{\underline{\ell}} \otimes 1) - {}^{\tau_1 \cdots \tau_i}(\underline{x}^{\underline{\ell}} \otimes 1)$ may be divided by $x_i \otimes 1 - 1 \otimes x_i$ on the left, by first factoring out $x_i^{\ell_i} \otimes 1 - 1 \otimes x_i^{\ell_i}$ on the left. Applying the twisting principle, one sees that this is indeed always a factor. One must include a coefficient given by the twisting principle, then use the identity

$$(x_i \otimes 1 - 1 \otimes x_i)^{-1} (x_i^{\ell_i} \otimes 1 - 1 \otimes x_i^{\ell_i}) = \sum_{r=1}^{\ell_i} x_i^{\ell_i - r} \otimes x_i^{r-1}.$$

For example, for $f = x_1 x_2^2$, let us compute $\partial_{[2]}(f)$. We have

$$\begin{aligned} {}^{\tau_1}(x_1 x_2^2 \otimes 1) &= q_{1,2}^2 x_2^2 \otimes x_1 = q_{1,2}^2 (x_2^2 \otimes 1)(1 \otimes x_1), \\ {}^{\tau_1 \tau_2}(x_1 x_2^2 \otimes 1) &= 1 \otimes x_1 x_2^2 = q_{1,2}^2 (1 \otimes x_2^2)(1 \otimes x_1), \end{aligned}$$

and so

$${}^{\tau_1}(x_1 x_2^2 \otimes 1) - {}^{\tau_1 \tau_2}(x_1 x_2^2 \otimes 1) = q_{1,2}^2 (x_2^2 \otimes 1 - 1 \otimes x_2^2)(1 \otimes x_1).$$

We obtain thus

$$\begin{aligned} \partial_{[2]}(f) &= (x_2 \otimes 1 - 1 \otimes x_2)^{-1} ({}^{\tau_1}(x_1 x_2^2 \otimes 1) - {}^{\tau_1 \tau_2}(x_1 x_2^2 \otimes 1)) \\ &= (x_2 \otimes 1 - 1 \otimes x_2)^{-1} (q_{1,2}^2 (x_2^2 \otimes 1 - 1 \otimes x_2^2)(1 \otimes x_1)) \\ &= q_{1,2}^2 (x_2 \otimes 1 + 1 \otimes x_2)(1 \otimes x_1) \\ &= q_{1,2}^2 x_2 \otimes x_1 + q_{1,2} \otimes x_1 x_2. \end{aligned}$$

In general, we have

$$\partial_{[j]}(\underline{x}^{\underline{\ell}}) = (\prod_{s=1}^{j-1} q_{s,j}^{\ell_s}) \sum_{r=1}^{\ell_j} (\prod_{s=1}^{j-1} \prod_{t=j+1}^{N} q_{s,t}^{\ell_s \ell_t}) (\prod_{t=j+1}^{N} q_{j,t}^{\ell_t (r-1)}) x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}.$$

That is, one has an extra coefficient $(\prod_{s=1}^{j-1} q_{s,j}^{\ell_s})$ as well as the coefficient included according to the twisting principle.

The chain contraction $t_p : S_{\mathbf{q}}(V) \otimes S_{\mathbf{q}}(V) \otimes \bigwedge^p(V) \to S_{\mathbf{q}}(V) \otimes S_{\mathbf{q}}(V) \otimes \bigwedge^{p+1}(V)$ may be expressed as

$$t_p(1 \otimes \underline{x}^{\underline{\ell}} \otimes x_{j_1} \wedge \cdots \wedge x_{j_p}) = (-1)^{p+1} \sum_{j_{p+1} = j_p + 1}^{N} (\prod_{t=1}^{N} q_{j_{p+1},t}^{\ell_t}) (\prod_{t=1}^{p} q_{j_{p+1}, j_t}) \partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}}) \otimes x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}.$$

This can be justified as follows: The coefficient in $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$ is nearly the coefficient needed by the twisting principle. The discrepancy is that $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$ has an extra factor $\prod_{t=1}^{j_{p+1}-1} q_{t,j_{p+1}}^{\ell_t}$, and we still need to insert $\prod_{t=j_{p+1}+1}^{N} q_{j_{p+1},t}^{\ell_t}$ and $\prod_{t=1}^{p} q_{j_{p+1},j_t}$ since the term $x_{j_{p+1}}$ in $x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}$ lies to the right of $x_{j_1} \wedge \cdots \wedge x_{j_p}$ and of $x_{j_{p+1}+1}^{\ell_{j_{p+1}+1}} \cdots x_N^{\ell_N}$ in $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$. Altogether, we need to include an extra factor of $\big(\prod_{t=1}^{N} q_{j_{p+1},t}^{\ell_t}\big)\big(\prod_{t=1}^{p} q_{j_{p+1},j_t}\big)$ in the coefficient in $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$.

The chain map $\Psi_p : S_{\mathbf{q}}(V) \otimes S_{\mathbf{q}}(V) \otimes \overline{S_{\mathbf{q}}(V)}^{\otimes p} \to S_{\mathbf{q}}(V) \otimes S_{\mathbf{q}}(V) \otimes \bigwedge^p(V)$ may be expressed as:

$$\Psi_p(1 \otimes 1 \otimes \underline{x}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{x}^{\underline{\ell}^p}) = \sum_{1 \le j_1 < \cdots < j_p \le N} \mu^{(\underline{\ell}^1, \cdots, \underline{\ell}^p)}_{(j_1, \cdots, j_p)} \big(\prod_{s=1}^{p} \partial_{[j_s]}(\underline{x}^{\underline{\ell}^s}) \big) \otimes x_{j_1} \wedge \cdots \wedge x_{j_p}, \tag{4.8}$$

where the scalar is defined according to the twisting principle by

$$\underline{x}^{\underline{\ell}^1} \cdots \underline{x}^{\underline{\ell}^p} = \mu^{(\underline{\ell}^1, \cdots, \underline{\ell}^p)}_{(j_1, \cdots, j_p)} \big(\prod_{s=1}^{p} \partial_{[j_s]}(\underline{x}^{\underline{\ell}^s}) \big)' x_{j_1} \cdots x_{j_p} \in S_{\mathbf{q}}(V). \tag{4.9}$$

Here in the above expression, the term $\big(\prod_{s=1}^p \partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\big)'$ is understood as follows: Suppose $\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s}) = a_s \otimes b_s$ (symbolically), then the product $\big(\prod_{s=1}^p \partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\big)'$ is $(\prod_s a_s)(\prod_s b_s) \in A$.

5. Gerstenhaber brackets for quantum symmetric algebras

The Schouten-Nijenhuis (Gerstenhaber) bracket on Hochschild cohomology of the symmetric algebra $S(V)$ is well known. In this section, we generalize it to the quantum symmetric algebras $S_{\mathbf{q}}(V)$. First we recall the definition of the Gerstenhaber bracket on Hochschild cohomology as defined on the normalized bar resolution of any $\Bbbk$-algebra A (associative with unity).

Let $f \in \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{\otimes p} \otimes A, A)$ and $f' \in \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{\otimes q} \otimes A, A)$. Define their bracket, $[f, f'] \in \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{(p+q-1)} \otimes A, A)$, by

$$[f, f'] = \sum_{k=1}^{p} (-1)^{(q-1)(k-1)} f \circ_k f' - (-1)^{(p-1)(q-1)} \sum_{k=1}^{q} (-1)^{(p-1)(k-1)} f' \circ_k f$$

where

$$\begin{aligned}&(f \circ_k f')(1 \otimes a_1 \otimes \cdots \otimes a_{p+q-1} \otimes 1)\\ &\quad = f(1 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes f'(1 \otimes a_k \otimes \cdots \otimes a_{k+q-1} \otimes 1) \otimes a_{k+q} \otimes \cdots \otimes a_{p+q-1} \otimes 1).\end{aligned}$$

In the above definition, the image of an element under f or f' is understood in $\overline{A}$, whenever required.

