FINITE GENERATION OF SOME COHOMOLOGY RINGS VIA TWISTED TENSOR PRODUCT AND ANICK RESOLUTIONS

VAN C. NGUYEN, XINGTING WANG, AND SARAH WITHERSPOON

ABSTRACT. Over a field of prime characteristic $p > 2$, we prove that the cohomology rings of some pointed Hopf algebras of dimension p^3 are finitely generated. These are Hopf algebras arising in the ongoing classification of finite dimensional pointed Hopf algebras in positive characteristic. They include bosonizations of Nichols algebras of Jordan type in a general setting. When $p = 3$, we also consider their Hopf algebra liftings, that is Hopf algebras whose associated graded algebra with respect to the coradical filtration is given by such a bosonization. Our proofs are based on an algebra filtration and a lemma of Friedlander and Suslin, drawing on both twisted tensor product resolutions and Anick resolutions to locate the needed permanent cocycles in May spectral sequences.

CONTENTS

1. INTRODUCTION

The cohomology ring of a finite dimensional Hopf algebra is conjectured to be finitely generated. Friedlander and Suslin [9] proved this for cocommutative Hopf algebras, generalizing earlier results of Evens [7], Golod [11], and Venkov [25] for finite group algebras and of Friedlander and Parshall [8] for restricted Lie algebras. There are many finite generation results as well for various types

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of noncocommutative Hopf algebras (see, e.g., [3, 10, 12, 15, 23]). Most of these results are in characteristic 0. In this paper, we prove finite generation for classes of noncocommutative Hopf algebras in prime characteristic $p > 2$. These are some of the pointed Hopf algebras arising in classification work of the first two authors.

Our main result is the following combination of Theorems 5.1.2 and 5.2.1 below:

Theorem. *Let $\Bbbk$ be an algebraically closed field of prime characteristic $p > 2$. Consider the following Hopf algebras over $\Bbbk$:*

(1) *the p^2q-dimensional bosonization $R\#\,\Bbbk\mathrm{G}$ of a rank two Nichols algebra R of Jordan type over a cyclic group G of order q, where q is divisible by p; and*
(2) *a lifting H of $R\#\,\Bbbk\mathrm{G}$ when $p = q = 3$.*

Then the cohomology rings of $R\#\,\Bbbk\mathrm{G}$ and of H are finitely generated.

Our theorem is exclusively an odd characteristic result since the Nichols algebra of Jordan type does not appear in characteristic 2. Instead there is another related Nichols algebra [5] that will require different techniques. Part (2) of our main theorem above is only stated for characteristic 3; this is because we use the classification of such Hopf algebras from [19]. There, a complete classification is given only in case $p = 3$ of the Hopf algebra liftings H of $R\#\Bbbk G$, that is the Hopf algebras whose associated graded algebra with respect to the coradical filtration is $R\#\Bbbk G$. We expect our homological techniques will be able to handle liftings in case $p > 3$ once more is known about their structure.

More specifically, we let $R\#\,\Bbbk\mathrm{G}$ be a p^2q-dimensional Hopf algebra given by the bosonization of a truncated Jordan plane R as introduced in [5] (see Section 2.2 below). We prove that the cohomology ring of such a Hopf algebra $R\#\Bbbk G$, that is $\mathrm{H}^*(R\#\Bbbk G, \Bbbk) := \mathrm{Ext}^*_{R\#\Bbbk G}(\Bbbk, \Bbbk)$, is finitely generated (this is part (1) of the main theorem). We also consider liftings H of $R\#\Bbbk G$ in the special case $p = q = 3$ (see Section 2.3 below). We prove that the cohomology ring $\mathrm{H}^*(H, \Bbbk)$ is finitely generated (this is part (2) of the main theorem).

The proof of the main theorem uses the May spectral sequence for the cohomology of a filtered algebra. Both results (1) and (2) rely on a spectral sequence lemma due to Friedlander and Suslin [9] whose application requires identifying some permanent cocycles in the spectral sequence. In either case (1) or (2) we choose a filtration for which the associated graded algebra is a truncated polynomial ring whose cohomology is straightforward. We find permanent cocycles by constructing two different resolutions of independent interest. The general definitions of the resolutions are not new, but we provide here some nontrivial examples, with our main theorem as an application. The first resolution is the twisted tensor product resolution of [21]. This resolution is recalled in Section 3.1, used in Section 3.2 to obtain a resolution for R, and used again in Section 3.3 to obtain a resolution for $R\#\Bbbk G$. This resolution features in the proof of part (1) of the main theorem in Section 5.1. Another resolution is the Anick resolution [1]. This is recalled in Section 4.1, and a general result for the Anick resolution of $\Bbbk$ over a truncated polynomial ring is in Section 4.2. The Anick resolution features in the proof of part (2) of the main theorem in Section 5.2. We could have chosen to work with just one type of resolution, either Anick or twisted tensor product, for proofs of parts (1) and (2) of our main theorem above. We instead chose to work with both to illustrate a wider variety of techniques available, each having its own advantages.

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2. Settings

Our results are for pointed Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic $p > 2$, hence we restrict to this assumption. The tensor product $\otimes$ is $\otimes_{\Bbbk}$, $\mathbf{1}$ denotes the identity map on any set, and all modules are left modules unless specified otherwise. In this section, we will define the Nichols algebras and pointed Hopf algebras that are featured in this paper, and summarize some structural results that will be needed.

2.1. **Pointed Hopf algebras.** Let H be any finite dimensional pointed Hopf algebra over $\Bbbk$. The *coradical* (the sum of all simple subcoalgebras) of H is $H_0 = \Bbbk G$, a Hopf subalgebra of H generated by the grouplike elements $G := \{g \in H \mid \Delta(g) = g \otimes g\}$, where Δ is the coproduct on H.

Let

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H$$

be the *coradical filtration* of H, where $H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H)$ inductively, see [18, Chapter 5]. Consider the associated graded Hopf algebra $\operatorname{gr} H = \bigoplus_{n \geq 0} H_n/H_{n-1}$, with the convention $H_{-1} = 0$. Note that the zero term of $\operatorname{gr} H$ equals its coradical, i.e., $(\operatorname{gr} H)_0 = H_0$. There is a projection $\pi : \operatorname{gr} H \to H_0$ and an inclusion $\iota : H_0 \to \operatorname{gr} H$ such that $\pi\iota = \mathbf{1}_{H_0}$. Let R be the algebra of coinvariants of π:

$$R := (\operatorname{gr} H)^{co\,\pi} = \{h \in \operatorname{gr} H \;:\; (\mathbf{1}_H \otimes \pi)\Delta(h) = h \otimes 1\}.$$

By results of Radford [24] and Majid [16], R is a Hopf algebra in the braided category ${}^G_G\mathcal{YD}$ of left Yetter-Drinfeld modules over $H_0 = \Bbbk G$. Moreover, $\operatorname{gr} H$ is the *bosonization* (or *Radford biproduct*) of R and H_0 so that $\operatorname{gr} H \cong R \# H_0$ with the Hopf structure given in [18, Theorem 10.6.5]. As an algebra, it is simply the smash product of H_0 with R, analogous to a semidirect product of groups.

2.2. **Our setting: Rank two Nichols algebra and its bosonization.** For Sections 3 and 5.1, we use the following setup. Let $G := \langle g \rangle \cong \mathbb{Z}/q\mathbb{Z}$ be a cyclic group whose order q is divisible by p. Consider

$$R := \Bbbk\langle x, y\rangle \Big/ \left(x^p,\ y^p,\ yx - xy - \frac{1}{2}x^2\right),$$

which is a p^2-dimensional algebra as described in [5, Theorem 3.5], with a vector space basis $\{x^i y^j \mid 0 \leq i, j \leq p-1\}$. Observe that R is the Nichols algebra of a rank two Yetter-Drinfeld module $V = \Bbbk x + \Bbbk y$ over G, where the G-action on V is given by

$${}^g x = x \quad \text{and} \quad {}^g y = x + y$$

(here, left superscript indicates group action), and the G-gradings of x and y are both given by g.

Let $R \# \Bbbk G$ be the bosonization of R and $\Bbbk G$. It is the corresponding $p^2 q$-dimensional pointed Hopf algebra as studied in [5, Corollary 3.14], [19, §3 (Case B)], and [20, §4]. Its Hopf structure is given by:

$$\begin{aligned}
\Delta(x) &= x \otimes 1 + g \otimes x, & \Delta(y) &= y \otimes 1 + g \otimes y, & \Delta(g) &= g \otimes g,\\
\varepsilon(x) &= 0, & \varepsilon(y) &= 0, & \varepsilon(g) &= 1,\\
S(x) &= -g^{-1}x, & S(y) &= -g^{-1}y, & S(g) &= g^{-1},
\end{aligned}$$

where Δ is the coproduct, ε is the counit, and S is the antipode map of $R \# \Bbbk G$.

Remark 2.2.1. These algebras have featured in the following papers:

(1) These $R, G, R \# \Bbbk G$ appear in [5, 19, 20] for various purposes. In their settings, G is a cyclic group of order p. Here, we consider a more general setting with G being cyclic of order q divisible by p.
(2) In [19], the first two authors classified p^3-dimensional pointed Hopf algebra over prime characteristic p. In their classification work, this p^2-dimensional Nichols algebra R of Jordan type is unique, up to isomorphism, and only occurs when $p > 2$.

(3) The authors of [5] used right modules in their settings, whereas left modules were used in [19], thus inducing a sign difference in the relation $yx - xy - \frac{1}{2}x^2$ of R there. Here, we adopt the relation described in [5]. In Section 5.2, when we study the cohomology of pointed Hopf algebras H lifted from the associated graded algebras $\operatorname{gr} H \cong R\# \Bbbk G$ with respect to the coradical filtration, we modify the relations of the lifting structure given in [19] accordingly, see the next Section 2.3.

2.3. **Our setting: A class of 27-dimensional pointed Hopf algebras.** Suppose that the field $\Bbbk$ has characteristic $p = 3$. Consider Hopf algebras $H(\epsilon, \mu, \tau)$ defined from three scalar parameters ϵ, μ, τ as in [19]: These are pointed Hopf algebras of dimension 27 whose associated graded algebra with respect to the coradical filtration is $R\# \Bbbk G$, the Hopf algebra described in Section 2.2, in this case $p = q = 3$.

