CLIFFORD CORRESPONDENCE FOR FINITE DIMENSIONAL HOPF ALGEBRAS

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Abstract. Let $B\#_\sigma H$ be a crossed product algebra over an algebraically closed field, with H a finite dimensional Hopf algebra. We give an explicit equivalence between the category of finite dimensional $B\#_\sigma H$-modules whose restriction to B is a direct sum of copies of a stable irreducible B-module, and the category of modules for a twisted product of H with the field. This describes all finite dimensional irreducible $B\#_\sigma H$-modules containing a stable irreducible B-submodule, and thus generalizes the classical stable Clifford correspondence for groups. In case H is cocommutative, we extend this correspondence to the nonstable case.

1. Introduction

One of the main results of classical Clifford theory is the Clifford correspondence, which explicitly describes a relationship between modules for a group and modules for a normal subgroup [4]. The Clifford correspondence has been generalized in various ways in [1, 5, 6, 8, 9, 12, 13]. In particular, Rieffel gives in [9] a generalization of the Clifford correspondence to some classes of ring extensions. When H is a Hopf algebra with bijective antipode, Schneider in [12] and van Oystaeyen and Zhang in [13] give a stable Clifford correspondence for a faithfully flat H-Galois extension (equivalently, an H-Galois extension with total integral). In this note, we prove a more explicit version of the stable Clifford correspondence, reminiscent of Clifford's original result, under stronger hypotheses. Further application of Schneider's results [12] shows that our formula also holds under his more general hypotheses. We then use results of Schneider in [10] to further extend this Clifford correspondence to the nonstable case when H is cocommutative.

Specifically, let k be an algebraically closed field, H a finite dimensional Hopf algebra over k, and $A = B\#_\sigma H$ a crossed product algebra of a k-algebra B with H. In particular if A is a finite dimensional Hopf algebra with a normal Hopf subalgebra B, then A is such a crossed product with $H = A/AB^+$ [11]. Let T be a finite dimensional irreducible left B-module, and $E = \mathrm{End}_A(A \otimes_B T)^{op}$, where $A \otimes_B T$ is the induced A-module.

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First suppose T is A-*stable* [10], as defined in Section 2. Then E is isomorphic to a twisted product $k\#_\alpha H = k_\alpha[H]$, analogous to the twisted group algebra appearing in the classical Clifford correspondence. In Theorem 2.2 we give the explicit structure of an A-module to the vector space $T \otimes_k U$ for any E-module U; we also point out that this structure may be shown to hold more generally using our ideas and work of Schneider [12]. In Theorem 3.1, we show that our construction provides an equivalence between the category of finite dimensional E-modules and the category of finite dimensional A-modules whose restriction to B is isomorphic to a direct sum of copies of T. Alternatively, this theorem follows from [12, Remark 5.8 (2)] or [13, Theorem 5.4]. In Lemma 3.2, we show that if V is a finite dimensional irreducible A-module whose restriction to B contains an irreducible A-stable submodule T, then V is semisimple on restriction to B; this generalizes a result of Clifford in the stable case. In Corollary 3.3 we show that every finite dimensional irreducible A-module containing T as a B-submodule has the form $T \otimes_k U$ for some irreducible E-module U; this generalizes the stable Clifford correspondence for groups.

We next consider the general case in which T is not necessarily A-stable, and we assume that H is cocommutative to obtain a Clifford correspondence. In this case H is also pointed as k is algebraically closed, and so the *stabilizer* H_{st} of T in H (defined in Section 3) is a Hopf subalgebra of H [10]. Letting $S = B\#_\sigma H_{st}$, and using the results of [10] and Section 2, we obtain in Theorem 3.4 a one-to-one correspondence between finite dimensional irreducible S-modules containing T as a B-submodule and finite dimensional irreducible A-modules containing T as a B-submodule. This correspondence is given by induction of modules from S to A. Further combining this correspondence with the results of Section 2, we obtain in Corollary 3.5 a generalization of the full Clifford correspondence for groups in the cocommutative case, given as a one-to-one correspondence of irreducible modules.

Throughout, all our modules (and module categories) will be left and finite dimensional. The field k is always assumed to be algebraically closed, and $\otimes = \otimes_k$.