Let $\bigwedge_{\mathbf{q}^{-1}}(V^*)$ be the quantum exterior algebra defined by the tuple $\mathbf{q}^{-1}$, that is, $\bigwedge_{\mathbf{q}^{-1}}(V^*)$ is the algebra generated by the dual basis $\{dx_1, \ldots, dx_N\}$ of V^* with respect to the basis $\{x_1, \ldots, x_N\}$ of V, subject to the relations $(dx_i)^2 = 0$ and $dx_i dx_j = -q_{i,j}^{-1} dx_j dx_i$ for all i, j. We denote the product on $\bigwedge_{\mathbf{q}^{-1}}(V^*)$ by $\wedge$. It is convenient to use abbreviated notation for monomials in this algebra: If I is the p-tuple $I = (i_1, \ldots, i_p)$, denote by dx_I the element $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ of $\bigwedge_{\mathbf{q}^{-1}}(V^*)$. We also write $\underline{x}^{\wedge I}$ for $x_{i_1} \wedge \cdots \wedge x_{i_p}$. Another notation we shall use is $dx_{\underline{b}}$, defined for any $\underline{b}$ in $\{0,1\}^N$ to be $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $i_1, \ldots, i_p$ are the positions of the entries 1 in $\underline{b}$, all other entries being 0. In this case we say the *length* of $\underline{b}$ is p, and write $|\underline{b}| = p$.

In [11, Corollary 4.3], the Hochschild cohomology of $S_{\mathbf{q}}(V)$ is given as the graded vector subspace of $S_{\mathbf{q}}(V) \otimes \bigwedge_{\mathbf{q}^{-1}}(V^*)$ that in degree m is

$$\mathrm{HH}^m(S_{\mathbf{q}}(V)) = \bigoplus_{\substack{\underline{b} \in \{0,1\}^N \\ |\underline{b}| = m}} \bigoplus_{\substack{\underline{a} \in \mathbb{N}^N \\ \underline{a} - \underline{b} \in C}} \mathrm{Span}_k\{\underline{x}^{\underline{a}} \otimes dx_{\underline{b}}\},$$

where

$$C = \{\gamma \in (\mathbb{N} \cup \{-1\})^N \mid \text{ for each } i \in \{1, \ldots, N\}, \prod_{s=1}^{N} q_{is}^{\gamma_s} = 1 \text{ or } \gamma_i = -1\}.$$

We wish to compute the bracket of two elements

$$\alpha = \underline{x}^{\underline{a}} \otimes dx_J \quad \text{ and } \quad \beta = \underline{x}^{\underline{b}} \otimes dx_L$$

where $J = (j_1, \dots, j_p)$ and $L = (l_1, \dots, l_q)$. We fix some notations. We denote by $J \sqcup L$ the reordered disjoint union of J and L (multiplicities counted if there are equal indices), so $dx_{J\sqcup L} = 0$ if $J \cap L \neq \varnothing$ and the entries of $J \sqcup L$ are in increasing order. For $1 \le k \le p$, set

$$I_k := (j_1, \dots, j_{k-1}, l_1, \dots, l_q, j_{k+1}, \dots, j_p),$$

although we do not have $j_1 < \dots < j_{k-1} < l_1 < \dots < l_q < j_{k+1} < \dots < j_p$ in general. So we have $dx_{I_k} = \operatorname{sgn}(\pi) q_\pi^{I_k}\ dx_{J_k \sqcup L}$, where $J_k = (j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_p)$. Similarly for $1 \le k \le q$, set

$$I'_k := (l_1, \dots, l_{k-1}, j_1, \dots, j_p, l_{k+1}, \dots, l_q).$$

Once we know the bracket of two elements of this form, others may be computed by extending bilinearly. The scalars arising in each term from the twisting principle are potentially different, so it is more convenient to express brackets in terms of these basis elements of Hochschild cohomology.

Theorem 5.1. *The graded Lie bracket of $\alpha = \underline{x}^{\underline{a}} \otimes dx_J$ and $\beta = \underline{x}^{\underline{b}} \otimes dx_L$ is*

$$\begin{aligned}[\alpha, \beta] \;=\; & \sum_{1\le k\le p} (-1)^{(q-1)(k-1)} \rho_k^{\underline{b};J,L}\ (\partial_{[j_k]}(\underline{x}^{\underline{b}})) \cdot \underline{x}^{\underline{a}} \otimes dx_{J_k \sqcup L} \\ & -(-1)^{(p-1)(q-1)} \sum_{1\le k\le q} (-1)^{(p-1)(k-1)} \rho_k^{\underline{a};L,J}\ (\partial_{[j_k]}(\underline{x}^{\underline{a}})) \cdot \underline{x}^{\underline{b}} \otimes dx_{J \sqcup L_k},\end{aligned}$$

for certain scalars $\rho_k^{\underline{b};J,L}$ and $\rho_k^{\underline{a};L,J}$, where $\partial_{[j_k]}(\underline{x}^{\underline{b}})$ is defined in (4.7) and $\partial_{[j_k]}(\underline{x}^{\underline{b}})) \cdot \underline{x}^{\underline{a}}$ is given by the A^e-module structure over A, that is, if $\partial_{[j_k]}(\underline{x}^{\underline{b}})) = \sum_i u_i \otimes v_i \in A \otimes A$, then $\partial_{[j_k]}(\underline{x}^{\underline{b}})) \cdot \underline{x}^{\underline{a}} = \sum_i u_i \underline{x}^{\underline{a}} v_i$.

Proof We denote by $\cdot$ the composition of two maps instead of $\circ$, in order to avoid confusion with the circle product. We compute the bracket using the formula

$$[\alpha, \beta] = [\alpha \cdot \Psi_p, \beta \cdot \Psi_q] \cdot \Phi_{p+q-1}.$$

The element $\alpha = \underline{x}^{\underline{a}} \otimes dx_J$ as a map from $A \otimes A \otimes \bigwedge^p(V)$ to A sends $1 \otimes 1 \otimes \underline{x}^{\wedge I}$ to $\delta_{IJ} \underline{x}^{\underline{a}}$ for $I = (i_1, \dots, i_p)$, similarly the element $\beta = \underline{x}^{\underline{b}} \otimes dx_L$ as a map from $A \otimes A \otimes \bigwedge^q(V)$ to A sends $1 \otimes 1 \otimes \underline{x}^{\wedge I}$ to $\delta_{IL} \underline{x}^{\underline{b}}$. By formula (4.8) for Ψ_p, the map $\alpha \cdot \Psi_p : A \otimes A \otimes \overline{A}^{\otimes p} \to A \otimes A \otimes \bigwedge^p(V) \to A$ is given by

$$\alpha \cdot \Psi_p(1 \otimes 1 \otimes \underline{x}^{\underline{m}^1} \otimes \cdots \otimes \underline{x}^{\underline{m}^p}) = \mu_{(j_1, \cdots, j_p)}^{(\underline{m}^1, \cdots, \underline{m}^p)} \Big(\prod_{s=1}^{p} (\partial_{[j_s]}(\underline{x}^{\underline{m}^s})) \Big) \cdot \underline{x}^{\underline{a}},$$

where the scalar coefficient is defined by (4.9). We have a similar formula for $\beta \cdot \Psi_q$.

For $1 \le k \le p$, $(\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q) : A \otimes A \otimes \overline{A}^{\otimes p+q-1} \to A$ sends $1 \otimes 1 \otimes \underline{x}^{\underline{m}^1} \otimes \cdots \otimes \underline{x}^{\underline{m}^{p+q-1}}$ to

$$\mu_k \mu_J^{(\underline{m}^1, \cdots, \underline{m}^{k-1}, \underline{\tilde{m}}^k, \underline{m}^{k+q}, \cdots, \underline{m}^{p+q-1})} \mu_L^{(\underline{m}^k, \cdots, \underline{m}^{k+q-1})}.$$

$$\big(\partial_{[j_1]}(\underline{x}^{\underline{m}^1}) \cdots \partial_{[j_{k-1}]}(\underline{x}^{\underline{m}^{k-1}}) \partial_{[j_k]}(\underline{x}^{\underline{\tilde{m}}^k}) \partial_{[j_{k+1}]}(\underline{x}^{\underline{m}^{k+q}}) \cdots \partial_{[j_p]}(\underline{x}^{\underline{m}^{p+q-1}})\big) \cdot \underline{x}^{\underline{a}},$$

where μ_k and $\underline{\tilde{m}}^k$ are defined by $\mu_k \underline{x}^{\underline{\tilde{m}}^k} = \big(\prod_{t=1}^q (\partial_{[l_t]} \underline{x}^{\underline{m}^{t+k-1}})\big) \cdot \underline{x}^{\underline{b}}$.