As an algebra, $H(\epsilon, \mu, \tau)$ is generated by g, x, y with relations

$$g^3 = 1, \qquad x^3 = \epsilon x, \qquad y^3 = -\epsilon y^2 - (\mu\epsilon - \tau - \mu^2)y,$$
$$yg - gy = xg + \mu(g - g^2), \qquad xg - gx = -\epsilon(g - g^2),$$
$$yx - xy = -x^2 + (\mu + \epsilon)x + \epsilon y + \tau(1 - g^2),$$

where $\epsilon \in \{0, 1\}$ and $\tau, \mu \in \Bbbk$ are arbitrary scalars. The coalgebra structure is the same as that of $R\#\Bbbk G$ described in Section 2.2. By setting $w = g - 1$, we obtain a new presentation in which the generators are w, x, y and the relations are:

$$w^3 = 0, \qquad x^3 = \epsilon x, \qquad y^3 = -\epsilon y^2 - (\mu\epsilon - \tau - \mu^2)y,$$
$$yw - wy = wx + x - (\mu - \epsilon)(w^2 + w), \qquad xw - wx = \epsilon(w^2 + w),$$
$$yx - xy = -x^2 + (\mu + \epsilon)x + \epsilon y - \tau(w^2 - w).$$

This choice of generating set will be convenient for our homological arguments later. As shown in [19], $H(\epsilon, \mu, \tau)$ has dimension 27. It has a vector space basis $\{w^i x^j y^k \mid 0 \le i, j, k \le 2\}$.

3. Twisted tensor product resolutions

In this section, we apply the construction of a twisted tensor product resolution introduced in [21] to our Nichols algebra R and its bosonization $R\# \Bbbk G$ that were defined in Section 2.2. This resolution will be used in the proof of Theorem 5.1.2 to identify some permanent cocycles in a May spectral sequence.

3.1. **The resolution construction.** We first recall from [4] the general definition of a twisted tensor product of algebras, and from [21] the general construction of a twisted tensor product resolution. Let A and B be associative algebras over $\Bbbk$ with multiplication maps $m_A : A \otimes A \to A$, $m_B : B \otimes B \to B$, and multiplicative identities 1_A, 1_B, respectively.

A *twisting map* $\tau : B \otimes A \to A \otimes B$ is a bijective $\Bbbk$-linear map for which $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$, for all $a \in A$ and $b \in B$, and

$$\tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1}) \circ (\tau \otimes \tau) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1}) \tag{3.1.1}$$

as maps $B \otimes B \otimes A \otimes A \to A \otimes B$. The *twisted tensor product algebra* $A \otimes_\tau B$ is the vector space $A \otimes B$ together with multiplication m_τ given by such a twisting map τ, that is, $m_\tau : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$ is given by $m_\tau = (m_A \otimes m_B) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1})$.

We now introduce compatibility conditions that are sufficient for constructing a resolution $Y_\bullet$ of $A \otimes_\tau B$-modules from known resolutions of A-modules and B-modules.

Definition 3.1.2. [21, Definition 5.1] Let M be an A-module with module structure map $\rho_{A,M} : A \otimes M \to M$. We say M is *compatible with the twisting map* τ if there is a bijective $\Bbbk$-linear map

$\tau_{B,M} : B \otimes M \to M \otimes B$ such that

$$\tau_{B,M} \circ (m_B \otimes \mathbf{1}) = (\mathbf{1} \otimes m_B) \circ (\tau_{B,M} \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \tau_{B,M}), \text{ and} \tag{3.1.3}$$

$$\tau_{B,M} \circ (\mathbf{1} \otimes \rho_{A,M}) = (\rho_{A,M} \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \tau_{B,M}) \circ (\tau \otimes \mathbf{1}) \tag{3.1.4}$$

as maps on $B \otimes B \otimes M$ and on $B \otimes A \otimes M$, respectively.

Let M be an A-module that is compatible with τ. We say a projective A-module resolution $P_\bullet(M)$ of M is *compatible with the twisting map* τ if each module $P_i(M)$ is compatible with τ via maps $\tau_{B,i}$ for which $\tau_{B,\bullet} : B \otimes P_\bullet(M) \to P_\bullet(M) \otimes B$ is a $\Bbbk$-linear chain map lifting $\tau_{B,M} : B \otimes M \to M \otimes B$. Let N be a B-module and let $P_\bullet(N)$ be a projective resolution of N over B. We put an $A \otimes_\tau B$-module structure on the bicomplex $P_\bullet(M) \otimes P_\bullet(N)$ by using maps $\tau_{B,\bullet}$.

Under such compatibility conditions, twisted tensor product resolutions for left modules over $A \otimes_\tau B$ were constructed in [21] satisfying the following theorem, which is a consequence of [21, Lemma 5.8, Lemma 5.9, Theorem 5.12].

Theorem 3.1.5. *Let A and B be $\Bbbk$-algebras with twisting map $\tau : B \otimes A \to A \otimes B$. Let $P_\bullet(M)$ be an A-projective resolution of M and $P_\bullet(N)$ be a B-projective resolution of N. Assume*

(a) *M and $P_\bullet(M)$ are compatible with τ, and*
(b) *$Y_{i,j} = P_i(M) \otimes P_j(N)$ is a projective $A \otimes_\tau B$-module, for all i, j.*

Then the twisted tensor product complex $Y_\bullet = \mathrm{Tot}(P_\bullet(M) \otimes P_\bullet(N))$ is a projective resolution of $M \otimes N$ as a module over the twisted tensor product $A \otimes_\tau B$.

By $\mathrm{Tot}(P_\bullet(M) \otimes P_\bullet(N))$ we mean the total complex of the bicomplex $P_\bullet(M) \otimes P_\bullet(N)$, that is the complex whose nth component is $Y_n = \oplus_{i+j=n}(P_i(M) \otimes P_j(N))$ and differential is $d_n = \sum_{i+j=n} d_{ij}$ where $d_{ij} = d_i \otimes \mathbf{1} + (-1)^i \mathbf{1} \otimes d_j$.

In comparison to [21, Theorem 5.12], we have replaced one of the hypotheses by our hypothesis (b) above, and in this case the proof is given by [21, Lemmas 5.8 and 5.9]. In some contexts, such as ours here, the hypothesis (b) can be checked directly, and thus this version of the theorem is sufficient.

3.2. **Resolution for the Nichols algebra R.** Let R be the Nichols algebra and $R\#\Bbbk G$ be its bosonization that were described in Section 2.2. We give the field $\Bbbk$ the structure of a trivial (left) R- or $(R\#\,\Bbbk\mathrm{G})$-module, that is, the action is given by the augmentation $\varepsilon(x) = 0$, $\varepsilon(y) = 0$, $\varepsilon(g) = 1$. We will construct a resolution of $\Bbbk$ as an R-module here, and as an $(R\#\,\Bbbk\mathrm{G})$-module in Section 3.3 via twisted tensor product constructions.

We start with a construction of a resolution of $\Bbbk$ over R. Using the relation (3.9) in [5, Lemma 3.8], we can view R as the twisted tensor product

$$R = \Bbbk\langle x, y\rangle \Big/ \left(x^p,\ y^p,\ yx - xy - \frac{1}{2}x^2\right) \cong A \otimes_\tau B,$$

where $A := \Bbbk[x]/(x^p)$, $B := \Bbbk[y]/(y^p)$. The twisting map $\tau : B \otimes A \to A \otimes B$ is defined by

$$\tau(y^r \otimes x^\ell) = \sum_{t=0}^{r} \binom{r}{t} \left(\frac{1}{2}\right)^t [\ell]^{[t]}\ x^{\ell+t} \otimes y^{r-t},$$

where we use the convention:

$$[\ell]^{[t]} = \ell(\ell+1)(\ell+2)\cdots(\ell+t-1),$$

with $[\ell]^{[0]} = 1$, for any ℓ.

Consider the following free resolutions of $\Bbbk$ as A-module and as B-module, respectively:

$$P_\bullet^A(\Bbbk): \qquad \cdots \xrightarrow{x^{p-1}\cdot} A \xrightarrow{x\cdot} A \xrightarrow{x^{p-1}\cdot} A \xrightarrow{x\cdot} A \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0$$

$$P_\bullet^B(\Bbbk): \qquad \cdots \xrightarrow{y^{p-1}\cdot} B \xrightarrow{y\cdot} B \xrightarrow{y^{p-1}\cdot} B \xrightarrow{y\cdot} B \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0.$$

The map ε on A (respectively, on B) takes x to 0 (respectively, y to 0). Consider the tensor product $P_\bullet^A(\Bbbk) \otimes P_\bullet^B(\Bbbk)$ as a graded vector space. The total complex is a complex of vector spaces with differential in degree n given by

$$d_n = \sum_{i+j=n} d_i \otimes \mathbf{1} + (-1)^i \mathbf{1} \otimes d_j.$$

We will now consider the total complex

$$K_\bullet := \mathrm{Tot}(P_\bullet^A(\Bbbk) \otimes P_\bullet^B(\Bbbk))$$

and show that it is a projective resolution of $\Bbbk$ as an R-module. We begin by establishing the compatibility conditions from Definition 3.1.2 for the A-module $\Bbbk$ and its resolution $P_\bullet^A(\Bbbk)$. This is hypothesis (a) of Theorem 3.1.5. That is, we define bijective $\Bbbk$-linear maps $\tau_{B,i}$ that we abbreviate here as τ_i, for which the diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\mathbf{1}\otimes x^{p-1}\cdot} & B\otimes A & \xrightarrow{\mathbf{1}\otimes x\cdot} & B\otimes A & \xrightarrow{\mathbf{1}\otimes x^{p-1}\cdot} & B\otimes A & \xrightarrow{\mathbf{1}\otimes x\cdot} & B\otimes A & \xrightarrow{\mathbf{1}\otimes \varepsilon} & B\otimes \Bbbk & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \cong} & & \\
\cdots & \xrightarrow{x^{p-1}\cdot\otimes\mathbf{1}} & A\otimes B & \xrightarrow{x\cdot\otimes\mathbf{1}} & A\otimes B & \xrightarrow{x^{p-1}\cdot\otimes\mathbf{1}} & A\otimes B & \xrightarrow{x\cdot\otimes\mathbf{1}} & A\otimes B & \xrightarrow{\varepsilon\otimes\mathbf{1}} & \Bbbk\otimes B & \longrightarrow & 0
\end{array}$$

commutes and conditions (3.1.3) and (3.1.4) hold. In Lemma 3.2.1 we show that the above chain map makes $\Bbbk$ and $P_\bullet^A(\Bbbk)$ compatible with τ, and so hypothesis (a) of Theorem 3.1.5 is satisfied. Then in Lemma 3.2.2 we show that the maps τ_i give each $P_i^A(\Bbbk) \otimes P_j^B(\Bbbk) = A \otimes B$ the structure of a free R-module, hence satisfying hypothesis (b) of Theorem 3.1.5.