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2. Construction of A-modules from a stable irreducible B-module

Let H be a finite dimensional Hopf algebra over k with comultiplication $\Delta : H \to H \otimes H$, counit $\epsilon : H \to k$, and antipode $S : H \to H$. We will use the summation notation $\Delta(h) = \sum h_1 \otimes h_2$ for the coproduct [7, 1.4.2]. Let $A = B\#_\sigma H$ be a crossed product algebra of a k-algebra B with H. That is, there is a convolution-invertible k-linear map $\sigma : H \otimes H \to B$, and a k-linear map $H \otimes B \to B$ denoted $h \otimes b \mapsto h \cdot b$, satisfying

$$\begin{aligned} h\cdot(\ell\cdot b) &= \sum \sigma(h_1,\ell_1)(h_2\ell_2\cdot b)\sigma^{-1}(h_3,\ell_3), \\ \sum[h_1\cdot\sigma(\ell_1,m_1)]\sigma(h_2,\ell_2m_2) &= \sum\sigma(h_1,\ell_1)\sigma(h_2\ell_2,m), \\ \sigma(h,1) &= \sigma(1,h) \;=\; \epsilon(h)1, \\ h\cdot(bc)=\sum(h_1\cdot b)(h_2\cdot c) \quad &, \quad h\cdot 1 \;=\; \epsilon(h)1\;,\; 1\cdot b \;=\; b, \end{aligned}$$

for all $h,\ell,m\in H$ and $b,c\in B$. In particular, H measures B and σ is a 2-cocycle. The *crossed product algebra*, denoted $B\#_\sigma H$, is the vector space $B\otimes H$, with the element $b\otimes h$ denoted $b\#h$, and multiplication

$$(b\#h)(c\#\ell)=\sum b(h_1\cdot c)\sigma(h_2,\ell_1)\#h_3\ell_2,$$

for all $b,c\in B$ and $h,\ell\in H$. This is an associative algebra with identity, and is a right H-comodule algebra via $b\#h\mapsto\sum(b\#h_1)\otimes h_2$ [7, §7]. We will also use the summation notation for comodules, $\sum a_0\otimes a_1=\sum(b\#h_1)\otimes h_2$ where $a=b\#h\in B\#_\sigma H$.

Let T be a finite dimensional irreducible left B-module with $\rho : B\to \mathrm{End}_k(T)$ expressing the action of B on T. Assume that T is A-*stable*, that is, there is a left B-linear and right H-colinear isomorphism

$$\Phi : A\otimes_B T\xrightarrow{\sim} T\otimes H,$$

where $A\otimes_B T$ is the A-module induced from T (with A acting as multiplication on the left factor) [10, p. 207]. Here the B-module and H-comodule structures of $A\otimes_B T$ and $T\otimes H$ are given by

$$\begin{aligned} b(a\otimes_B t):=ba\otimes_B t \quad &\text{and} \quad a\otimes_B t\mapsto\sum a_0\otimes_B t\otimes a_1, \\ b(t\otimes h):=bt\otimes h \quad &\text{and} \quad t\otimes h\mapsto t\otimes\Delta(h), \end{aligned}$$

for all $a\in A$, $b\in B$, $h\in H$, and $t\in T$. This generalizes the standard definition of a stable module in group representation theory, in which a module is stable if it is isomorphic to all its conjugate modules. We may assume that $\Phi(1\otimes_B t)=t\otimes 1$ for all $t\in T$ [10, p. 208].

We point out that in case the measuring of B by H is trivial (that is $h\cdot b=\epsilon(h)b$ for all $h\in H$ and $b\in B$), so that $A=B_\sigma[H]$ is a twisted product, then every B-module is A-stable. The map Φ is given simply by $\Phi((b\#h)\otimes_B t)=\rho(b)(t)\otimes h$, which is well-defined as $b\#1$ commutes with $1\#h$ in this case. In general Φ will be more complicated.

Let

$$E:=\mathrm{End}_A(A\otimes_B T)^{op} \text{ and } E':=\mathrm{End}_B(T)^{op}.$$

Then $E\cong\mathrm{Hom}_B(T,A\otimes_B T)$ [2, Proposition 2.8.3], and by Schur's Lemma, $E'\cong k$. By [10, Theorem 3.6], $k\cong E'\subseteq E$ is an H-crossed product, with right H-colinear and convolution invertible map $J : H \to E$ defined by

$$J(h)(1 \otimes_B t) := \Psi(t \otimes h),$$

where $\Psi = \Phi^{-1} : T \otimes H \xrightarrow{\sim} A \otimes_B T$. Note that under our hypotheses, J is bijective. Also note that any measuring of the field k by H is necessarily trivial, so E is in fact isomorphic to a twisted product $k_\alpha[H] = k\#_\alpha H$. By [7, 7.2.5], $\alpha : H \otimes H \to k$ is given by

$$\alpha(h, \ell) = \sum J(h_1)J(\ell_1)J^{-1}(h_2\ell_2) \tag{2.1}$$

for all $h, \ell \in H$, where J^{-1} is the convolution inverse of J (denoted J' in [10]).