For $I = (i_1, \dots, i_{p+q-1})$ with $1 \le i_1 < \cdots < i_{p+q-1} \le N$, let us compute $((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1}(1 \otimes 1 \otimes \underline{x}^{\wedge I})$. Indeed, by (4.3) and our identifications,

$$\Phi_{p+q-1}(1 \otimes 1 \otimes \underline{x}^{\wedge I}) = \sum_{\pi \in \mathrm{Sym}_{p+q-1}} \operatorname{sgn}(\pi) q_\pi^I \otimes 1 \otimes x_{i_{\pi(1)}} \otimes \cdots \otimes x_{i_{\pi(p+q-1)}}.$$

Now for a fixed $\pi \in \mathrm{Sym}_{p+q-1}$, as input into the formula of the previous paragraph, we have

$$\underline{m}^1 = e_{i_{\pi(1)}},\ \dots\ , \underline{m}^{p+q-1} = e_{i_{\pi(p+q-1)}},$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, the 1 in the ith position, and since $\partial_{[j]}(x_i) = \delta_{ij} \otimes 1$, the factor

$$\big(\partial_{[j_1]}(\underline{x}^{\underline{m}^1}) \cdots \partial_{[j_{k-1}]}(\underline{x}^{\underline{m}^{k-1}}) \partial_{[j_k]}(\underline{x}^{\underline{\tilde{m}}^k}) \partial_{[j_{k+1}]}(\underline{x}^{\underline{m}^{k+q}}) \cdots \partial_{[j_p]}(\underline{x}^{\underline{m}^{p+q-1}})\big) \cdot \underline{x}^{\underline{a}}$$

vanishes unless

$$j_1 = i_{\pi(1)}, \ldots, j_{k-1} = i_{\pi(k-1)}, l_1 = i_{\pi(k)}, \ldots, l_q = i_{\pi(k+q-1)}, j_{k+1} = i_{\pi(k+q)}, \ldots, j_p = i_{\pi(p+q-1)},$$

that is, when $I_k = \pi(I) := (i_{\pi(1)}, \cdots, i_{\pi(p+q-1)})$ or equivalently $I = J_k \sqcup L$. As long as $J_k \cap L = \varnothing$, there exist unique I and permutation $\pi_k \in \mathrm{Sym}_{p+q-1}$ satisfying this property. In this case,

$$\mu_k \underline{x}^{\underline{\tilde{m}}^k} = \Big(\prod_{t=1}^{q} \partial_{[l_t]}(\underline{x}^{\underline{m}^{t+q-1}})\Big) \cdot \underline{x}^{\underline{b}} = \underline{x}^{\underline{b}},$$

so that $\mu_k = 1$ and $\underline{\tilde{m}}^k = \underline{b}$. Consequently, the map $((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1}$ sends $1 \otimes 1 \otimes \underline{x}^{\wedge I}$ to $\delta_{I, J_k \sqcup L} \rho_k^{\underline{b}; J, L} \partial_{[j_k]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}}$ where

$$\rho_k^{\underline{b}; J, L} = \mathrm{sgn}(\pi_k) q^I_{\pi_k} \mu_J^{(e_{j_1}, \ldots, e_{j_{k-1}}, \underline{b}, e_{j_{k+1}}, \ldots, e_{j_p})} \mu_L^{(e_{\ell_1}, \ldots, e_{\ell_q})}$$

is determined by the permutation π_k as described above and the scalars defined by (4.4) and (4.9). Therefore,

$$((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1} = \rho_k^{\underline{b}; J, K} \partial_{[j_k]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} \otimes dx_{J_k \sqcup L}.$$

The formula in the statement can be obtained accordingly.

□

6. Gerstenhaber brackets for group extensions of quantum symmetric algebras

Let G be a finite group for which $|G| \neq 0$ in $\Bbbk$, acting linearly on a finite dimensional vector space V, thus inducing an action on the symmetric algebra $S(V)$ by automorphisms. In case the action preserves the relations on the quantum symmetric algebra $S_{\mathbf{q}}(V)$ as defined by (4.1), there is also an action on this algebra. This is always the case, for example, if G acts diagonally on the chosen basis $x_1, \ldots, x_N$ of V. We shall first recall the definition of a group extension, $S_{\mathbf{q}}(V) \rtimes G$, of $S_{\mathbf{q}}(V)$, and explain how the Koszul resolution of $S_{\mathbf{q}}(V) \rtimes G$ is related to that of $S_{\mathbf{q}}(V)$. In fact this works for an arbitrary Koszul algebra, as we shall explain next. Although this is well known, we include details for completeness.

Let $R \subseteq V \otimes V$ be a G-invariant subspace. Let $T_{\Bbbk}(V)$ denote the tensor algebra of V over $\Bbbk$. Suppose that $A = T_{\Bbbk}(V)/(R)$ is a *Koszul algebra* over $\Bbbk$, with the induced action of G. That is, the complex $K_\bullet(A)$ in which $K_0(A) = A \otimes A$, $K_1(A) = A \otimes V \otimes A$, and

$$K_i(A) = \bigcap_{j=0}^{i-2} (A \otimes V^{\otimes j} \otimes R \otimes V^{\otimes (i-2-j)} \otimes A),$$

for $i \geq 2$, is a free A-bimodule resolution of A under the differential from the bar resolution. In case $A = S_{\mathbf{q}}(V)$, this can be shown to be equivalent to the Koszul resolution given in Section 4. The group extension $A \rtimes G$ of A, or *skew group algebra*, is the tensor product $A \otimes \Bbbk G$ as a vector space, with multiplication given by $(a \otimes g)(b \otimes h) = a({}^g b) \otimes gh$ for all $a, b \in A$ and $g, h \in G$ (where we have used a left superscript to denote the group action). We shall denote elements of $A \rtimes G$ by $a \sharp g$, in place of $a \otimes g$, for $a \in A$ and $g \in G$, to indicate that they are elements of this skew group algebra. In this section we adapt and generalize the techniques of [7, 14] from $S(V) \rtimes G$ to $S_{\mathbf{q}}(V) \rtimes G$, explaining how to compute the Gerstenhaber bracket via the Koszul resolution and our chain maps from Section 4. In the next section we focus on some special cases to give explicit results.

We know that $A \rtimes G$ is a Koszul ring over $\Bbbk G$ (see [2, Definition 1.1.2 and Section 2.6]). In fact let $V \otimes \Bbbk G$ be the $\Bbbk G$-bimodule under the actions $g \cdot (v \otimes h) = {}^g v \otimes gh$ and $(v \otimes h) \cdot g = v \otimes hg$ for all $v \in V$ and $g, h \in G$. Then there is an algebra isomorphism

$$T_{\Bbbk G}(V \otimes \Bbbk G) \simeq T_{\Bbbk}(V) \rtimes G$$

sending $(v_1 \otimes g_1) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (v_{m-1} \otimes g_{m-1}) \otimes_{\Bbbk G} (v_m \otimes g_m)$ to $(v_1 \otimes {}^{g_1} v_2 \otimes \cdots \otimes {}^{g_1 \cdots g_{m-1}} v_m) \sharp g_1 \cdots g_m$, and the inverse isomorphism sends $(v_1 \otimes \cdots \otimes v_m) \sharp g$ to $(v_1 \otimes e_G) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (v_{m-1} \otimes e_G) \otimes_{\Bbbk G} (v_m \otimes g)$, where we write e_G or e for the unit element of G. Via this isomorphism, $R \otimes \Bbbk G$ becomes a $\Bbbk G$-subbimodule of $(V \otimes \Bbbk G) \otimes_{\Bbbk G} (V \otimes \Bbbk G) \simeq V \otimes V \otimes \Bbbk G$, and it induces an isomorphism of algebras, $A \rtimes G \simeq T_{\Bbbk G}(V \otimes \Bbbk G)/(R \otimes \Bbbk G)$.