Lemma 3.2.1. *For any integer $i \geq 0$, let $\tau_i : B \otimes A \to A \otimes B$ be defined as follows:*

$$\tau_i(y^r \otimes x^\ell) = \begin{cases} \tau(y^r \otimes x^\ell), & i \text{ is even} \\ \sum_{t=0}^{r} \binom{r}{t} \left(\frac{1}{2}\right)^t [\ell+1]^{[t]}\ x^{\ell+t} \otimes y^{r-t}, & i \text{ is odd.} \end{cases}$$

Then

(a) *τ_i is a bijective $\Bbbk$-linear map whose inverse is*

$$\tau_i^{-1}(x^\ell \otimes y^r) = \begin{cases} \sum_{t=0}^{r} \binom{r}{t} \left(-\frac{1}{2}\right)^t [\ell]^{[t]}\ y^{r-t} \otimes x^{\ell+t}, & i \text{ is even} \\ \sum_{t=0}^{r} \binom{r}{t} \left(-\frac{1}{2}\right)^t [\ell+1]^{[t]}\ y^{r-t} \otimes x^{\ell+t}, & i \text{ is odd.} \end{cases}$$

(b) *τ_i satisfies conditions* (3.1.3) *and* (3.1.4)*. In particular,*

$$\tau_i \circ (m_B \otimes \mathbf{1}) = (\mathbf{1} \otimes m_B) \circ (\tau_i \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \tau_i) \text{ and}$$
$$\tau_i \circ (\mathbf{1} \otimes m_A) = (m_A \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \tau_i) \circ (\tau \otimes \mathbf{1}),$$

as maps on $B \otimes B \otimes A$ and on $B \otimes A \otimes A$, respectively.

(c) *Each square in the above diagram commutes.*

Consequently, $\Bbbk$ and its resolution $P_\bullet^A(\Bbbk)$ are compatible with τ.

Proof. Straightforward calculations show that each τ_i defined as in the lemma has the claimed properties. We include one such verification as an example: To show that the maps τ_i satisfy conditions (3.1.3) and (3.1.4), observe that if i is even, then both conditions hold from the definition of τ, as $\tau_i = \tau$ in this case and satisfies (3.1.1). It remains to check the case when i is odd. Let us verify (3.1.3). The left hand side applied to $y^{r_1} \otimes y^{r_2} \otimes x^\ell$ in $B \otimes B \otimes A$ is

$$\tau_i \circ (m_B \otimes \mathbf{1})(y^{r_1} \otimes y^{r_2} \otimes x^\ell) = \tau_i(y^{r_1+r_2} \otimes x^\ell) = \sum_{t=0}^{r_1+r_2} \binom{r_1+r_2}{t} \left(\frac{1}{2}\right)^t [\ell+1]^{[t]}\ x^{\ell+t} \otimes y^{r_1+r_2-t},$$

while the right hand side is

$$\begin{aligned}
&(\mathbf{1} \otimes m_B) \circ (\tau_i \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \tau_i)(y^{r_1} \otimes y^{r_2} \otimes x^\ell)\\
&= (\mathbf{1} \otimes m_B) \circ (\tau_i \otimes \mathbf{1}) \left(y^{r_1} \otimes \sum_{k=0}^{r_2} \binom{r_2}{k} \left(\frac{1}{2}\right)^k [\ell+1]^{[k]}\ x^{\ell+k} \otimes y^{r_2-k} \right)\\
&= (\mathbf{1} \otimes m_B) \left[\sum_{k=0}^{r_2} \binom{r_2}{k} \left(\frac{1}{2}\right)^k [\ell+1]^{[k]} \left(\sum_{s=0}^{r_1} \binom{r_1}{s} \left(\frac{1}{2}\right)^s [\ell+k+1]^{[s]}\ x^{\ell+k+s} \otimes y^{r_1-s} \right) \otimes y^{r_2-k} \right]\\
&= \sum_{t=0}^{r_1+r_2} \left(\sum_{k+s=t} \binom{r_2}{k}\binom{r_1}{s} \left(\frac{1}{2}\right)^t [\ell+1]^{[k]}\, [\ell+k+1]^{[s]} \right)\ x^{\ell+t} \otimes y^{r_1+r_2-t}.
\end{aligned}$$

It is straightforward to check that $[\ell+1]^{[k]}\, [\ell+k+1]^{[s]} = [\ell+1]^{[k+s]}$, and

$$\sum_{k=0}^{t} \binom{r_2}{k}\binom{r_1}{t-k} = \binom{r_1+r_2}{t}.$$

This gives us the equality (3.1.3) as desired. ∎

Lemma 3.2.2. *The vector space $P_i^A(\Bbbk) \otimes P_j^B(\Bbbk) = A \otimes B$ is a free $(A \otimes_\tau B)$-module of rank one, generated by $1 \otimes 1$, via the $(A \otimes_\tau B)$-module isomorphism $\varphi : A \otimes_\tau B \to A \otimes B$ given by*

$$\varphi(x^\ell \otimes y^r) = \begin{cases} x^\ell \otimes y^r, & i \text{ is even} \\ \sum_{t=0}^{r} \binom{r}{t} \frac{t!}{2^t}\ x^{\ell+t} \otimes y^{r-t}, & i \text{ is odd,} \end{cases}$$

whose inverse is given by

$$\varphi^{-1}(x^\ell \otimes y^r) = \begin{cases} x^\ell \otimes y^r, & i \text{ is even} \\ x^\ell \otimes y^r - \frac{r}{2} x^{\ell+1} \otimes y^{r-1}, & i \text{ is odd.} \end{cases}$$

Proof. For each i, the map τ_i is used to give $P_i^A(\Bbbk) \otimes P_j^B(\Bbbk) = A \otimes B$ the structure of a left $(A \otimes_\tau B)$-module. In the case when i is even, this is the usual $(A \otimes_\tau B)$-module structure. In the case when i is odd, the $(A \otimes_\tau B)$-module structure via τ_i is given by

$$(A \otimes_\tau B) \otimes (A \otimes B) \xrightarrow{\mathbf{1} \otimes \tau_i \otimes \mathbf{1}} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

We now show that $A \otimes B$ is indeed a free module via the maps φ and φ^{-1} defined in the statement of the lemma.

When i is even, the map φ is clearly bijective. We check that φ is a bijection with the stated inverse when i is odd:

$$\begin{aligned}
\varphi \circ \varphi^{-1}(x^\ell \otimes y^r) &= \varphi\left(x^\ell \otimes y^r - \frac{r}{2} x^{\ell+1} \otimes y^{r-1} \right)\\
&= \sum_{t=0}^{r} \binom{r}{t} \frac{t!}{2^t}\ x^{\ell+t} \otimes y^{r-t} - \frac{r}{2} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{k!}{2^k}\ x^{\ell+1+k} \otimes y^{r-1-k}
\end{aligned}$$

$$
\begin{aligned}
&= x^\ell \otimes y^r + \sum_{t=1}^{r} \binom{r}{t} \frac{t!}{2^t}\, x^{\ell+t} \otimes y^{r-t} - \sum_{t=1}^{r} r \binom{r-1}{t-1} \frac{(t-1)!}{2^t}\, x^{\ell+t} \otimes y^{r-t} \\
&= x^\ell \otimes y^r + \sum_{t=1}^{r} \left(\binom{r}{t} t - r \binom{r-1}{t-1} \right) \frac{(t-1)!}{2^t}\, x^{\ell+t} \otimes y^{r-t} \\
&= x^\ell \otimes y^r + \sum_{t=1}^{r} (0) \frac{(t-1)!}{2^t}\, x^{\ell+t} \otimes y^{r-t} = x^\ell \otimes y^r .
\end{aligned}
$$

By (3.1.4), φ is a module isomorphism. Therefore $A \otimes B$ is free as an $(A \otimes_\tau B)$-module. □

In particular, the following is useful for our computations later:

$$
\varphi^{-1}(1 \otimes y) = \begin{cases} 1 \otimes y, & i \text{ is even} \\ 1 \otimes y - \frac{1}{2} x \otimes 1, & i \text{ is odd,} \end{cases}
\tag{3.2.3}
$$

$$
\varphi^{-1}(1 \otimes y^{p-1}) = \begin{cases} 1 \otimes y^{p-1}, & i \text{ is even} \\ 1 \otimes y^{p-1} + \frac{1}{2} x \otimes y^{p-2}, & i \text{ is odd.} \end{cases}
$$

By Lemma 3.2.1, Lemma 3.2.2, and Theorem 3.1.5, the complex $K_\bullet$ is a free resolution of $\Bbbk$ as $A \otimes_\tau B$-module. For each $i, j \geq 0$, let $\phi_{i,j}$ denote the free generator $1 \otimes 1$ of $P_i^A(\Bbbk) \otimes P_j^B(\Bbbk)$ as an $A \otimes_\tau B$-module. Then as an R-module:

$$
K_n = \bigoplus_{i+j=n} R\, \phi_{i,j}.
$$

Recall that the differentials of this total complex are $d_n = \sum_{i+j=n} (d_i \otimes \mathbf{1} + (-1)^i \mathbf{1} \otimes d_j)$. As $P_i^A(\Bbbk) \otimes P_j^B(\Bbbk)$ is free as an $A \otimes_\tau B$-module with generator $\phi_{i,j}$, we can write the image of $\phi_{i,j}$ under the differential map via the action of $A \otimes_\tau B$ on $\phi_{i,j}$, using values of the inverse map φ^{-1} defined in (3.2.3) where needed. We express the differential on elements via this notation:

$$
d(\phi_{i,j}) = \begin{cases}
x^{p-1}\phi_{i-1,j} + y^{p-1}\phi_{i,j-1} & \text{if } i, j \text{ are even,} \\
x^{p-1}\phi_{i-1,j} + y\phi_{i,j-1} & \text{if } i \text{ is even and } j \text{ is odd,} \\
x\phi_{i-1,j} - (y^{p-1} + \frac{1}{2} x y^{p-2})\phi_{i,j-1} & \text{if } i \text{ is odd and } j \text{ is even,} \\
x\phi_{i-1,j} - (y - \frac{1}{2} x)\phi_{i,j-1} & \text{if } i, j \text{ are odd.}
\end{cases}
$$

We interpret $\phi_{i,j}$ to be 0 if either i or j is negative.