Let $\gamma : H \to A = B\#_\sigma H$ be given by $\gamma(h) = 1\#h$. Then γ has convolution inverse γ^{-1} given by $\gamma^{-1}(h) = \sum \sigma^{-1}(Sh_2, h_3)\#Sh_1$ [7, Proposition 7.2.7]. Let

$$q := (\mathrm{id} \otimes \epsilon) \circ \Phi : A \otimes_B T \to T.$$

We claim that

$$J^{-1}(h)(1 \otimes_B t) = \sum \gamma^{-1}(h_1) \otimes_B q(\gamma(h_2) \otimes_B t). \tag{2.2}$$

This may be checked directly by calculations similar to that in [10, pp. 211–212], or by using the formula $J^{-1}(h)(1 \otimes_B t) = \sum r_i(h) \otimes_B q(\ell_i(h) \otimes t)$, where $r_i(h), \ell_i(h)$ are determined by the equation $1 \otimes h = \sum r_i(h)\ell_i(h)_0 \otimes \ell_i(h)_1$ in $A \otimes H$ [10, pp. 209 and 211]. Here $\sum \gamma^{-1}(h_1)\gamma(h_2)_0 \otimes \gamma(h_2)_1 = \sum \gamma^{-1}(h_1)\gamma(h_2) \otimes h_3 = 1 \otimes h$, and so $\sum r_i(h) \otimes \ell_i(h) = \sum \gamma^{-1}(h_1) \otimes \gamma(h_2)$.

Let $\beta : H \to \mathrm{End}_k(T)$ be defined by

$$\beta(h)(t) := q(\gamma(h) \otimes_B t).$$

In the lemma below, we give some formulas involving the maps β, J, and J^{-1}. Note that part (ii) of the lemma generalizes the fact that in case H is a group algebra kG, for $g \in G$, $\beta(g)$ is an isomorphism between T and its conjugate module $g \cdot T$.

Lemma 2.1. *Let $h, \ell \in H$, $b \in B$, and $t \in T$.*

(i) $J^{-1}(h)(1 \otimes_B t) = \sum \gamma^{-1}(h_1) \otimes_B \beta(h_2)(t)$ *and* $1 \otimes_B \beta(h)(t) = \sum J^{-1}(h_2)(\gamma(h_1) \otimes_B t)$.

(ii) $\beta(h)\rho(b) = \sum \rho(h_1 \cdot b)\beta(h_2)$.

(iii) $J(h)J(\ell) = \sum \alpha(h_1, \ell_1)J(h_2\ell_2)$ *and* $J^{-1}(\ell)J^{-1}(h) = \sum \alpha^{-1}(h_2, \ell_2)J^{-1}(h_1\ell_1)$.

(iv) $\beta(h)\beta(\ell) = \sum \alpha^{-1}(h_3, \ell_3)\rho(\sigma(h_1, \ell_1))\beta(h_2\ell_2)$.

(v) $J(Sh) = \sum \alpha(h_2, Sh_3)J^{-1}(h_1)$.

Proof. (i) The first equation follows from the definition of β and formula (2.2) for J^{-1}. The second follows from the first, as $J^{-1}(h_2)$ is A-linear, and γ and γ^{-1} are convolution inverses.

(ii) By part (i),

$$\begin{aligned}
1 \otimes_B \beta(h)\rho(b)(t) &= \sum J^{-1}(h_2)(\gamma(h_1) \otimes_B \rho(b)(t)) \\
&= \sum J^{-1}(h_2)((1\#h_1)(b\#1) \otimes_B t) \\
&= \sum J^{-1}(h_3)(((h_1 \cdot b)\#h_2) \otimes_B t) \\
&= \sum ((h_1 \cdot b)\#1) J^{-1}(h_3)(\gamma(h_2) \otimes_B t) \\
&= \sum ((h_1 \cdot b)\#1)(1 \otimes_B \beta(h_2)(t)) \\
&= \sum 1 \otimes_B \rho(h_1 \cdot b)\beta(h_2)(t).
\end{aligned}$$

Therefore $\beta(h)\rho(b) = \sum \rho(h_1 \cdot b)\beta(h_2)$.

(iii) Convolve both sides of the defining relation (2.1) for α with $J(h\ell)$ to obtain the first equation. For the second equation, convolve both sides of (2.1) with $J^{-1}(h)$, $J^{-1}(\ell)$, and $\alpha^{-1}(h, \ell)$.