The Koszul resolution $K_\bullet(A \rtimes G)$ of $A \rtimes G$ as a Koszul ring over $\Bbbk G$ is related to the Koszul resolution of A as follows:

$$\begin{aligned} K_0(A \rtimes G) &= (A \rtimes G) \otimes_{\Bbbk G} (A \rtimes G) \simeq A \otimes A \otimes \Bbbk G = K_0(A) \otimes \Bbbk G, \\ K_1(A \rtimes G) &= (A \rtimes G) \otimes_{\Bbbk G} (V \otimes \Bbbk G) \otimes_{\Bbbk G} (A \rtimes G) \simeq A \otimes V \otimes A \otimes \Bbbk G = K_1(A) \otimes \Bbbk G, \end{aligned}$$

and for $i \geq 2$,

$$\begin{aligned} &K_i(A \rtimes G) \\ &= (A \rtimes G) \otimes_{\Bbbk G} \bigcap_{j=0}^{i-2} ((V \otimes \Bbbk G)^{\otimes_{\Bbbk G} j} \otimes_{\Bbbk G} (R \otimes \Bbbk G) \otimes_{\Bbbk G} (V \otimes \Bbbk G)^{\otimes_{\Bbbk G} (i-2-j)}) \otimes_{\Bbbk G} (A \rtimes G) \\ &\simeq (A \rtimes G) \otimes_{\Bbbk G} \big(\bigcap_{j=0}^{i-2} (V^{\otimes j} \otimes R \otimes V^{\otimes (i-2-j)}) \otimes \Bbbk G \big) \otimes_{\Bbbk G} (A \rtimes G) \\ &\simeq \big(A \otimes \bigcap_{j=0}^{i-2} (V^{\otimes j} \otimes R \otimes V^{\otimes (i-2-j)}) \otimes A \big) \otimes \Bbbk G \\ &\simeq K_i(A) \otimes \Bbbk G. \end{aligned}$$

Notice that the above isomorphism is induced by the map sending

$$(a_0 \sharp g_0) \otimes_{\Bbbk G} ((a_1 \otimes g_1) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (a_p \otimes g_p)) \otimes_{\Bbbk G} (a_{p+1} \sharp g_{p+1})$$

to

$$(a_0 \otimes ({}^{g_0}a_1 \otimes \cdots \otimes {}^{g_0 \cdots g_{p-1}}a_p) \otimes {}^{g_0 \cdots g_p}a_{p+1}) \otimes (g_0 \cdots g_{p+1}).$$

The inverse isomorphism sends $(a_0 \otimes (a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1}) \sharp g$ to

$$(a_0 \sharp e) \otimes_{\Bbbk G} ((a_1 \otimes e) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (a_p \otimes e)) \otimes_{\Bbbk G} (a_{p+1} \sharp g).$$

One may check that this isomorphism commutes with the differentials. Therefore as complexes of $A \rtimes G$-bimodules,

$$K_\bullet(A \rtimes G) \simeq K_\bullet(A) \otimes \Bbbk G.$$

Under this isomorphism, the $A \rtimes G$-bimodule structure of $K_p(A) \otimes \Bbbk G$, for each $p \geq 0$, is given by

$$(b \sharp h)\big((a_0 \otimes (a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1}) \otimes g\big)(c \sharp k) = \big(b\,{}^h a_0 \otimes ({}^h a_1 \otimes \cdots \otimes {}^h a_p) \otimes {}^h a_{p+1}\,{}^{hg}c\big) \otimes hgk.$$

Similar statements apply to the normalized bar resolution:

$$B_\bullet(A \rtimes G) \simeq B_\bullet(A) \otimes \Bbbk G,$$

where the former involves tensor products over $\Bbbk G$, and the latter over $\Bbbk$.

Now we consider the case of $A := S_{\mathbf{q}}(V)$, under the condition that the action of G on V preserves the relations of $S_{\mathbf{q}}(V)$. The differentials on $K_\bullet(A \rtimes G)$ (respectively, $B_\bullet(A \rtimes G)$) are those induced by the Koszul resolution (respectively, bar resolution) of $S_{\mathbf{q}}(V)$, under the exact functor $- \otimes \Bbbk G$. Therefore the contracting homotopy and chain maps for $S_{\mathbf{q}}(V)$ may be extended to the corresponding complexes for $S_{\mathbf{q}}(V) \rtimes G$:

$$\Phi_\bullet \otimes \Bbbk G : K_\bullet(A \rtimes G) \simeq K_\bullet(A) \otimes \Bbbk G \to B_\bullet(A) \otimes \Bbbk G \simeq B_\bullet(A \rtimes G)$$

and

$$\Psi_\bullet \otimes \Bbbk G : B_\bullet(A \rtimes G) \simeq B_\bullet(A) \otimes \Bbbk G \to K_\bullet(A) \otimes \Bbbk G \simeq K_\bullet(A \rtimes G).$$

However, since $\Phi_\bullet$ and $\Psi_\bullet$ are in general not G-invariant, there is no reason to expect that $\Phi_\bullet \otimes \Bbbk G$ and $\Psi_\bullet \otimes \Bbbk G$ should be chain maps of complexes of $(A \rtimes G)^e$-modules. Since $|G|$ is invertible in $\Bbbk$, we can apply the Reynolds operator (that averages over images of group elements) to obtain chain maps of complexes of $(A \rtimes G)^e$-modules, which are denoted by $\tilde{\Phi}_\bullet$ and $\tilde{\Psi}_\bullet$ respectively. We have thus quasi-isomorphisms

$$\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G) \underset{\tilde{\Phi}^\bullet}{\overset{\tilde{\Psi}^\bullet}{\rightleftarrows}} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A)\otimes \Bbbk G, A\rtimes G)\ .$$

We shall use the complex on the left side to compute Lie brackets, via the chain maps $\tilde{\Psi}^\bullet$ and $\tilde{\Phi}^\bullet$. Notice that for $A = S_{\mathbf{q}}(V)$, we have

$$\begin{aligned}\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G) &\simeq \mathrm{Hom}_{\Bbbk G^e}(\textstyle\bigwedge^\bullet(V)\otimes \Bbbk G, A\rtimes G)\\ &\simeq \mathrm{Hom}_{\Bbbk G}(\textstyle\bigwedge^\bullet(V), A\rtimes G)\\ &\simeq \big(A\rtimes G\otimes \textstyle\bigwedge^\bullet(V^*)\big)^G.\end{aligned}$$

We wish to express the Lie bracket at the chain level, on elements of $\big(A\rtimes G\otimes \bigwedge^\bullet(V^*)\big)^G$. The method consists of the following steps (cf. [7, 14]).

(i) Compute the cohomology groups of the complexes $\big((A\rtimes G)\otimes \bigwedge^\bullet(V^*)\big)^G$. In case the action of G on V is diagonal, this computation is done in [11, Section 4].

(ii) Give a precise formula for the chain map Θ that is the composition

$$\Theta : \big((A\rtimes G)\otimes \textstyle\bigwedge^\bullet(V^*)\big)^G \xrightarrow{\sim} \mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G)$$

$$\xrightarrow{\tilde{\Psi}^\bullet} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A)\otimes \Bbbk G, A\rtimes G) \xrightarrow{\sim} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A\rtimes \Bbbk G), A\rtimes G).$$

(iii) Give a precise formula for the chain map Γ that is the composition

$$\Gamma : \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A\rtimes \Bbbk G), A\rtimes G) \xrightarrow{\sim} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A)\otimes \Bbbk G, A\rtimes G)$$

$$\xrightarrow{\tilde{\Phi}^\bullet} \mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G) \xrightarrow{\sim} \big((A\rtimes G)\otimes \textstyle\bigwedge^\bullet(V^*)\big)^G.$$

(iv) Use the formulae in the previous two steps to compute the Lie bracket of two cocycles given by Step (i).