3.3. **G-action on the resolution for R.** We retain the notations R and G as in Section 3.2. The group action gives a twisting map under which $R \# \Bbbk G$ may be viewed as a twisted tensor product of R and $\Bbbk G$, and will allow us to form a resolution $Y_\bullet$ of the trivial $(R \# \Bbbk G)$-module $\Bbbk$ via another application of Theorem 3.1.5. Specifically, $Y_\bullet$ will be obtained as the twisted tensor product resolution of $K_\bullet$ with the bar resolution $P_\bullet^{\Bbbk G}(\Bbbk)$. In order to do this, we use the group action to give us a twisting map on the complex $K_\bullet$ analogous to the twisting map defining a skew group algebra. We give this group action after Lemma 3.3.1. We then show in Lemma 3.3.3 that $K_\bullet$ is G-equivariant, that is, the group action commutes with the differentials. This observation will lead us to a chain map τ_i required for application of Theorem 3.1.5.

We obtain the following relations in R by straightforward calculations:

- For any integer $\ell \geq 0$, $y x^\ell = x^\ell y + \dfrac{\ell}{2} x^{\ell+1}$.
- For any integer $n \geq 1$, $(x+y)^n = \displaystyle\sum_{i=0}^{n} \binom{n}{i} \frac{(i+1)!}{2^i}\, x^i y^{n-i}$.

Further calculations show the following.

Lemma 3.3.1. *Define an element*

$$\alpha := -y^{p-2} + \sum_{i=1}^{p-2} (-1)^{i+1} \frac{(i+1)!}{2^{i+1}} \, x^i y^{p-2-i} \in R.$$

Then α satisfies the following:

(a) $x\alpha = (x+y)^{p-1} - y^{p-1} + \frac{1}{2}x[(x+y)^{p-2} - y^{p-2}]$,

(b) $(x+y)\alpha = -y^{p-1} - \frac{1}{2}xy^{p-2}$,

(c) $\alpha x = (x+y)^{p-1} - y^{p-1}$,

(d) $\alpha(y - \frac{1}{2}x) = -(x+y)^{p-1}$.

Recall that $G = \langle g \rangle \cong \mathbb{Z}/q\mathbb{Z}$ acts on R by ${}^g x = x$ and ${}^g y = x + y$. We define an action of G on the complex $K_\bullet$, constructed in Section 3.2, as follows: for all $0 \le s \le q-1$,

$${}^{g^s}\phi_{i,j} = \begin{cases} \phi_{i,j}, & \text{if } i \text{ is odd} \\ \phi_{i,j} + s\,\phi_{i+1,j-1}, & \text{if } i \text{ is even and } j \text{ is odd} \\ \phi_{i,j} + ({}^{(g^{s-1}+\cdots+g+1)}\alpha)\,\phi_{i+1,j-1}, & \text{if } i,j \text{ are even.} \end{cases}$$

Moreover, it is straightforward to check the following for all $1 \le s \le q$,

$$\text{(3.3.2)} \qquad {}^{g^{-s}}\phi_{i,j} = \begin{cases} \phi_{i,j}, & \text{if } i \text{ is odd} \\ \phi_{i,j} - s\,\phi_{i+1,j-1}, & \text{if } i \text{ is even and } j \text{ is odd} \\ \phi_{i,j} - ({}^{(g^{-s}+\cdots+g^{-1})}\alpha)\,\phi_{i+1,j-1}, & \text{if } i,j \text{ are even.} \end{cases}$$

Lemma 3.3.3. *With the above G-action, the complex $K_\bullet$ is G-equivariant.*

Proof. We first show that this G-action is well-defined, that is, ${}^{g^q}\phi_{i,j} = \phi_{i,j}$.

It is clear that when i is odd, or when i is even and j is odd that ${}^{g^q}\phi_{i,j} = \phi_{i,j}$, as q is divisible by the characteristic p of the field $\Bbbk$. When i,j are both even, we need to show that

$${}^{g^q}\phi_{i,j} = \phi_{i,j} + ({}^{(g^{q-1}+\cdots+g+1)}\alpha)\,\phi_{i+1,j-1} = \phi_{i,j},$$

that is, we need to show ${}^{(g^{q-1}+\cdots+g+1)}\alpha = 0$.

From Lemma 3.3.1(c), we have $\alpha x = (x+y)^{p-1} - y^{p-1}$. Now apply g^s to both sides to obtain:

$$\begin{aligned} {}^{g^s}\alpha\ {}^{g^s}x &= ({}^{g^s}x + {}^{g^s}y)^{p-1} - ({}^{g^s}y)^{p-1}, \\ {}^{g^s}\alpha\ x &= (x+y+sx)^{p-1} - (y+sx)^{p-1}. \end{aligned}$$

Thus, summing over all $0 \le s \le q-1$:

$$\left(\sum_{s=0}^{q-1} {}^{g^s}\alpha\right) x = \sum_{s=0}^{q-1} \left([y+(s+1)x]^{p-1} - (y+sx)^{p-1}\right) = (y+qx)^{p-1} - y^{p-1} = 0.$$

So $\sum_{s=0}^{q-1} {}^{g^s}\alpha = 0$ in the domain $\Bbbk\langle x,y\rangle/(yx - xy - \frac{1}{2}x^2)$ and hence is also 0 in R. Therefore, in all cases, we have ${}^{g^q}\phi_{i,j} = \phi_{i,j}$ and the above G-action is well-defined.

To check that the complex $K_\bullet$ is G-equivariant, we need to check such G-action is compatible with the differential maps in each degree, that is, $d({}^g\phi_{i,j}) = {}^g d(\phi_{i,j})$, for all $i,j \ge 0$. When i and j are both even:

$$\begin{aligned} d({}^g\phi_{i,j}) &= d(\phi_{i,j} + \alpha\phi_{i+1,j-1}) \\ &= x^{p-1}\phi_{i-1,j} + y^{p-1}\phi_{i,j-1} + \alpha\left(x\phi_{i,j-1} - (y - \frac{1}{2}x)\phi_{i+1,j-2}\right), \\ {}^g d(\phi_{i,j}) &= {}^g x^{p-1}\ {}^g\phi_{i-1,j} + {}^g y^{p-1}\ {}^g\phi_{i,j-1} \\ &= x^{p-1}\phi_{i-1,j} + (x+y)^{p-1}(\phi_{i,j-1} + \phi_{i+1,j-2}). \end{aligned}$$

Comparing, we see that coefficients for the terms $\phi_{i,j-1}$ and $\phi_{i+1,j-2}$ are exactly identities (c) and (d) in Lemma 3.3.1, respectively. The other cases are similar by applying Lemma 3.3.1, and we have $d({}^g\phi_{i,j}) = {}^g d(\phi_{i,j})$ for all i, j, g. Thus, the complex $K_\bullet$ is G-equivariant. □

We use this G-action next to form a twisted tensor product resolution of $\Bbbk$ as an $R\#\,\Bbbk\mathrm{G}$-module. Let the twisting map $\tau'_n : \Bbbk\mathrm{G} \otimes K_n \to K_n \otimes \Bbbk\mathrm{G}$ be given by the action of G on K_n, so that

$$\tau'_{i+j}(g \otimes \phi_{i,j}) = \begin{cases} \phi_{i,j} \otimes g, & \text{if } i \text{ is odd} \\ (\phi_{i,j} + \phi_{i+1,j-1}) \otimes g, & \text{if } i \text{ is even and } j \text{ is odd} \\ (\phi_{i,j} + \alpha\,\phi_{i+1,j-1}) \otimes g, & \text{if } i, j \text{ are even.} \end{cases}$$

Then $K_\bullet$ is compatible with τ' (giving the twisting that governs the smash product construction) via the maps τ'_n, and hypothesis (a) of Theorem 3.1.5 is satisfied.

3.4. **Resolution for the bosonization** $R\#\Bbbk G$**.** Let $P_\bullet^{\Bbbk G}(\Bbbk)$ be the following free resolution of the trivial $\Bbbk G$-module $\Bbbk$:

$$P_\bullet^{\Bbbk G}(\Bbbk): \quad \cdots \xrightarrow{(\sum_{s=0}^{q-1} g^s)\cdot} \Bbbk\mathrm{G} \xrightarrow{(g-1)\cdot} \Bbbk\mathrm{G} \xrightarrow{(\sum_{s=0}^{q-1} g^s)\cdot} \Bbbk\mathrm{G} \xrightarrow{(g-1)\cdot} \Bbbk\mathrm{G} \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0,$$

where ε takes g to 1.

Let

$$Y_\bullet := \mathrm{Tot}(K_\bullet \otimes P_\bullet^{\Bbbk G}(\Bbbk)).$$

The modules are free $(R\#\,\Bbbk\mathrm{G})$-modules by a similar argument to what we used earlier: In each degree we have a direct sum of modules of the form $R \otimes \Bbbk\mathrm{G}$. Each such is freely generated by some $\phi_{i,j} \otimes \phi_k$, where ϕ_k denotes the free generator for $P_k^{\Bbbk G}(\Bbbk) = \Bbbk\mathrm{G}$. Thus hypothesis (b) of Theorem 3.1.5 is satisfied, and we saw above that hypothesis (a) is satisfied via maps τ'_n. So by Theorem 3.1.5, $Y_\bullet$ is a free resolution of the $(R\#\,\Bbbk\mathrm{G})$-module $\Bbbk$.