(iv) By part (i),

$$\begin{aligned}
1 \otimes_B \beta(h)\beta(\ell)(t) &= \sum J^{-1}(h_2)(\gamma(h_1) \otimes_B \beta(\ell)(t)) \\
&= \sum \gamma(h_1) J^{-1}(h_2)(1 \otimes_B \beta(\ell)(t)) \\
&= \sum \gamma(h_1) J^{-1}(h_2)(J^{-1}(\ell_2)(\gamma(\ell_1) \otimes_B t)) \\
&= \sum \gamma(h_1)\gamma(\ell_1)(J^{-1}(\ell_2) J^{-1}(h_2))(1 \otimes_B t),
\end{aligned}$$

as $J^{-1}(h_2)$ and $J^{-1}(\ell_2)$ are A-maps and the multiplication in E is opposite that of endomorphisms. Now,

$$\gamma(h_1)\gamma(\ell_1) = (1\#h_1)(1\#\ell_1) = \sum \sigma(h_1, \ell_1)\#h_2\ell_2 = \sum (\sigma(h_1, \ell_1)\#1)\gamma(h_2\ell_2).$$

Using this and parts (iii) and (i), the above expression becomes

$$\begin{aligned}
1 \otimes_B \beta(h)\beta(\ell)(t) &= \sum \left(\alpha^{-1}(h_4, \ell_4)\sigma(h_1, \ell_1)\#1\right) J^{-1}(h_3\ell_3)\left(\gamma(h_2\ell_2) \otimes_B t\right) \\
&= \sum \left(\alpha^{-1}(h_3, \ell_3)\sigma(h_1, \ell_1)\#1\right)\left(1 \otimes_B \beta(h_2\ell_2)(t)\right) \\
&= \sum 1 \otimes_B \alpha^{-1}(h_3, \ell_3)\rho(\sigma(h_1, \ell_1))\beta(h_2\ell_2)(t).
\end{aligned}$$

Therefore $\beta(h)\beta(\ell) = \sum \alpha^{-1}(h_3, \ell_3)\rho(\sigma(h_1, \ell_1))\beta(h_2\ell_2)$.

(v) First note that $J(1)$ is the identity map on $A \otimes_B T$ as $\Phi(1 \otimes_B t) = t \otimes 1$ for all $t \in T$. By part (iii) then, we have

$$\begin{aligned}
\sum J(h_1)J(Sh_2) &= \sum \alpha(h_1, Sh_4) J(h_2 Sh_3) \\
&= \sum \alpha(h_1, Sh_2) \cdot \mathrm{id}_{A \otimes_B T}\,.
\end{aligned}$$

By convolving with $J^{-1}(h)$, we obtain $J(Sh) = \sum \alpha(h_2, Sh_3)J^{-1}(h_1)$. □

Next we will use these relations to give $T \otimes U$ the structure of an A-module whenever U is an E-module. On the other hand, if V is an A-module, we will give the obvious E-module structure to the vector space $\mathrm{Hom}_B(T, V) \cong \mathrm{Hom}_A(A \otimes_B T, V)$.

Theorem 2.2. *Let T be an irreducible A-stable B-module and $E = \mathrm{End}_A(A \otimes_B T)^{op}$.*

(i) *Let U be a left E-module. Then $T \otimes_k U$ is an A-module where*

$$a \cdot (t \otimes u) := \sum q(a_0 \otimes_B t) \otimes J(a_1) \cdot u.$$

Further, the restriction of $T \otimes_k U$ to B is isomorphic to a direct sum of copies of T.

(ii) *Let V be an A-module. Then the space* $\mathrm{Hom}_B(T, V) \cong \mathrm{Hom}_A(A \otimes_B T, V)$ *is a left E-module where $f \cdot g := g \circ f$.*

Proof. (i) First we note that, for $a = b\#h$, the definitions of β and ρ, and the fact that q is B-linear imply

$$\sum q(a_0 \otimes_B t) \otimes J(a_1) \cdot u = \sum \rho(b)\beta(h_1)(t) \otimes J(h_2) \cdot u.$$

We will use this formula to show that

$$(aa') \cdot (t \otimes u) = a \cdot (a' \cdot (t \otimes u))$$

for all $a, a' \in A$, $t \in T$, and $u \in U$. Letting $a = b\#h$ and $a' = c\#\ell$, the left hand side is

$$\begin{aligned} (aa') \cdot (t \otimes u) &= \sum (b(h_1 \cdot c)\sigma(h_2, \ell_1)\#h_3\ell_2) \cdot (t \otimes u) \\ &= \sum \rho(b)\rho(h_1 \cdot c)\rho(\sigma(h_2, \ell_1))\beta(h_3\ell_2)(t) \otimes J(h_4\ell_3) \cdot u. \end{aligned}$$