We obtain thus

Theorem 6.1. *Let $\alpha,\beta \in ((A\rtimes G)\otimes \bigwedge^\bullet(V^*))^G$ be two cocycles. Then the Lie bracket of the two corresponding cohomological classes is represented by the cocycle*

$$[\alpha,\beta] = \Gamma\big([\Theta(\alpha),\Theta(\beta)]\big).$$

We see that the actual computations are rather hard and we shall perform these computations for the diagonal action case in the next section.

7. Diagonal actions

Assume now that G acts diagonally on the basis $\{x_1,\dots,x_N\}$ of V, in which case the action extends to an action of G on $S_{\mathbf{q}}(V)$ by automorphisms. Let $\chi_i : G \to \Bbbk^\times$ be the character of G corresponding to its action on x_i, that is

$$g\cdot x_i = \chi_i(g)x_i$$

for all $g\in G$, and $i = 1,\cdots,N$. For $I = (i_1,\cdots,i_p)$ with $1\le i_1<\cdots<i_p\le N$, define $\chi_I(g) = \prod_{j=1}^p \chi_{i_j}(g)$, and for $\underline{\ell}\in \mathbb{N}^N$, define $\chi_{\underline{\ell}}(g) = \prod_{1\le i\le N}\chi_i^{\ell_i}(g)$, for $g\in G$.

Let us make precise the action of G on $(A\rtimes G)\otimes \bigwedge^\bullet(V^*)$, occurring in the isomorphism of the previous section,

$$\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G) \simeq \big((A\rtimes G)\otimes \textstyle\bigwedge^\bullet(V^*)\big)^G.$$

Letting $g,h\in G$, $\underline{\ell}\in \mathbb{N}^N$, and $I = (i_1<\cdots<i_p)$, we have

$${}^h(\underline{x}^{\underline{\ell}}\sharp g\otimes dx_I) = {}^h(\underline{x}^{\underline{\ell}})\sharp {}^hg \ \otimes\ {}^h(dx_I) = \chi_{\underline{\ell}}(h)\chi_I(h^{-1})\ \underline{x}^{\underline{\ell}}\ \sharp\ hgh^{-1}\otimes dx_I.$$

In [11, Section 4], the authors compute homology of this chain complex $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ with the differential

$$d_p(\underline{x}^{\ell} \sharp g \otimes dx_I) = \sum_{i \notin I} (-1)^{\#\{s: i_s < i\}} \Big(\big(\prod_{s: i_s < i} q_{i_s, i} \big) x_i \underline{x}^{\ell} - \big(\prod_{s: i_s > i} q_{i, i_s} \big) \underline{x}^{\ell}\ {}^{g}x_i \Big) \sharp g \otimes dx_{I+e_i},$$

where e_i is the ith element of the canonical basis of $\mathbb{N}^N$, and $I+e_i$ is the sequence of $p+1$ integers obtained by inserting 1 in the ith position. Since the action of G is diagonal, this differential is G-equivariant. So the Reynolds operator is a chain map from $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ to $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$ which realizes $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$ as a direct summand of $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ as complexes. We shall see that in fact, the induced structure of $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$, as a complex, is the same as the one induced from the isomorphism

$$\mathrm{Hom}_{(A \rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) \simeq \big((A \rtimes G) \otimes \bigwedge\nolimits^\bullet (V^*)\big)^G.$$

We shall prove this fact in the first step below.

We follow the step-by-step outline given towards the end of Section 6. As we shall use the result of the second step in the first one, we begin with the second step.

Step (ii). As shown in the previous section, we have a series of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{(A \rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) &\simeq \mathrm{Hom}_{(\Bbbk G)^e}(\textstyle\bigwedge^\bullet(V) \otimes \Bbbk G, A \rtimes G) \\ \simeq \mathrm{Hom}_{\Bbbk G}(\textstyle\bigwedge^\bullet(V), A \rtimes G) &\simeq \big((A \rtimes G) \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G. \end{aligned}$$

A map $f \in \mathrm{Hom}_{(A \rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ corresponds to $f_1 \in \mathrm{Hom}_{\Bbbk G^e}(\bigwedge^p V \otimes \Bbbk G, A \rtimes G)$ via

$$f_1(\underline{x}^{\wedge I} \otimes g) = f(1 \otimes \underline{x}^{\wedge I} \otimes 1 \otimes g)$$

and

$$f(a_0 \otimes \underline{x}^{\wedge I} \otimes a_{p+1} \otimes g) = (a_0 \sharp e) f_1(\underline{x}^{\wedge I} \otimes g)({}^{g^{-1}}a_{p+1} \sharp e).$$

The element $f_1 \in \mathrm{Hom}_{\Bbbk G^e}(\bigwedge^p V \otimes \Bbbk G, A \rtimes G)$ corresponds to $f_2 \in \mathrm{Hom}_{\Bbbk G}(\bigwedge^p V, A \rtimes G)$ via

$$f_2(\underline{x}^{\wedge I}) = f_1(\underline{x}^{\wedge I} \otimes e)$$

and

$$f_1(\underline{x}^{\wedge I} \otimes g) = f_2(\underline{x}^{\wedge I})(1 \sharp g).$$

Finally, $f_2 \in \mathrm{Hom}_{\Bbbk G}(\bigwedge^p V, A \rtimes G)$ corresponds to $f_3 \in \big((A \rtimes G) \otimes \bigwedge^p(V^*)\big)^G$ via

$$f_3 = \sum_{|I|=p} f_2(\underline{x}^{\wedge I}) \otimes dx_I,$$

and for $f_3 = \sum_{|J|=p} \sum_{g \in G} (a_{J,g} \sharp g) \otimes dx_J \in \big(A \rtimes G \otimes \bigwedge^p(V^*)\big)^G$, the corresponding $f_2 \in \mathrm{Hom}_{\Bbbk G}(\bigwedge^p V, A \rtimes G)$ sends $\underline{x}^{\wedge I}$ to $\sum_{g \in G} a_{I,g} \sharp g$.

Altogether, $f \in \mathrm{Hom}_{(A \rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ corresponds to $f_3 \in \big(A \rtimes G \otimes \bigwedge^p V^*\big)^G$ via

$$f_3 = \sum_{|I|=p} f(1 \otimes \underline{x}^{\wedge I} \otimes 1 \otimes e) \otimes dx_I$$

and for $f_3 = \sum_{|J|=p} \sum_{g \in G}\ a_{J,g} \sharp g\ \otimes dx_J \in \big(A \rtimes G \otimes \bigwedge^p(V^*)\big)^G$,

$$f(a_0 \otimes \underline{x}^{\wedge I} \otimes a_{p+1} \otimes g) = \sum_{h \in G} (a_0 \sharp e)(a_{I,h} \sharp h)(1 \sharp g)({}^{g^{-1}}a_{p+1} \sharp e) = \sum_{h \in G}\ a_0 a_{I,h}{}^{h}(a_{p+1})\ \sharp hg.$$

Now for $\alpha = a \sharp g \otimes dx_J \in A \rtimes G \otimes \bigwedge^p(V^*)$, the Reynolds operator

$$\mathcal{R} : A \rtimes G \otimes \bigwedge\nolimits^p(V^*) \to (A \rtimes G \otimes \bigwedge\nolimits^p(V^*))^G$$

gives $f_3 = \frac{1}{|G|} \sum_{h \in G} \chi_J(h^{-1})\ {}^{h}a \sharp\ hgh^{-1} \otimes dx_J$ and thus α corresponds to $f \in \mathrm{Hom}_{(A \rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ sending $a_0 \otimes \underline{x}^{\wedge I} \otimes a_{p+1} \otimes k$ to $\delta_{IJ} \frac{1}{|G|} \sum_{h \in G} \chi_J(h^{-1})\ a_0({}^{h}a)({}^{hgh^{-1}}a_{p+1})\ \sharp\ hgh^{-1}k$.