For each $i, j, k \geq 0$, let $\phi_{i,j,k}$ denote the free generator $\phi_{i,j} \otimes \phi_k$ of $K_{i+j} \otimes P_k^{\Bbbk G}(\Bbbk)$ as an $(R\#\,\Bbbk\mathrm{G})$-module. We set $\phi_{i,j,k} = 0$ if one of i, j, k is negative. Then, for all $n \geq 0$, as an $(R\#\,\Bbbk\mathrm{G})$-module,

$$Y_n = \bigoplus_{i+j+k=n} (R\#\,\Bbbk\mathrm{G})\,\phi_{i,j,k}.$$

We express the differentials on elements as follows, where $d(\phi_{i,j})$ is given at the end of Section 3.2:

$$d(\phi_{i,j,k}) = d(\phi_{i,j}) \otimes \phi_k$$
$$+ (-1)^{i+j} \begin{cases} (g-1)\,\phi_{i,j,k-1}, & \text{if } i, k \text{ are odd} \\ (g-1)\,\phi_{i,j,k-1} - g\,\phi_{i+1,j-1,k-1}, & \text{if } i \text{ is even and } j, k \text{ are odd} \\ (g-1)\,\phi_{i,j,k-1} - \alpha g\,\phi_{i+1,j-1,k-1}, & \text{if } i, j \text{ are even and } k \text{ is odd} \\ \left(\sum_{s=0}^{q-1} g^s\right)\phi_{i,j,k-1}, & \text{if } i \text{ is odd and } k \text{ is even} \\ \left(\sum_{s=0}^{q-1} g^s\right)\phi_{i,j,k-1} - \left(\sum_{s=0}^{q-1} s g^s\right)\,\phi_{i+1,j-1,k-1}, & \text{if } i, k \text{ are even and } j \text{ is odd} \\ \left(\sum_{s=0}^{q-1} g^s\right)\phi_{i,j,k-1} & \\ \quad - \sum_{s=1}^{q-1} ({}^{(g^{s-1}+\cdots+g+1)}\alpha)\, g^s \phi_{i+1,j-1,k-1}, & \text{if } i, j, k \text{ are even.} \end{cases}$$

We will give partial verification of the above differentials in view of (3.3.2).

The case where i, k are even and j is odd:

$$d(\phi_{i,j,k}) = d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j}\phi_{i,j} \otimes d(\phi_k) = d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j}\phi_{i,j}\left(\sum_{s=0}^{q-1} g^s\right) \otimes \phi_{k-1}$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \sum_{s=0}^{q-1} g^s \left({}^{g^{-s}}\phi_{i,j}\right) \otimes \phi_{k-1}$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \sum_{s=0}^{q-1} g^s \left(\phi_{i,j} - s\phi_{i+1,j-1}\right) \otimes \phi_{k-1}$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \left[\left(\sum_{s=0}^{q-1} g^s\right) \phi_{i,j,k-1} - \left(\sum_{s=0}^{q-1} sg^s\right) \phi_{i+1,j-1,k-1} \right].$$

The case where i, j, k are even:

$$d(\phi_{i,j,k}) = d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j}\phi_{i,j} \otimes d(\phi_k) = d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j}\phi_{i,j} \left(\sum_{s=0}^{q-1} g^s\right) \otimes \phi_{k-1}$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \sum_{s=0}^{q-1} g^s \left({}^{g^{-s}}\phi_{i,j}\right) \otimes \phi_{k-1}$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \left[\left(\phi_{i,j} + \sum_{s=1}^{q-1} g^s (\phi_{i,j} - {}^{(g^{-1}+\cdots+g^{-s})}\alpha\, \phi_{i+1,j-1}) \right) \otimes \phi_{k-1} \right]$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \left[\left(\sum_{s=0}^{q-1} g^s\right) \phi_{i,j,k-1} - \left(\sum_{s=1}^{q-1} g^s \left({}^{(g^{-1}+\cdots+g^{-s})}\alpha\right)\right) \phi_{i+1,j-1,k-1} \right]$$

$$= d(\phi_{i,j}) \otimes \phi_k + (-1)^{i+j} \left[\left(\sum_{s=0}^{q-1} g^s\right) \phi_{i,j,k-1} - \left(\sum_{s=1}^{q-1} {}^{(g^{s-1}+\cdots+g+1)}\alpha\, g^s\right) \phi_{i+1,j-1,k-1} \right].$$

Other verifications are similar. Note that in the above formulas for the differentials, we have $\varepsilon\left(\sum_{s=0}^{q-1} g^s\right) = 0 = \varepsilon\left(\sum_{s=0}^{q-1} sg^s\right)$ in characteristic p, since p divides q. Among these differentials of free $R\#\,\Bbbk G$-module basis elements, the only terms in the outcomes $d(\phi_{i,j,k})$ that do not have coefficients in the augmentation ideal $\mathrm{Ker}(\varepsilon)$ are the terms $-g\phi_{i+1,j-1,k-1}$, occurring when i is even and j,k are odd. Consequently, letting $n = i+j+k$ and letting $\phi^*_{i,j,k}$ be the dual basis vector to $\phi_{i,j,k}$ in the Hom space $\mathrm{Hom}_{\Bbbk}\left(\bigoplus_{i'+j'+k'=n} \Bbbk\phi_{i',j',k'}, \Bbbk\right) \cong \mathrm{Hom}_{R\#\,\Bbbk G}(Y_n, \Bbbk)$, we have

$$d^*(\phi^*_{i,j,k}) = \begin{cases} -\phi^*_{i-1,j+1,k+1}, & \text{if } i \text{ is odd and } j,k \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

The cocycles are thus all the $\phi^*_{i,j,k}$ except those for which i is odd and j,k are even. The coboundaries are the $\phi^*_{i,j,k}$ for which i is even and j,k are odd. Therefore, for all $n \geq 0$, as a vector space,

$$\mathrm{H}^n(R\#\,\Bbbk G, \Bbbk) \cong \begin{cases} \mathrm{Span}_{\Bbbk}\{\phi^*_{i,j,k} \mid i+j+k=n\} \\ \quad - \mathrm{Span}_{\Bbbk}\{\phi^*_{i,j,k} \mid i \text{ is even and } j,k \text{ are odd}\}, & \text{if } n \text{ is even} \\ \mathrm{Span}_{\Bbbk}\{\phi^*_{i,j,k} \mid i+j+k=n\} \\ \quad - \mathrm{Span}_{\Bbbk}\{\phi^*_{i,j,k} \mid i \text{ is odd and } j,k \text{ are even}\}, & \text{if } n \text{ is odd.} \end{cases}$$

4. Anick resolutions

In this section, we recall the Anick resolution and make some additional observations about it in our setting. This resolution will be used in the proof of Theorem 5.2.1 to identify some permanent cocycles in a May spectral sequence. Specifically, setting $A = H(\epsilon, \mu, \tau)$, the 27-dimensional Hopf algebra of Section 2.3, we view A as a quotient of a free algebra by an ideal of relations. This viewpoint leads to the construction of the Anick resolution, recalled in some generality in Section 4.1.

We will consider a filtration on $H(\epsilon, \mu, \tau)$ under which it has the associated graded algebra given by a truncated polynomial ring. In the proof of Theorem 5.2.1, we connect the Anick resolutions of $\Bbbk$ over $A = H(\epsilon, \mu, \tau)$ and over its associated graded algebra $\operatorname{gr} A$ in a May spectral sequence. To exploit this connection, we recognize the Anick resolution of $\Bbbk$ over $\operatorname{gr} A$ in Lemma 4.2.3 as a standard resolution for a truncated polynomial ring, relying on some explicit calculations of differentials in the Anick resolution in Lemma 4.2.2.

4.1. **The resolution construction.** We generally construct the Anick resolution [1] as envisioned by Cojocaru and Ufnarovski [6], adapted here to left modules under some conditions. An algorithmic description using Gröbner bases is given by Green and Solberg [13]. The construction of the resolution also serves as a proof of exactness, since the differentials are defined recursively in each degree, making use of a contracting homotopy in the previous degree that is constructed recursively as well. See Theorem 4.1.2 below, due to Anick. We include a proof in our setting because we will use some details of the construction in Sections 4.2 and 5.2.

Let $A = T(V)/(I)$ where V is a finite dimensional vector space, $T(V) = T_{\Bbbk}(V)$ is the tensor algebra on V over $\Bbbk$, and I is a set of relations generating an ideal (I). We denote the image of an element v of V in A also by v when it causes no confusion. We assume that A is augmented by an algebra homomorphism $\varepsilon : A \to \Bbbk$ with $\varepsilon(v) = 0$ for all $v \in V$. Fix a totally ordered basis $v_1, \ldots, v_n$ of V (say $v_1 < \cdots < v_n$) and consider the degree lexicographic ordering on words in $v_1, \ldots, v_n$. That is, we give each v_i the degree 1, and monomials (words) are ordered first according to total degree, then monomials having the same degree (i.e. word length) are ordered as in a dictionary.

A *normal word* (called an element of an *order ideal of monomials*, or *o.i.m.* in [1]) is a monomial (considered as an element of $T(V)$) that cannot be written as a linear combination of smaller words in A. As a vector space, A has a basis in one-to-one correspondence with the set of normal words.

A *tip* (called an *obstruction* in [1]) is a word (considered as an element in $T(V)$) that is not normal but for which any proper subword is normal. It follows that the tips correspond to the relations: Let u be a tip and write the image of u in A as a linear combination $u = \sum a_i t_i$, where each t_i is a normal word and a_i is a scalar. Then, viewed as an element of $T(V)$, $u - \sum a_i t_i$ is in the ideal of relations, (I). It also follows that the tips are in one-to-one correspondence with a Gröbner basis of (I) [13].

We will construct the Anick resolution from the chosen sets of generators and relations in A as follows. We reindex in comparison to [1] so that indices for spaces correspond to homological degrees, and indices for functions correspond to homological degrees of their domains.

The Anick resolution is a free resolution of $\Bbbk$ considered to be an A-module under the augmentation ε. We will first describe a free basis C_n in each homological degree n of the resolution. We will write the resolution as:

$$\cdots \xrightarrow{d_3} A \otimes \Bbbk C_2 \xrightarrow{d_2} A \otimes \Bbbk C_1 \xrightarrow{d_1} A \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0,$$

where $\Bbbk C_n$ denotes the vector space with basis C_n. We adapt the degree lexicographic ordering on monomials in $T(V)$ to each A-module $A \otimes \Bbbk C_n$ by giving an element $s \otimes t$, where s is a normal word and $t \in C_n$, the degree of st viewed as an element of $T(V)$.