On the other hand, using Lemma 2.1 (ii), (iii), and (iv), $a \cdot (a' \cdot (t \otimes u))$ is equal to

$$\begin{aligned} &\sum (b\#h) \cdot (\rho(c)\beta(\ell_1)(t) \otimes J(\ell_2) \cdot u) \\ &= \sum \rho(b)\beta(h_1)\rho(c)\beta(\ell_1)(t) \otimes J(h_2) \cdot (J(\ell_2) \cdot u) \\ &= \sum \rho(b)\rho(h_1 \cdot c)\beta(h_2)\beta(\ell_1)(t) \otimes (J(h_3)J(\ell_2)) \cdot u \\ &= \sum \rho(b)\rho(h_1 \cdot c)\alpha^{-1}(h_4, \ell_3)\rho(\sigma(h_2, \ell_1))\beta(h_3\ell_2)(t) \otimes \alpha(h_5, \ell_4)J(h_6\ell_5) \cdot u \\ &= \sum \rho(b)\rho(h_1 \cdot c)\rho(\sigma(h_2, \ell_1))\beta(h_3\ell_2)(t) \otimes J(h_4\ell_3) \cdot u, \end{aligned}$$

as α and α^{-1} are convolution inverses with values in k. (As U is an E-module, we have $e \cdot (f \cdot u) = (ef) \cdot u$ for all $e, f \in E, u \in U$, even though as *endomorphisms*, e and f multiply in the opposite order.) Comparison of the two calculations shows that $(aa') \cdot (t \otimes u) = a \cdot (a' \cdot (t \otimes u))$. It may be checked that 1_A acts as the identity on $T \otimes U$. Therefore this formula gives $T \otimes U$ the structure of an A-module.

Restricting $T \otimes U$ to B, we have $(b\#1) \cdot (t \otimes u) = \rho(b)\beta(1)t \otimes J(1) \cdot u = \rho(b)t \otimes u$. Therefore the restricted module $(T \otimes U) \downarrow_B$ is isomorphic to $T^{\oplus n}$ where $n = \dim_k U$.

(ii) This is clear given that multiplication in E is opposite that of the endomorphisms. □

We remark that results of Schneider [12] may be used to show that the formula in part (i) of the theorem holds more generally (Schneider's results are stated in terms of right modules, but we translate here into left modules): Let H be a Hopf algebra with bijective antipode, $B \subset A$ a right faithfully flat H-Galois extension, T an irreducible A-stable B-module, $E' = \mathrm{End}_B(T)^{op}$, and other notation as in this section. As in [12, Remark 5.8 (2)], the map $T \otimes_{E'} E \to A \otimes_B T$ given by $t \otimes f \mapsto f(1 \otimes_B t)$ is a right E-module isomorphism. It may be checked that $(\mathrm{id} \otimes J) \circ \Phi$ is the inverse of this map. It follows that $T \otimes_{E'} U \cong (A \otimes_B T) \otimes_E U$ for all E-modules U. This may be used to give $T \otimes_{E'} U$ an A-module structure, via the standard A-module action on $(A \otimes_B T) \otimes_E U$ by multiplication on the left-most factor. The resulting A-module structure on $T \otimes_{E'} U$ is given by a formula precisely as in Theorem 2.2 (i).

3. The Clifford correspondence

In this section we give a stable Clifford correspondence in the general case, and a nonstable Clifford correspondence in case H is cocommutative. We start with the same assumptions and notation as in Section 2. In particular, T is a finite dimensional irreducible A-stable B-module, and $E = \mathrm{End}_A(A \otimes_B T)^{op}$.

Let F be the functor defined as follows, from the category of finite dimensional E-modules to the category of finite dimensional A-modules whose restriction to B is isomorphic to a direct sum of copies of T. If U is an E-module, let $F(U) = T \otimes U$ with A-module structure as in Theorem 2.2 (i). If $f : U \to V$ is an E-linear map, let $F(f) = \mathrm{id}_T \otimes f$. Let G be the functor defined as follows, from the category of finite dimensional A-modules whose restriction to B is isomorphic to a direct sum of copies of T, to the category of finite dimensional E-modules. If V is an A-module, let $G(V) = \mathrm{Hom}_B(T, V)$ with E-module structure as in Theorem 2.2 (ii). If $f : U \to V$ is an A-linear map, let $G(f) = f_*$ where $f_*(\phi) = f \circ \phi$ for any $\phi \in \mathrm{Hom}_B(T, U)$. In the next theorem, we show that F and G provide a category equivalence.