We shall compute $\Theta\mathcal{R}(\alpha) \in \mathrm{Hom}_{\Bbbk}((A \rtimes G)^{\otimes p}, A \rtimes G)$ corresponding to f with $a = \underline{x}^{\underline{\ell}}$, which is the composition

$$\begin{aligned}
&\underline{x}^{\underline{\ell}^1} \sharp g_1 \otimes \cdots \otimes \underline{x}^{\underline{\ell}^p} \sharp g_p \\
&\mapsto \underline{x}^{\underline{\ell}^1} \otimes {}^{g_1}(\underline{x}^{\underline{\ell}^2}) \otimes \cdots \otimes {}^{g_1\cdots g_{p-1}}(\underline{x}^{\underline{\ell}^p}) \ \sharp \ g_1\cdots g_p \\
&= \chi_{\underline{\ell}^2}(g_1)\cdots\chi_{\underline{\ell}^p}(g_1\cdots g_{p-1})\underline{x}^{\underline{\ell}^1} \otimes \ \cdots \otimes \underline{x}^{\underline{\ell}^p} \ \sharp \ g_1\cdots g_p \\
&\mapsto \chi_{\underline{\ell}^2}(g_1)\cdots\chi_{\underline{\ell}^p}(g_1\cdots g_{p-1}) \sum_{|I|=p} \sum_{\substack{0\le r_s\le \ell^s_{i_s}-1 \\ s=1,\cdots,p}} \mu \ \underline{x}^{\underline{Q}} \otimes x^{\wedge I} \otimes \underline{x}^{\underline{\widehat{Q}}} \ \otimes \ g_1\cdots g_p \quad (\text{use } \Psi_\bullet) \\
&\mapsto \frac{1}{|G|}\chi_{\underline{\ell}^2}(g_1)\cdots\chi_{\underline{\ell}^p}(g_1\cdots g_{p-1}) \sum_{h\in G} \sum_{\substack{0\le r_s\le \ell^s_{j_s}-1 \\ s=1,\cdots,p}} \lambda\mu\cdot \\
&\qquad \chi_J(h^{-1})\chi_{\underline{\ell}}(h)\chi_{\underline{\widehat{Q}}}(hgh^{-1}) \ \ \underline{x}^{\underline{\ell}^1+\cdots+\underline{\ell}^p+\underline{\ell}-J} \ \sharp \ hgh^{-1}g_1\cdots g_p,
\end{aligned}$$

where, as in (4.5),

$$\begin{aligned}
\mu &= \mu^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)} \\
\underline{Q} &= \underline{Q}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}, \\
\underline{\widehat{Q}} &= \underline{\widehat{Q}}^{(\underline{\ell}^1,\cdots,\underline{\ell}^p;\ j_1,\cdots,j_p)}_{(r_1,\cdots,r_p)}, \\
\lambda\ \underline{x}^{\underline{Q}}\underline{x}^{\underline{\ell}}\underline{x}^{\underline{\widehat{Q}}} &= \underline{x}^{\underline{\ell}^1+\cdots+\underline{\ell}^p+\underline{\ell}-I} \in S_{\mathbf{q}}(V).
\end{aligned}$$

This completes the second step.

Step (i). We shall identify the cohomology groups of the complexes $\big(A \rtimes G \otimes \bigwedge^\bullet(V^*)\big)^G$ with the computation in [11, Section 4]. It suffices to see that the map

$$A \rtimes G \otimes \bigwedge\nolimits^\bullet(V^*) \xrightarrow{\mathcal{R}} \big(A \rtimes G \otimes \bigwedge\nolimits^\bullet(V^*)\big)^G \xrightarrow{\sim} \mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G)$$

is a chain map, where $A \rtimes G \otimes \bigwedge^\bullet(V^*)$ is endowed with the differential given in [11, Section 4] and $\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G)$ with the differential induced from that of $K_\bullet(A)$. We shall use the computations in the second step to prove this statement.

In fact, given $a\sharp g \otimes dx_I \in A \rtimes G \otimes \bigwedge^p(V^*)$, by the second step, it corresponds to the map $f \in \mathrm{Hom}_{(A\rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ sending $a_0 \otimes \underline{x}^{\wedge J} \otimes a_{p+1} \otimes k$ to

$$\delta_{IJ}\frac{1}{|G|}\sum_{h\in G}\chi_I(h^{-1}) \ a_0({}^h a)({}^{hgh^{-1}}a_{p+1}) \ \sharp \ hgh^{-1}k.$$

Now df is the composition (for $k \in G$ and $L = (l_1, \cdots, l_{p+1})$)

$$\begin{aligned}
&1 \otimes \underline{x}^{\wedge L} \otimes 1 \otimes k \mapsto \sum_{j=1}^{p+1}(-1)^{j-1}\big(\big(\prod_{s=1}^{j} q_{l_s,l_j}\big)x_{l_j} \otimes \underline{x}^{\wedge(L-e_{l_j})} \otimes 1 \otimes k - \big(\prod_{s=j}^{p+1} q_{l_j,l_s}\big)1 \otimes \underline{x}^{\wedge(L-e_{l_j})} \otimes x_{l_j} \otimes k\big) \\
&\mapsto \frac{1}{|G|}\sum_{h\in G}\sum_{j=1}^{p+1}(-1)^{j-1}\delta_{I,L-e_{l_j}}\chi_I(h^{-1})\big(\big(\prod_{s=1}^{j} q_{l_j,l_j}\big)x_{l_j} \ {}^h a - \big(\prod_{s=j}^{p+1} q_{l_j,l_j}\big)\chi_{l_j}(hgh^{-1})^h \ a x_{l_j}\big)\sharp hgh^{-1}k\big).
\end{aligned}$$

On the other hand, by [11, Section 4],

$$d_p(\underline{x}^{\underline{\ell}}\sharp g \otimes dx_I) = \sum_{i\notin I}(-1)^{\#\{s:i_s<i\}}\big(\big(\prod_{s:i_s<i} q_{i_s,i}\big)x_i\underline{x}^{\underline{\ell}} - \big(\prod_{s:i_s>i} q_{i,i_s}\big)\underline{x}^{\underline{\ell}} \ {}^g x_i\big)\sharp g \otimes dx_{I+e_i},$$

which corresponds to the map sending $1 \otimes \underline{x}^{\wedge L} \otimes 1 \otimes k$ to

$$\frac{1}{|G|}\sum_{h\in G}\sum_{i\notin I}(-1)^{\#\{s:i_s<i\}}\big(\big(\prod_{s:i_s<i} q_{i_s,i}\big)\chi_L(h^{-1})\delta_{L,I+e_i}\chi_i(h)x_i \ {}^h a - \big(\prod_{s:i_s>i} q_{i,i_s}\big)\chi_i(hg) \ {}^h a x_i\big)\sharp hgh^{-1}k.$$

One sees readily that these two expressions are the same.

Let us recall the result of [11, Section 4]. For $g \in G$, define

$$C_g = \{\underline{c} \in (\mathbb{N} \cup \{-1\})^N \mid \text{for each } i \in \{1, \cdots, N\}, \prod_{s=1}^{N} q_{i,s}^{c_s} = \chi_i(g) \text{ or } c_i = -1\}.$$

For $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^N$, the authors of [11] introduced certain subcomplexes $K^{\bullet}_{g,\gamma}$ of $(A \rtimes G) \otimes \bigwedge^p(V^*)$ with $(A \rtimes G) \otimes \bigwedge^p(V^*) = \bigoplus_{g,\gamma} K^{\bullet}_{g,\gamma}$. They also proved that if $\gamma \in C_g$, the subcomplex $K^{\bullet}_{g,\gamma}$ has zero differential and if $\gamma \notin C_g$, the subcomplex $K^{\bullet}_{g,\gamma}$ is acyclic. (We do not define $K^{\bullet}_{g,\gamma}$ here as we shall not need the details.) Using this information, for $m \in \mathbb{N}$, [11, Theorem 4.1] gives

$$\mathrm{H}^m\left((A \rtimes G) \otimes \bigwedge^p(V^*)\right) \simeq \mathrm{HH}^m(A, A \rtimes G) \simeq \bigoplus_{g \in G} \bigoplus_{\substack{\underline{b} \in \{0,1\}^N \\ |\underline{b}| = m}} \bigoplus_{\substack{\underline{a} \in \mathbb{N}^N \\ \underline{a} - \underline{b} \in C_g}} \mathrm{span}_{\mathbb{k}}\{\underline{x}^{\underline{a}} \sharp g \otimes dx_{\underline{b}}\}.$$

We shall use these notations when expressing the Lie bracket of two cohomological classes. This completes the first step.