Let

$$C_1 = \{v_1, \ldots, v_n\},$$

that is, C_1 is the chosen set of generators. Let C_2 be the set of tips (or obstructions). The remaining sets C_n will be defined as sets of paths of length n in a directed graph (or quiver) associated to the generators and tips as follows [6]. The graph will have at most one directed arrow joining two vertices, and paths will be denoted by the product of their vertices in $T(V)$, written from right to left, for example, if f, g are vertices and there is an arrow from f to g, we denote the arrow by gf, and if there is a further arrow from g to h, then hgf denotes the path

$$f \to g \to h$$

starting at f, passing through g, and ending at h.

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V, equipped with the ordering $v_1 < \cdots < v_n$ as above, so that we may identify $\mathcal{B}$ with C_1. Let $\mathcal{T}$ be the set of tips. Let $\mathcal{R}$ be the set of all proper prefixes (that is, left factors) of the tips considered as elements of $T(V)$. (Note that $\mathcal{B} \subset \mathcal{R}$.) Let $\mathbf{Q} = \mathbf{Q}(\mathcal{B}, \mathcal{T})$ be the following quiver. The vertex set is $\{1\} \cup \mathcal{R}$. The arrows are all $1 \to v_i$ for $v_i \in \mathcal{B}$ and all $f \to g$ for which the word gf (viewed as an element of $T(V)$) uniquely contains a tip, and that tip is a prefix (possibly coinciding with gf).

The set C_n consists of all paths of length n starting from 1 in the quiver. In this context, the path $1 \to f \to g$ is identified with the product gf. (Note that $f \to g$ does not occur on its own as an element of any C_i if $f \neq 1$, so for our purposes there will be no confusion in denoting paths this way.) For use in constructing the chains C_n, we observe that we only use the vertices that are in the connected component of $\mathbf{Q}$ containing 1. Let $\overline{\mathbf{Q}} = \overline{\mathbf{Q}}(\mathcal{B}, \mathcal{T})$ be the connected component of 1 in $\mathbf{Q}$, called the *reduced quiver* of $\mathcal{B}$ and $\mathcal{T}$.

The differentials d are defined recursively, with a simultaneous recursive definition of a $\Bbbk$-linear contracting homotopy s:

$$\cdots \underset{s_2}{\overset{d_3}{\rightleftarrows}} A \otimes \Bbbk C_2 \underset{s_1}{\overset{d_2}{\rightleftarrows}} A \otimes \Bbbk C_1 \underset{s_0}{\overset{d_1}{\rightleftarrows}} A \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \Bbbk \longrightarrow 0, \tag{4.1.1}$$

where η is the unit map (taking the multiplicative identity of $\Bbbk$ to the multiplicative identity of A). We give these definitions next in our setting, simultaneously proving the following theorem. Examples are given in [1, 6] and below in Sections 4.2 and 5.2.

Theorem 4.1.2. [1, Theorem 1.4] *There are maps d_n, s_n for which $(A \otimes \Bbbk C_\bullet, d_\bullet)$ is a free resolution of $\Bbbk$ as an A-module and $s_\bullet$ is a contracting homotopy.*

Proof. We first define the maps d_n, s_{n-1} for $n = 1, 2$ to illustrate the general method. We then use induction on n.

Degree 1: We take $n = 1$ and let

$$d_1(1 \otimes v_i) = v_i$$

for all v_i in $\mathcal{B}$ and extend d_1 so that it is a left A-module homomorphism. To define the $\Bbbk$-linear map $s_0 : A \to A \otimes \Bbbk C_1$, first write elements of A as $\Bbbk$-linear combinations of normal words (which form the chosen vector space basis of A). Define s_0 on A via its values on all normal words, which are as follows. Set $s_0(1) = 0$ and $s_0(uv_i) = u \otimes v_i$ for all normal words of the form uv_i for some word u and v_i in $\mathcal{B}$. Extend s_0 so that it is a $\Bbbk$-linear map on A, and note that it will not be an A-module homomorphism in general. We now see that by construction,

$$a = (d_1 s_0 + \eta\varepsilon)(a)$$

for all $a \in A$. It also follows that $A \otimes \Bbbk C_1 = \mathrm{Ker}(d_1) \oplus \mathrm{Im}(s_0)$. To see this, let $b \in A \otimes \Bbbk C_1$ and write $b = (b - s_0 d_1(b)) + s_0 d_1(b)$. One checks that $b - s_0 d_1(b) \in \mathrm{Ker}(d_1)$; by definition, $s_0 d_1(b) \in \mathrm{Im}(s_0)$. The intersection of these two spaces is 0 by the above equation and definitions: If $b \in \mathrm{Ker}(d_1) \cap \mathrm{Im}(s_0)$, write $b = s_0(c)$. Then

$$c = (d_1 s_0 + \eta\varepsilon)(c) = \eta\varepsilon(c),$$

which implies $c \in \Bbbk$ so that $b = s_0(c) = 0$.

Degree 2: We take $n = 2$ and define $d_2(1 \otimes u)$ for u in C_2 as follows. By definition of C_2, we may write $u = rv_i$ uniquely in $T(V)$ for a word r in $\mathcal{R}$ and $v_i \in C_1$. Consider $r \otimes v_i$ as an element of $A \otimes \Bbbk C_1$, and further take its image under the A-module homomorphism d_1:

$$d_1(r \otimes v_i) = r d_1(1 \otimes v_i) = rv_i.$$

Define

$$d_2(1 \otimes u) = r \otimes v_i - s_0(d_1(r \otimes v_i)) = r \otimes v_i - s_0(rv_i),$$

and extend d_2 so that it is an A-module homomorphism on $A \otimes \Bbbk C_2$. By its definition, rv_i when considered as an element of $T(V)$ is a tip (not a normal word), and considered here as an element of A, it must be rewritten as a $\Bbbk$-linear combination of normal words before applying s_0 (since s_0 is a $\Bbbk$-linear map but not an A-module homomorphism). Now the definitions of d_1 and s_0 imply $d_1 s_0|_{\mathrm{Ker}(\varepsilon)} = \mathbf{1}_{\mathrm{Ker}(\varepsilon)}$, the identity map on $\mathrm{Ker}(\varepsilon)$. It also follows that $d_1 d_2 = 0$.

We wish to define s_1 so that $d_2 s_1|_{\mathrm{Ker}(d_1)} = \mathbf{1}_{\mathrm{Ker}(d_1)}$ and more generally so that

$$d_2 s_1 + s_0 d_1 = \mathbf{1}_{A\otimes \Bbbk C_1}.$$

First define s_1 on elements in $\mathrm{Ker}(d_1)$ by induction on their degrees, starting with those that are least in the ordering, which are elements of $A \otimes \Bbbk C_1$ corresponding to relations: Ordering the elements of C_2 as $u_1, \ldots, u_\ell$, with u_1 least, we define $s_1(d_2(1 \otimes u_1)) = 1 \otimes u_1$. Recall that we have chosen a total order on a vector space basis of $A \otimes \Bbbk C_1$, given by elements $r \otimes v_i$ where r is a normal word and $v_i \in \mathcal{B}$, to coincide with the order on the corresponding words rv_i in $T(V)$. Assume s_1 has been defined on elements of $\mathrm{Ker}(d_1)$ with highest term of degree (i.e. position in the total order) less than n. Let $\sum_{j=1}^{m} a_{i_j} r_{i_j} \otimes v_{i_j} \in \mathrm{Ker}(d_1)$ for some nonzero $a_{i_j} \in \Bbbk$, $r_{i_j} \in A$, with $v_{i_1}, \ldots, v_{i_m}$ distinct elements of $\mathcal{B}$, and terms ordered so that $r_{i_1} \otimes v_{i_1}$ is greatest and has degree n. Since $d_1(\sum a_{i_j} r_{i_j} \otimes v_{i_j}) = 0$ in A by assumption, and $r_{i_1} \otimes v_{i_1}$ is greatest, the monomial $r_{i_1} v_{i_1}$ in $T(V)$ must contain a tip. Since r_{i_1} is a nonzero normal word, the tip must be a suffix (that is, right factor) of $r_{i_1} v_{i_1}$, say $r_{i_1} v_{i_1} = v'u'$ in $T(V)$ with u' a tip. Since u' is a tip, there is an element in the ideal (I) of the form $u' + \sum_{k=1}^{\ell} b_k t_k$ for some normal words t_k and scalars b_k. Write each $t_k = t'_k v_{i_k}$ for words t'_k. Set $\alpha = -\sum_{k=1}^{\ell} a_{i_1} b_k v' t'_k \otimes v_{i_k} + \sum_{j=2}^{m} a_{i_j} r_{i_j} \otimes v_{i_j}$ and

$$s_1\left(\sum_{j=1}^{m} a_{i_j} r_{i_j} \otimes v_{i_j}\right) = a_{i_1} v' \otimes u' + s_1(\alpha).$$

Note that α consists of terms of lower degree than $r_{i_1} \otimes v_{i_1}$, and $\alpha \in \mathrm{Ker}(d_1)$ by construction, so we may now apply the induction hypothesis to define $s_1(\alpha)$. Recall that $A \otimes \Bbbk C_1 = \mathrm{Ker}(d_1) \oplus \mathrm{Im}(s_0)$. We define s_1 on $\mathrm{Im}(s_0)$ to be 0. We claim that by these definitions,

$$d_2 s_1 + s_0 d_1 = \mathbf{1}_{A\otimes \Bbbk C_1}.$$

To see this, we check separately for elements of $\mathrm{Ker}(d_1)$ and of $\mathrm{Im}(s_0)$. If $x = \sum_{j=1}^{m} a_{i_j} r_{i_j} \otimes v_{i_j} \in \mathrm{Ker}(d_1)$ as above, then by the inductive definition of s_1, we have

$$(d_2 s_1 + s_0 d_1)(x) = d_2 s_1(x) = x.$$

If $x \in \mathrm{Im}(s_0)$ then

$$(d_2 s_1 + s_0 d_1)(x) = s_0 d_1(x) = x$$

since $d_1 s_0 + \eta\varepsilon = \mathbf{1}_A$. It follows that $A \otimes \Bbbk C_2 = \mathrm{Ker}(d_2) \oplus \mathrm{Im}(s_1)$: If $b \in A \otimes \Bbbk C_2$ then $b = (b - s_1 d_2(b)) + s_1 d_2(b)$, with $b - s_1 d_2(b)$ in $\mathrm{Ker}(d_2)$ and $s_1 d_2(b)$ in $\mathrm{Im}(s_1)$. If $b \in \mathrm{Ker}(d_2) \cap \mathrm{Im}(s_1)$, write $b = s_1(c)$, and we have $c = (d_2 s_1 + s_0 d_1)(c) = s_0 d_1(c)$. Then $b = s_1(c) = s_1 s_0 d_1(c) = 0$ since $s_1 s_0 = 0$ by definition of s_1.