Theorem 3.1. *Let T be a finite dimensional irreducible A-stable B-module, and $E = \mathrm{End}_A(A \otimes_B T)^{op}$. The category of finite dimensional E-modules is equivalent to the category of finite dimensional A-modules whose restriction to B is isomorphic to a direct sum of copies of T. The equivalence is given by sending an E-module U to $T \otimes_k U$, and an A-module V to $\mathrm{Hom}_B(T, V)$, with module structures as given in Theorem 2.2.*

Proof. Let M be an A-module such that $M \downarrow_B \cong T^{\oplus n}$, where $M \downarrow_B$ denotes the module M restricted to B. Define $\eta_M : T \otimes \mathrm{Hom}_B(T, M) \to M$ by $\eta_M(t \otimes \phi) = \phi(t)$. As $M \downarrow_B \cong T^{\oplus n}$, we have $\mathrm{Hom}_B(T, M) \cong k^{\oplus n}$. It follows from this and the definition of

η_M that η_M is surjective. Therefore η_M is bijective, as these modules have the same k-dimension.

We show that η_M is A-linear. The isomorphism $\mathrm{Hom}_B(T,M)\cong\mathrm{Hom}_A(A\otimes_B T,M)$ allows us to identify $\phi\in\mathrm{Hom}_B(T,M)$ with $\phi\in\mathrm{Hom}_A(A\otimes_B T,M)$ by $\phi(1\otimes_B t)=\phi(t)$ for all $t\in T$. Then by Theorem 2.2 and Lemma 2.1 (i), we have

$$\begin{aligned}
\eta_M\left((b\#h)\cdot(t\otimes\phi)\right) &= \sum \eta_M\left(\rho(b)\beta(h_1)(t)\otimes J(h_2)\cdot\phi\right)\\
&= \sum (J(h_2)\cdot\phi)\left(1\otimes_B \rho(b)\beta(h_1)(t)\right)\\
&= \sum \phi\circ J(h_2)\left(1\otimes_B \rho(b)\beta(h_1)(t)\right)\\
&= \sum \phi\left((b\#1)J(h_2)(1\otimes_B \beta(h_1)(t))\right)\\
&= \sum \phi\left((b\#1)J(h_3)(J^{-1}(h_2)(\gamma(h_1)\otimes_B t))\right)\\
&= \sum \phi\left((b\#1)(J^{-1}(h_2)J(h_3))(\gamma(h_1)\otimes_B t)\right)\\
&= \phi((b\#h)\otimes_B t)\\
&= (b\#h)\cdot\phi(1\otimes_B t)\\
&= (b\#h)\cdot\eta_M(t\otimes\phi),
\end{aligned}$$

for all $b\in B$, $h\in H$, $t\in T$, and $\phi\in\mathrm{Hom}_A(A\otimes_B T,M)\cong\mathrm{Hom}_B(T,M)$. Therefore η_M is A-linear. We have already seen that η_M is bijective, and so for each M, η_M is an isomorphism of A-modules. Further, it is straightforward to check that $f\circ\eta_M=\eta_N\circ FG(f)$ whenever $f:M\to N$ is A-linear.

Let U be an E-module. Define $\eta_U:U\to\mathrm{Hom}_B(T,T\otimes U)$ by $\eta_U(u)(t)=t\otimes u$, for all $t\in T$ and $u\in U$. This defines a B-linear map $\eta_U(u)$ for each $u\in U$, as B acts trivially on the second factor U in $T\otimes U$. Identifying $\eta_U(u)\in\mathrm{Hom}_B(T,T\otimes U)$ with $\eta_U(u)\in\mathrm{Hom}_A(A\otimes_B T,T\otimes U)$ where $\eta_U(u)(1\otimes_B t)=\eta_U(u)(t)$, the map $\eta_U(u)$ becomes A-linear. Clearly η_U is injective. As $\mathrm{Hom}_B(T,T\otimes U)\cong\mathrm{Hom}_B(T,T^{\oplus n})\cong k^{\oplus n}$ where $\dim_k(U)=n$, η_U must be bijective.

We show that η_U is E-linear. As H is finite dimensional, S is bijective [7, Theorem 2.1.3]. Therefore Lemma 2.1 (v) and the surjectivity of J imply $J^{-1}:H\to E$ is also surjective. So it suffices to show that $\left(J^{-1}(h)\cdot\eta_U(u)\right)(1\otimes_B t)=\eta_U(J^{-1}(h)\cdot u)(1\otimes_B t)$

for all $h \in H$, $t \in T$, and $u \in U$. By Lemma 2.1 (i) and Theorem 2.2, we have

$$\begin{aligned}(J^{-1}(h)\cdot\eta_U(u))(1\otimes_B t) &= \eta_U(u)(J^{-1}(h)(1\otimes_B t))\\ &= \sum \eta_U(u)(\gamma^{-1}(h_1)\otimes_B \beta(h_2)(t))\\ &= \sum \gamma^{-1}(h_1)\eta_U(u)(1\otimes_B \beta(h_2)(t))\\ &= \sum \gamma^{-1}(h_1)\cdot(\beta(h_2)(t)\otimes u)\\ &= \sum(\sigma^{-1}(Sh_2,h_3)\#Sh_1)\cdot(\beta(h_4)(t)\otimes u)\\ &= \sum \rho(\sigma^{-1}(Sh_3,h_4))\beta(Sh_2)\beta(h_5)(t)\otimes J(Sh_1)\cdot u.\end{aligned}$$