Step (iii). Now given a map $f \in \mathrm{Hom}_{\mathbb{k}}((A \rtimes G)^{\otimes \bullet}, A \rtimes G)$, we compute the corresponding $\Gamma(f) \in \left((A \rtimes G) \otimes \bigwedge^p(V^*)\right)^G$. Direct inspection gives

$$\Gamma(f) = \sum_{|I|=p} \sum_{\pi \in \mathrm{Sym}_p} \mathrm{sgn}\pi \; q_\pi^I \; f(x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(p)}} \sharp e) \otimes dx_I,$$

where $q_\pi^I = q_\pi^{i_1, \cdots, i_p}$ is defined in (4.4), and e denotes the identity group element.

Step (iv). We can now compute the Lie bracket of two cohomological classes.

Let

$$\alpha = \underline{x}^{\underline{a}} \sharp g \otimes dx_J \quad \text{and} \quad \beta = \underline{x}^{\underline{b}} \sharp h \otimes dx_L$$

for some group elements $g, h \in G$, where $J = (j_1, \ldots, j_p)$ and $L = (l_1, \ldots, l_q)$ and such that $\underline{a} - J \in C_g$ and $\underline{b} - K \in C_h$. Then α and β are cocycles for the complex $A \rtimes G \otimes \bigwedge^{\bullet}(V^*)$, because the subcomplex $K^{\bullet}_{g,\gamma}$ of $\mathrm{Hom}_{A^e}(K_{\bullet}(A), A \rtimes G)$ is a complex with zero differential whenever $\gamma \in C_g$ (for details, see [11, Section 4]). Consequently, $\mathcal{R}\alpha$ and $\mathcal{R}\beta$ are G-invariant cocycles where, as before, $\mathcal{R}$ is the Reynold's operator. The bracket operation on Hochschild cohomology is determined by its values on cocycles of this form.

Theorem 7.1. *In case G acts diagonally on the basis $x_1, \ldots, x_N$, the graded Lie bracket of $\mathcal{R}\alpha$ and $\mathcal{R}\beta$, where $\alpha = \underline{x}^{\underline{a}} \sharp g \otimes dx_J$ and $\beta = \underline{x}^{\underline{b}} \sharp h \otimes dx_L$, is*

$$\begin{aligned}
[\mathcal{R}\alpha, \mathcal{R}\beta] \;\; = \;\; & \sum_{1 \le s \le p} (-1)^{(q-1)(s-1)} \frac{1}{|G|^2} \sum_{k,\ell \in G} \rho_s^{\alpha,\beta} \; \partial_{[j_s]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} \; \sharp \; kgk^{-1}\ell h \ell^{-1} \otimes dx_{J_s \sqcup L} \\
& -(-1)^{(p-1)(q-1)} \sum_{1 \le s \le q} (-1)^{(p-1)(s-1)} \frac{1}{|G|^2} \sum_{k,\ell \in G} \rho_s^{\beta,\alpha} \; \partial_{[l_s]}(\underline{x}^{\underline{a}}) \cdot \underline{x}^{\underline{b}} \; \sharp \; \ell h \ell^{-1} kgk^{-1} \otimes dx_{J \sqcup L_s},
\end{aligned}$$

for certain coefficients $\rho_s^{\alpha,\beta}$ and $\rho_s^{\beta,\alpha}$.

Remark 7.2. This formula generalizes Theorem 5.1 (which is the case $G = 1$) and [14, Corollary 7.3] (which is the case $q_{i,j} = 1$ for all i, j).

Proof We may compute $[\mathcal{R}(\alpha), \mathcal{R}(\beta)]$ as $\Gamma([\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)])$.

Now by the third step,

$$\Gamma([\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)]) = \sum_{|I|=p+q-1} \sum_{\pi \in \mathrm{Sym}_{p+q-1}} \mathrm{sgn}(\pi) \; q_\pi^I \; [\Theta(\mathcal{R}\alpha), \Theta(\mathcal{R}\beta)](x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(p+q-1)}} \sharp e) \otimes dx_I.$$

Note that Ψ_p applied to an element of the form $1\otimes x_{c_1}\otimes\cdots\otimes x_{c_p}\otimes 1$ is $1\otimes x_{c_1}\wedge\cdots\wedge x_{c_p}\otimes 1$ if $1\leq c_1<\cdots<c_p\leq N$, and is 0 otherwise. This observation will simplify considerably the computation of $[\Theta\mathcal{R}(\alpha),\Theta\mathcal{R}(\beta)](x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp e)$. For $1\leq s\leq p$, we have

$$\begin{aligned}&(\Theta\mathcal{R}(\alpha)\circ_s\Theta\mathcal{R}(\beta)(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes x_{j_{\pi(p)}}\sharp e)\\ &=\Theta\mathcal{R}(\alpha)(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes\Theta\mathcal{R}(\beta)(x_{i_{\pi(s)}}\sharp e\otimes\cdots\otimes x_{i_{\pi(s+q-1)}}\sharp e)\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp e).\end{aligned}$$

By the second step, a simple computation shows that $\Theta\mathcal{R}(\beta)(x_{i_{\pi(s)}}\sharp e\otimes\cdots\otimes x_{i_{\pi(s+q-1)}}\sharp e)$ is nonzero only when

$$i_{\pi(s)}=l_1,\ldots,i_{\pi(s+q-1)}=l_q,$$

in which case it is equal to $\frac{1}{|G|}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\ \underline{x}^{\underline{b}}\sharp\ell h\ell^{-1}$. Therefore, when

$$i_{\pi(s)}=l_1,\ldots,i_{\pi(s+q-1)}=l_q,$$

we have

$$\begin{aligned}&\Theta\mathcal{R}(\alpha)(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes\Theta\mathcal{R}(\beta)(x_{i_{\pi(s)}}\sharp e\otimes\cdots\otimes x_{i_{\pi(s+q-1)}}\sharp e)\otimes x_{i_{\pi(s+q)}}\sharp e\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp e)\\ &=\Theta\mathcal{R}(\alpha)(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes\big(\frac{1}{|G|}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\underline{x}^{\underline{b}}\ \sharp\ \ell h\ell^{-1}\otimes x_{i_{\pi(s+q)}}\sharp e\big)\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp e)\\ &=\frac{1}{|G|}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\Theta\mathcal{R}(\alpha)(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes\underline{x}^{\underline{b}}\ \sharp\ \ell h\ell^{-1}\otimes x_{i_{\pi(s+q)}}\sharp e)\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp e).\end{aligned}$$

Applying the second step, in order that the above expression be nonzero, the following condition must hold:

$$j_1=i_{\pi(1)},\ldots,j_{s-1}=i_{\pi(s-1)},j_{s+1}=i_{\pi(s+q)},\ldots,j_p=i_{\pi(p+q-1)}.$$

When

$$i_{\pi(s)}=l_1,\ldots,i_{\pi(s+q-1)}=l_q,j_1=i_{\pi(1)},\ldots,j_{s-1}=i_{\pi(s-1)},j_{s+1}=i_{\pi(s+q)},\ldots,j_p=i_{\pi(p+q-1)},$$

we have

$$\begin{aligned}&(\Theta\mathcal{R}(\alpha)\circ_s\Theta\mathcal{R}(\beta)(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes x_{j_{\pi(p)}}\sharp e)\\ &=\frac{1}{|G|^2}\sum_{k\in G}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\chi_{j_{s+1}}(\ell h\ell^{-1})\cdots\chi_{j_p}(\ell h\ell^{-1})\cdot\\ &\sum_{0\leq r\leq b_{j_s}-1}\lambda\mu\chi_J(k^{-1})\chi_{\underline{a}}(k)\chi_{\hat{Q}}(kgk^{-1})\ \underline{x}^{\underline{a}+\underline{b}-e_{j_s}}\sharp kgk^{-1}\ell h\ell^{-1},\end{aligned}$$

where

$$\begin{aligned}&\underline{x}^{Q}=x_{j_s}^{r}\ x_{j_s+1}^{b_{j_s+1}}\cdots x_N^{b_N},\\ &\underline{x}^{\widehat{Q}}=x_1^{b_1}\cdots x_{j_s-1}^{b_{j_s-1}}\ x_{j_s}^{b_{j_s}-r+1},\\ &\mu\ \underline{x}^{Q}\underline{x}^{\widehat{Q}}=x_{j_1}\cdots x_{j_{s-1}}\ \underline{x}^{\underline{b}}\ x_{j_{s+1}}\cdots x_{j_p}\in S_{\mathbf{q}}(V),\\ &\lambda\ \underline{x}^{Q}\underline{x}^{\underline{a}}\underline{x}^{\widehat{Q}}=\underline{x}^{\underline{a}+\underline{b}-e_{j_s}}\in S_{\mathbf{q}}(V).\end{aligned}$$