Degree at least 3: We take $n \geq 3$ and assume that A-module homomorphisms $d_1, \ldots, d_{n-1}$ and $\Bbbk$-linear maps $s_0, \ldots, s_{n-2}$ have been defined so that $d_{i-1} d_i = 0$, $s_{i-1} s_{i-2} = 0$, and $d_i s_{i-1} + s_{i-2} d_{i-1} = \mathbf{1}_{A\otimes \Bbbk C_{i-1}}$ for $1 \leq i \leq n-1$. It follows by an argument similar to the above that

$$A \otimes \Bbbk C_i = \mathrm{Ker}(d_i) \oplus \mathrm{Im}(s_{i-1})$$

for all $1 \leq i \leq n-1$. In particular, $A \otimes \Bbbk C_{n-1} = \mathrm{Ker}(d_{n-1}) \oplus \mathrm{Im}(s_{n-2})$. We will define d_n and s_{n-1}. The map d_n is defined first as follows. Let $u \in C_n$. We may write uniquely $u = ru'$ for $u' \in C_{n-1}$ and r in $\mathcal{R}$ by construction of the quiver $\mathbf{Q}$. Let

$$d_n(1 \otimes u) = r \otimes u' - s_{n-2} d_{n-1}(r \otimes u'). \tag{4.1.3}$$

Now $d_{n-1}(r\otimes u') = rd_{n-1}(1\otimes u')$ since d_{n-1} is an A-module homomorphism, and in order to apply s_{n-2} to this element of $A\otimes \Bbbk C_{n-2}$, any elements of A will need to be rewritten as linear combinations of normal words before applying s_{n-2} (since s_{n-2} is $\Bbbk$-linear but not an A-module homomorphism in general). It follows directly from the definition of d_n and the induction hypothesis that $d_{n-1}d_n = 0$.

We wish to define s_{n-1} so that $d_n s_{n-1}|_{\mathrm{Ker}(d_{n-1})} = \mathbf{1}_{\mathrm{Ker}(d_{n-1})}$ and more generally so that

$$d_n s_{n-1} + s_{n-2}d_{n-1} = \mathbf{1}_{A\otimes \Bbbk C_{n-1}}.$$

The map s_{n-1} is defined inductively as follows. Let $\sum_{i=1}^m a_i r_i\otimes u_i \in \mathrm{Ker}(d_{n-1})$ for some $a_i\in\Bbbk$, normal words $r_i\in A$, and $u_i\in C_{n-1}$. Recall that we have chosen a total order on a vector space basis of $A\otimes \Bbbk C_{n-1}$, given by elements $r\otimes u$ where r is a normal word and $u\in C_{n-1}$, to coincide with the order on the corresponding words ru in $T(V)$. We may assume $r_1\otimes u_1$ is the highest term among all $r_i\otimes u_i$. Write $u_1 = u'u''$, uniquely, where $u''\in C_{n-2}$. Then by definition of d_{n-1} (replacing n by $n-1$ in equation (4.1.3)), we have

$$0 = d_{n-1}\left(\sum_{i=1}^m a_i r_i\otimes u_i\right) = a_1 r_1 u'\otimes u'' + \beta,$$

where $\beta = -s_{n-3}d_{n-2}(a_1r_1u'\otimes u'') + d_{n-1}(\sum_{i=2}^m a_ir_i\otimes u_i)$, and when the term $d_{n-1}(\sum a_ir_i\otimes u_i)$ is expanded, due to cancellation, the resulting expression for β consists of terms lower in the order than $r_1\otimes u_1$. Since $0 = a_1r_1u'\otimes u''+\beta$, considering r_1u' as a word in $T(V)$, there is a tip v'' that is a factor of r_1u' in $T(V)$. To make a unique choice of such a tip, write $r_1 = v_{j_1}\cdots v_{j_\ell}$ as a word in the letters in $\mathcal{B}$. Now u' is not a tip, but r_1u' contains a tip, and so there is a largest k ($k\le \ell$) for which $v_{j_k}\cdots v_{j_\ell}u'$ (uniquely) contains a tip, and by construction this tip will then be a prefix. Thus we may write, uniquely, $r_1u' = v'tu'$ where $t\in\mathcal{R}$ and tu' uniquely contains a tip that is a prefix. So there is an arrow $u'\to t$ in the reduced quiver $\overline{\mathbf{Q}}$ by definition. Therefore $tu_1 = tu'u''\in C_n$. We may thus set

$$s_{n-1}\left(\sum_{i=1}^m a_ir_i\otimes u_i\right) = a_1v'\otimes tu_1 + s_{n-1}(\gamma), \tag{4.1.4}$$

where

$$\gamma = \sum_{i=1}^m a_ir_i\otimes u_i - d_n(a_1v'\otimes tu_1)$$

has highest term that is lower in the order than $r_1\otimes u_1$. (To obtain the above expression, in the argument $\sum a_ir_i\otimes u_i$ of s_{n-1}, we have added and subtracted $d_n(a_1v'\otimes tu_1)$.) Now continue in the same fashion to obtain $s_{n-1}(\gamma)$ in terms of elements lower in the total order, and so on. Since the chosen basis of $A\otimes\Bbbk C_{n-1}$ is well-ordered, we eventually reach an expression involving $s_{n-1}(0)=0$.

As before, we define s_{n-1} on $\mathrm{Im}(s_{n-2})$ to be 0, so that $s_{n-1}s_{n-2}=0$. A calculation as before now shows that

$$d_n s_{n-1} + s_{n-2}d_{n-1} = \mathbf{1}_{A\otimes \Bbbk C_{n-1}}.$$

Now by its definition and the above arguments, $s_\bullet$ is a contracting homotopy for the complex (4.1.1), implying that the complex is exact. Thus, $(A\otimes\Bbbk C_\bullet, d_\bullet)$ is a free resolution of $\Bbbk$ as an A-module. $\square$

4.2. **A truncated polynomial ring.** In this section, let $\Bbbk$ be any field and $m_1,m_2,m_3\ge 2$ be integers. We look closely at the Anick resolution of $\Bbbk$ over the algebra $A=\Bbbk[w,x,y]/(w^{m_1},x^{m_2},y^{m_3})$, which in the next section will be identified with an associated graded algebra of the Hopf algebra $H(\epsilon,\mu,\tau)$ defined in Section 2.3. This connection will be used in the proof of Theorem 5.2.1.

Choose generating set $\mathcal{B}=\{w,x,y\}$ and relations

$$I = \{w^{m_1},\ x^{m_2},\ y^{m_3}, wx-xw,\ wy-yw,\ xy-yx\}. \tag{4.2.1}$$

Then A has basis $\{w^i x^j y^k \mid 0 \le i \le m_1 - 1,\ 0 \le j \le m_2 - 1,\ 0 \le k \le m_3 - 1\}$. We choose the ordering $w < x < y$. So, for example, the degree lex ordering on basis elements in degrees 0, 1, 2 is

$$1 < w < x < y < w^2 < wx < wy < x^2 < xy < y^2.$$

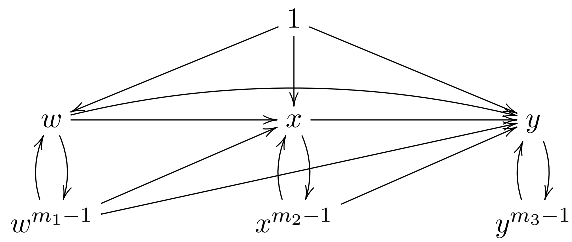
Note that xy is a normal word, while yx is a tip (or obstruction). Generally, the normal words correspond to the PBW basis of A, and the tips are

$$\mathcal{T} := \{w^{m_1},\ x^{m_2},\ y^{m_3},\ xw,\ yw,\ yx\}.$$

The proper prefixes of the tips are

$$\mathcal{R} = \{w^i,\ x^j,\ y^k \mid 0 \le i \le m_1 - 1,\ 0 \le j \le m_2 - 1,\ 0 \le k \le m_3 - 1\}.$$

The corresponding reduced quiver $\overline{\mathbf{Q}}$, as defined in Section 4.1, is as follows. (The nonreduced quiver $\mathbf{Q}$ contains additional vertices and arrows if $m > 3$, but we do not need this here.) If $m_1 = 2$, the vertices w and w^{m_1-1} are identified, and there is a loop at that vertex. Similarly for m_2, m_3.



We have

$$\begin{aligned} C_1 &= \{w,\ x,\ y\}, \\ C_2 &= \{w^{m_1},\ x^{m_2},\ y^{m_3},\ xw,\ yw,\ yx\}, \\ C_3 &= \{w^{m_1+1},\ x^{m_2+1},\ y^{m_3+1}, xw^{m_1},\ yw^{m_1},\ yx^{m_2},\ x^{m_2}w,\ y^{m_3}w,\ y^{m_3}x,\ yxw\}, \end{aligned}$$

and similarly we may find C_n for $n > 3$.

A free basis of C_n is all $1 \otimes u$ where u is a path of length n starting at 1 in the above reduced quiver $\overline{\mathbf{Q}}$. Fix such a free basis element $1 \otimes u$. Suppose $n = i + j + k$ and the first i vertices in the path u are in the set $\{w, w^{m_1-1}\}$, the second j vertices of u are in the set $\{x, x^{m_2-1}\}$, and the third k vertices are in the set $\{y, y^{m_3-1}\}$. Write $u = u_{i,j,k}$ and note that the triple of indices i, j, k uniquely determines the path. For convenience, we set $C_0 = \{1\}$ and $u_{000} = 1$, identifying A with $A \otimes \Bbbk C_0$. We set $u_{ijk} = 0$ if i, j, or k is negative.