By Lemma 2.1 (iv), this becomes

$$\sum \rho(\sigma^{-1}(Sh_5,h_6))\alpha^{-1}(Sh_2,h_9)\rho(\sigma(Sh_4,h_7))\beta(Sh_3h_8)(t)\otimes J(Sh_1)\cdot u.$$

Now the factors involving σ and σ^{-1} cancel, giving

$$\begin{aligned}(J^{-1}(h)\cdot\eta_U(u))(1\otimes_B t) &= \sum \alpha^{-1}(Sh_2,h_5)\beta(Sh_3h_4)(t)\otimes J(Sh_1)\cdot u\\ &= \sum \alpha^{-1}(Sh_2,h_3)t\otimes J(Sh_1)\cdot u\\ &= \sum \alpha^{-1}(Sh_4,h_5)\alpha(h_2,Sh_3)t\otimes J^{-1}(h_1)\cdot u,\end{aligned}$$

by Lemma 2.1 (v). By [7, p. 109], we have $\sum \alpha^{-1}(Sh_4,h_5)\alpha(h_2,Sh_3)=\epsilon(h_2)$, so now

$$\begin{aligned}\left(J^{-1}(h)\cdot\eta_U(u)\right)(1\otimes_B t) &= t\otimes J^{-1}(h)\cdot u\\ &= \eta_U(J^{-1}(h)\cdot u)(1\otimes_B t).\end{aligned}$$

Therefore η_U is E-linear. We have already seen that η_U is bijective, and so for each U, η_U is an isomorphism of E-modules. Further, it is straightforward to check that $\eta_V\circ f = GF(f)\circ\eta_U$ whenever $f : U\to V$ is E-linear. □

As pointed out at the end of Section 2, $T\otimes U\cong(A\otimes_B T)\otimes_E U$. Therefore our Theorem 3.1 may alternatively be derived from [12, Remark 5.8 (2)] or [13, Theorem 5.4].

We next show that in fact any finite dimensional irreducible A-module containing T as a B-submodule is isomorphic to a direct sum of copies of T as a B-module. Thus we may apply the theorem to such an A-module to obtain the stable Clifford correspondence.

Lemma 3.2. *Let V be a finite dimensional irreducible A-module whose restriction $V\downarrow_B$ to B contains an irreducible A-stable B-submodule T. Then $V\downarrow_B$ is isomorphic to a direct sum of copies of T. In particular, $V\downarrow_B$ is semisimple.*

Proof. Consider the map $A\otimes_B T\to V$ given by $a\otimes_B t\mapsto a\cdot t$ (as $T\subseteq V$, $a\cdot t\in V$ is defined for all $a\in A, t\in T$). This is a nonzero A-linear map, surjective as V is irreducible. As T is A-stable, $(A\otimes_B T)\downarrow_B\cong T\otimes H$ is isomorphic to a direct sum of copies of T, and in particular is a semisimple B-module. This and the Krull-Schmidt-Azumaya Theorem now force $V\downarrow_B$ to be semisimple and isomorphic to a direct sum of copies of T as well. □

This lemma generalizes a special case of the classical Clifford theory result that the restriction of an irreducible module from a group to a normal subgroup is semisimple. For a generalization in a different direction, see [10, Corollary 2.2].

The stable Clifford correspondence follows immediately from Theorem 3.1 and Lemma 3.2.

Corollary 3.3 (Stable Clifford correspondence). *Let T be a finite dimensional irreducible A-stable B-module, and $E = \mathrm{End}_A(A \otimes_B T)^{op}$. There is a one-to-one correspondence between isomorphism classes of finite dimensional irreducible E-modules and finite dimensional irreducible A-modules containing T as a B-submodule. This correspondence is given by sending an E-module U to $T \otimes_k U$, with module structure as given in Theorem 2.2.*