We see that in this case we have $I=J_s\sqcup L$. Furthermore, if this is the case, there is a unique permutation $\pi_s\in\mathrm{Sym}_{p+q-1}$ such that

$$j_1=i_{\pi_s(1)},\ldots,j_{s-1}=i_{\pi_s(s-1)},i_{\pi_s(s)}=l_1,\ldots,i_{\pi_s(s+q-1)}=l_q,j_{s+1}=i_{\pi_s(s+q)},\ldots,j_p=i_{\pi_s(p+q-1)},$$

that is, $\pi_s(I)=J_s\sqcup L$ as introduced before Theorem 5.1. We obtain that when $I=J_s\sqcup L$ and $\pi=\pi_s$ for $1\leq s\leq p$,

$$(\Theta\mathcal{R}(\alpha)\circ_s\Theta\mathcal{R}(\beta)(x_{i_{\pi_s(1)}}\sharp e\otimes\cdots\otimes x_{i_{\pi_s(p+q-1)}}\sharp e)=\frac{1}{|G|^2}\sum_{k,\ell\in G}\rho_s^{\alpha,\beta}\ \partial_{[j_s]}(\underline{x}^{\underline{b}})\cdot\underline{x}^{\underline{a}}\ \sharp\ kgk^{-1}\ell h\ell^{-1},$$

for a certain coefficient $\rho_s^{\alpha,\beta}$ determined by the above data.

Finally

$$
\begin{aligned}
&\Gamma([\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)]) \\
&= \sum_{|I|=p+q-1} \sum_{\pi\in \mathrm{Sym}_{p+q-1}} \mathrm{sgn}(\pi)\ q_\pi^I\ [\Theta(\mathcal{R}\alpha), \Theta(\mathcal{R}\beta)](x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp e)\otimes dx_I \\
&= \frac{1}{|G|^2}\sum_{k,\ell\in G}\sum_{1\le s\le p}(-1)^{(q-1)(s-1)}\rho_s^{\alpha,\beta}\ \partial_{[j_s]}(\underline{x}^{\underline{b}})\cdot \underline{x}^{\underline{a}}\ \sharp\ kgk^{-1}\ell h\ell^{-1}\otimes dx_I \\
&\quad -(-1)^{(p-1)(q-1)}\frac{1}{|G|^2}\sum_{k,\ell\in G}\sum_{1\le s\le q}(-1)^{(p-1)(s-1)}\rho_s^{\beta,\alpha}\ \partial_{[\ell_s]}(\underline{x}^{\underline{a}})\cdot \underline{x}^{\underline{b}}\ \sharp\ \ell h\ell^{-1}kgk^{-1}\otimes dx_I.
\end{aligned}
$$

□

In this diagonal case, the following corollary is immediate, since the difference operators in the bracket formula take 1 to 0. It generalizes [14, Theorem 8.1].

Corollary 7.3. *Assume G acts diagonally on the chosen basis $x_1,\dots,x_N$ of V, and let $\alpha = 1\sharp g\otimes dx_J$ and $\beta = 1\sharp h\otimes dx_L$. Then $[\mathcal{R}\alpha,\mathcal{R}\beta] = 0\in HH^\bullet(A\rtimes G)$.*

In fact, this result can be seen to hold in the nondiagonal case as well, even without an explicit description of Hochschild cocycles in that case. Nonetheless we may still use a general argument for those cocycles having a particular form.

Corollary 7.4. *Assume G acts on V, not necessarily diagonally. Let α and β be cocycles in $(A\rtimes G\otimes\bigwedge^\bullet(V^*))^G$ for which α (respectively, β) is a linear combination of elements of the form $1\sharp g\otimes dx_J$ (respectively, $1\sharp h\otimes dx_L$). Then $[\alpha,\beta] = 0\in HH^\bullet(A\rtimes G)$. In particular, if α is a 2-cocycle, then it is a noncommutative Poisson structure.*

Proof The proof is similar to that of Theorem 7.1. However, rather than computing explicitly, we shall only explain why the bracket is 0.

We compute $[\alpha,\beta]$ using Theorem 6.1. Consider α as a homomorphism in $\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G)$, then it maps into $\Bbbk\otimes\Bbbk G\subset A\rtimes G$. Now by Theorem 6.1

$$[\alpha,\beta] = [\alpha\cdot\tilde\Psi_\bullet, \beta\cdot\tilde\Psi_\bullet]\cdot\tilde\Phi_\bullet.$$

Here $\tilde\Phi_\bullet$ and $\tilde\Psi_\bullet$ are chain maps of complexes of $(A\rtimes G)^e$-modules obtained by applying the Reynolds operator (that averages over images of group elements) to $\Phi_\bullet$ and $\Psi_\bullet$ respectively. So one needs to consider certain terms like $(\alpha\cdot{}^a\Psi)\circ_k(\beta\cdot{}^b\Psi)$ applied to ${}^c\Phi(1\otimes 1\otimes \underline{x}^{\wedge I})$ for $k\ge 1$, and $a,b,c\in G$.

Recall that, if $I=(i_1,\dots,i_p)$, then

$$\Phi(1\otimes1\otimes\underline{x}^{\wedge I}) = \sum_{\pi\in\mathrm{Sym}_p}(\mathrm{sgn}\pi)q_\pi^{i_1,\dots,i_p}\otimes x_{i_{\pi(1)}}\otimes\cdots\otimes x_{i_{\pi(p)}}\otimes 1.$$

So ${}^c\Phi(1\otimes1\otimes\underline{x}^{\wedge I})$ is a linear combination of terms of the form $1\otimes x_{j_1}\otimes\cdots\otimes x_{j_p}\otimes 1$ for $1\le j_1,\cdots,j_p\le N$. In applying $(\alpha\cdot{}^a\Psi)\circ_k(\beta\cdot{}^b\Psi)$ to each term above, one first applies ${}^b\Psi$ to $1\otimes x_{j_k}\otimes\cdots\otimes x_{j_{k+m-1}}\otimes 1$, if the degree of β is m. By (4.5),

$$\Psi_m(1\otimes x_{j_k}\otimes\cdots\otimes x_{j_{k+m-1}}\otimes 1) = \mu\otimes x_{j_k}\wedge\cdots\wedge x_{j_{k+m-1}}\otimes 1$$

for some scalar μ and so ${}^b\Psi_m(1\otimes x_{j_k}\otimes\cdots\otimes x_{j_{k+m-1}}\otimes 1)$ is a linear combination of terms of the form $1\otimes x_{\ell_1}\wedge\cdots\wedge x_{\ell_m}\otimes 1$ with $1\le \ell_1<\cdots<\ell_m\le N$.

Applying β to the result, we obtain 0 unless $L=(\ell_1,\cdots,\ell_m)$ for some L for which $1\sharp h\otimes dx_L$ has a nonzero coefficient in the expression β, in which case we obtain a nonzero scalar multiple of $1\sharp h$ for that term. After factoring h to the right, this becomes 0 as an element of the normalized bar resolution. The same argument applies to each term in $[\alpha,\beta]$, and so $[\alpha,\beta]=0$.

For the last statement, recall that a noncommutative Poisson structure is simply a Hochschild 2-cocycle whose square bracket is a coboundary.

□

Compare to the proof of [12, Theorem 4.6], of which the above corollary is a consequence via the alternative route of algebraic deformation theory.

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