For the rest of this section we assume H is *cocommutative*. Note that as k is algebraically closed, this implies H is *pointed* as well [7, p. 76], that is every simple subcoalgebra of H is one dimensional. Let T be a finite dimensional irreducible left B-module that is not necessarily A-stable. If $C \subseteq H$ is a subcoalgebra, let $A(C) := \Delta_A^{-1}(A \otimes C)$, where $\Delta_A : A \to A \otimes H$ is the right H-comodule map arising from the crossed product $A = B\#_\sigma H$. As in [10, p. 216], we say that C *stabilizes* T if

$$T \otimes C \cong A(C) \otimes_B T$$

as left B-modules and right C-comodules. The *stabilizer* H_{st} of T in H is defined by

$$H_{st} := \sum_C C,$$

the sum over all subcoalgebras $C \subseteq H$ such that C stabilizes T. By [10, Theorem 4.4], H_{st} stabilizes T and is a Hopf subalgebra of H. (In case H is not pointed, H_{st} is a subcoalgebra of H, but not necessarily a Hopf subalgebra.) Write σ also for the restricted cocycle $\sigma|_{H_{st}\otimes H_{st}}$, and let

$$S := A(H_{st}) = B\#_\sigma H_{st}.$$

Then T is S-stable by definition, so Theorem 3.1 and Corollary 3.3 apply, with S in place of A.

Theorem 3.4. *Let H be cocommutative, $A = B\#_\sigma H$, and T a finite dimensional irreducible B-module. There is a one-to-one correspondence between isomorphism classes of finite dimensional irreducible S-modules containing T as a B-submodule, and finite dimensional irreducible A-modules containing T as a B-submodule. This correspondence is given by sending an S-module N to $A \otimes_S N$.*

Proof. Let N be a finite dimensional irreducible S-module containing T as a B-submodule. By Corollary 3.2, $N\downarrow_B$ is isomorphic to a direct sum of copies of T. By [10, Corollary 5.6 (4)], $A\otimes_S N$ is an irreducible A-module. As $A\otimes_S N$ contains $S\otimes_S N\cong N$ as an S-submodule, and $N\downarrow_B$ contains T as a B-submodule, the irreducible A-module $A\otimes_S N$ contains T as a B-submodule.

On the other hand, let M be a finite dimensional irreducible A-module containing T as a B-submodule. Consider the S-linear map $f : S\otimes_B T\to M\downarrow_S$ defined by $f(s\otimes_B t)=s\cdot t$ for all $s\in S$, $t\in T$. Let $N'=\mathrm{Im}(f)$, an S-submodule of $M\downarrow_S$. As T is S-stable, $(S\otimes_B T)\downarrow_B\cong T\otimes H_{st}$ is a direct sum of copies of T. In particular, $(S\otimes_B T)\downarrow_B$ is semisimple, and so its quotient $N'\downarrow_B$ is also semisimple, and is a direct sum of copies of T. Let N be an irreducible S-submodule of N'. Then $N\downarrow_B\subseteq N'\downarrow_B$ must be a direct sum of copies of T.

We claim that $M\cong A\otimes_S N$. By [10, Corollary 5.6 (4)], $A\otimes_S N$ is an irreducible A-module. Consider the nonzero A-linear map $A\otimes_S N\to M$ given by $a\otimes_S n\mapsto a\cdot n$ for all $a\in A$, $n\in N$. As M and $A\otimes_S N$ are both irreducible, this map is an isomorphism. By [10, Theorem 5.4 (2) (a)], $S\otimes_S N\cong N$ is isomorphic to the T-socle of $A\otimes_S N$, that is of M. As $N'\downarrow_B$ is a direct sum of copies of T as well, this forces $N=N'$, and N is the unique irreducible S-submodule of $M\downarrow_S$ containing T. □

We do not know if the correspondence in the theorem arises from a category equivalence, as is true in the stable case (Theorem 3.1) and in case H is a group algebra [8, Theorem 1.3].

Letting $E=\mathrm{End}_S(S\otimes_B T)^{op}$, note that $E\cong\mathrm{End}_A(A\otimes_B T)^{op}$ as $\mathrm{End}_A(A\otimes_B T)\cong\mathrm{Hom}_B(T,A\otimes_B T)$ and $S\otimes_B T$ is isomorphic to the T-socle of $A\otimes_B T$ [10, Theorem 5.4 (1) (a)]. The next result follows immediately from Corollary 3.3 and Theorem 3.4.

Corollary 3.5 (Cocommutative Clifford correspondence)**.** *Let H be cocommutative, $A=B\#_\sigma H$, and T a finite dimensional irreducible B-module. There is a one-to-one correspondence between isomorphism classes of finite dimensional irreducible E-modules, and finite dimensional irreducible A-modules containing T as a B-submodule. This correspondence is given by sending an E-module U to $A\otimes_S(T\otimes_k U)$.*

As any A-module contains some irreducible B-submodule, it follows from the corollary that any finite dimensional irreducible A-module has the form $A\otimes_S(T\otimes U)$ for some irreducible B-module T and irreducible E-module U. As $E\cong k_\alpha[H]$ is a twisted product, E-modules are equivalent to projective representations of H, which may be studied by the methods of Boca in [3] as H is cocommutative.

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