Lemma 4.2.2. *Let $A = \Bbbk[w, x, y]/(w^{m_1}, x^{m_2}, y^{m_3})$, and let $P_\bullet^A(\Bbbk)$ denote the Anick resolution of $\Bbbk$ over A with respect to the chosen generators w, x, y and relations (4.2.1). Then*

$$d_n(1 \otimes u_{ijk}) = y^{\sigma_3(k)} \otimes u_{i,j,k-1} + (-1)^k x^{\sigma_2(j)} \otimes u_{i,j-1,k} + (-1)^{j+k} w^{\sigma_1(i)} \otimes u_{i-1,j,k}$$

where $n = i + j + k$, and $\sigma_a(\ell) = \begin{cases} 1, & \text{if } \ell \text{ is odd,} \\ m_a - 1, & \text{if } \ell \text{ is even.} \end{cases}$

Proof. We will prove the formula for d_n by induction on n. By definition, $d_1(1 \otimes u_{100}) = w \otimes u_{000}$, $d_1(1 \otimes u_{010}) = y \otimes u_{000}$, and $d_1(1 \otimes u_{001}) = x \otimes u_{000}$, and these values agree with the claimed formula for d_1.

Assume the formula holds for d_{n-1}. We first consider the case $j = k = 0$ and $i > 0$:

$$\begin{aligned} d_n(1 \otimes u_{i00}) &= w^{\sigma_1(i)} \otimes u_{i-1,0,0} - s_{n-2} d_{n-1}(w^{\sigma_1(i)} \otimes u_{i-1,0,0}) \\ &= w^{\sigma_1(i)} \otimes u_{i-1,0,0} - s_{n-2}(w^{\sigma_1(i)}(w^{\sigma_1(i-1)} \otimes u_{i-2,0,0})) \\ &= w^{\sigma_1(i)} \otimes u_{i-1,0,0} - s_{n-2}(0) \quad = \quad w^{\sigma_1(i)} \otimes u_{i-1,0,0}, \end{aligned}$$

since $w^{\sigma_1(i)} w^{\sigma_1(i-1)} = 0$ in the algebra A. This outcome agrees with the stated formula for d_n.

Next consider the case $k = 0$ and $j > 0$, applying the construction of s_{n-2} described in the proof of Theorem 4.1.2: The vertex $x^{\sigma_2(j)}$ is last in the path u_{ij0}, so by induction,

$$\begin{aligned} d_n(1 \otimes u_{ij0}) &= x^{\sigma_2(j)} \otimes u_{i,j-1,0} - s_{n-2}d_{n-1}(x^{\sigma_2(j)} \otimes u_{i,j-1,0}) \\ &= x^{\sigma_2(j)} \otimes u_{i,j-1,0} - s_{n-2}((-1)^{j-1}w^{\sigma_1(i)}x^{\sigma_2(j)} \otimes u_{i-1,j-1,0}) \\ &= x^{\sigma_2(j)} \otimes u_{i,j-1,0} + (-1)^j w^{\sigma_1(i)} \otimes u_{i-1,j,0}, \end{aligned}$$

since $x^{\sigma_2(j)}x^{\sigma_2(j-1)} = 0$ in A. This agrees with the stated formula for d_n.

In case $k > 0$, since the vertex labeled $y^{\sigma_3(k)}$ is the last in the path u_{ijk}, by induction, since $y^{\sigma_3(k)}y^{\sigma_3(k-1)} = 0$,

$$\begin{aligned} &d_n(1 \otimes u_{ijk}) = y^{\sigma_3(k)} \otimes u_{i,j,k-1} - s_{n-2}d_{n-1}(y^{\sigma_3(k)} \otimes u_{i,j,k-1}) \\ &= y^{\sigma_3(k)} \otimes u_{i,j,k-1} - s_{n-2}(y^{\sigma_3(k)}((-1)^{k-1}x^{\sigma_2(j)} \otimes u_{i,j-1,k-1} + (-1)^{j+k-1}w^{\sigma_1(i)} \otimes u_{i-1,j,k-1})) \\ &= y^{\sigma_3(k)} \otimes u_{i,j,k-1} - s_{n-2}((-1)^{k-1}x^{\sigma_2(j)}y^{\sigma_3(k)} \otimes u_{i,j-1,k-1} + (-1)^{j+k-1}w^{\sigma_1(i)}y^{\sigma_3(k)} \otimes u_{i-1,j,k-1}). \end{aligned}$$

Compare the two terms comprising the argument of s_{n-2}; they are $x^{\sigma_2(j)}y^{\sigma_3(k)} \otimes u_{i,j-1,k-1}$ and $w^{\sigma_1(i)}y^{\sigma_3(k)} \otimes u_{i-1,j,k-1}$, up to sign. These terms have the same total degree, since each arises via an application to $u_{i,j,k}$ of some differential maps (that do not change total degree). Thus we must compare them lexicographically, and we see that $x^{\sigma_2(j)}y^{\sigma_3(k)} \otimes u_{i,j-1,k-1}$ is the higher of the two terms. The first step of applying s_{n-2} thus involves the term $x^{\sigma_2(j)} \otimes u_{i,j-1,k}$ corresponding to this expression as the first term on the right side of equation (4.1.4). Continuing by working inductively, with appropriate signs, we obtain

$$d_n(1 \otimes u_{ijk}) = y^{\sigma_3(k)} \otimes u_{i,j,k-1} + (-1)^k x^{\sigma_2(j)} \otimes u_{i,j-1,k} + (-1)^{j+k} w^{\sigma_1(i)} \otimes u_{i-1,j,k},$$

as desired. □

Next we see that the Anick resolution is isomorphic to a twisted tensor product resolution for this small example. This is surely known to experts, but we include a proof for completeness.

Lemma 4.2.3. *Let $A = \Bbbk[w, x, y]/(w^{m_1}, x^{m_2}, y^{m_3})$. The Anick resolution $P_\bullet := P^A_\bullet(\Bbbk)$ of $\Bbbk$ over A is equivalent to the total complex $X_\bullet$ of $\Bbbk$ over the tensor product of the minimal resolutions of $A_w = \Bbbk[w]/(w^{m_1})$, $A_x = \Bbbk[x]/(x^{m_2})$, and $A_y = \Bbbk[y]/(y^{m_3})$, that is, for each n there is an A-module isomorphism $\psi_n : P_n \to X_n$ and $\psi_\bullet$ is a chain map lifting the identity map on $\Bbbk$.*

Proof. Let $P^{A_w}_\bullet(\Bbbk)$ be the following free resolution of $\Bbbk$ as an A-module:

$$P^{A_w}_\bullet(\Bbbk): \quad \cdots \xrightarrow{(w^{m_1-1})\cdot} A_w \xrightarrow{w\cdot} A_w \xrightarrow{(w^{m_1-1})\cdot} A_w \xrightarrow{w\cdot} A_w \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0.$$

Let $P^{A_x}_\bullet(\Bbbk)$ and $P^{A_y}_\bullet(\Bbbk)$ be similar free resolutions of $\Bbbk$ as an A_x-module and as an A_y-module, respectively. Let $X_\bullet = \mathrm{Tot}(P^{A_y}_\bullet(\Bbbk) \otimes P^{A_x}_\bullet(\Bbbk) \otimes P^{A_w}_\bullet(\Bbbk))$, be the total complex of the tensor product of these three complexes.

We will show that $P_n \cong X_n$ as an A-module for each n and that such isomorphisms may be chosen so as to constitute a chain map between $P_\bullet$ and $X_\bullet$. We will prove this by induction on n, beginning with $n = 0$ and $n = 1$. For $n = 0$, note that $P_0 = A \cong A_y \otimes A_x \otimes A_w = X_0$ and each maps onto $\Bbbk$ via ε. We take ψ_0 to be this isomorphism.

For $n = 1$, note that $P_1 = A \otimes \Bbbk\{w, x, y\}$, while X_1 is equal to

$$(P^{A_y}_1 \otimes P^{A_x}_0 \otimes P^{A_w}_0) \oplus (P^{A_y}_0 \otimes P^{A_x}_1 \otimes P^{A_w}_0) \oplus (P^{A_y}_0 \otimes P^{A_x}_0 \otimes P^{A_w}_1).$$

To keep track of degrees, let ϕ_{100} denote $1 \otimes 1 \otimes 1$ in $P^{A_y}_1 \otimes P^{A_x}_0 \otimes P^{A_w}_0$ and similarly ϕ_{010}, ϕ_{001}. Let $\psi_1 : P_1 \to X_1$ be defined by

$$\psi_1(1 \otimes w) = \phi_{100}, \quad \psi_1(1 \otimes x) = \phi_{010} \quad \text{and} \quad \psi_1(1 \otimes y) = \phi_{001}.$$

More generally let ϕ_{ijk} denote $1\otimes 1\otimes 1$ in $P_k^{A_y}\otimes P_j^{A_x}\otimes P_i^{A_w}$. Recall similar notation u_{ijk} for free basis elements of P_n described above. Define $\psi_n : P_n \to X_n$ as follows:

$$\psi_n(1\otimes u_{ijk}) = \phi_{ijk}.$$

Extend ψ_n to an A-module isomorphism.

The differential on $X_\bullet$ may be written as

$$d_n(\phi_{ijk}) = y^{\sigma_3(k)}\phi_{i,j,k-1} + (-1)^k x^{\sigma_2(j)}\phi_{i,j-1,k} + (-1)^{j+k} w^{\sigma_1(i)}\phi_{i-1,j,k}.$$

Comparing with Lemma 4.2.2, we see that $\psi_\bullet$ is a chain map. □

5. Finite generation of some cohomology rings

We now apply the constructions of twisted tensor product and Anick resolutions discussed in Sections 3 and 4 to prove that the cohomology rings of the Hopf algebras in our settings (see Sections 2.2 and 2.3) are finitely generated.

5.1. **Cohomology of the Nichols algebra and its bosonization.** Let R, G be defined as in Section 2.2. Recall that we have used a twisted tensor product to construct a resolution $K_\bullet$ for $\Bbbk$ as an R-module in Section 3.2 and further to construct a resolution $Y_\bullet$ for $\Bbbk$ as a module over the bosonization $R\#\,\Bbbk G$ in Section 3.4.

By examining the expression for the cohomology $\mathrm{H}^2(R\#\,\Bbbk G, \Bbbk)$ given at the end of Section 3.4, we see that it includes nonzero elements represented by the 2-cocycles $\phi^*_{2,0,0}$, $\phi^*_{0,2,0}$, $\phi^*_{0,0,2}$. We find their cup products, which will be used in the proof of Theorem 5.1.2 below.

To simplify notation, let

$$\xi_x = \phi^*_{2,0,0}, \qquad \xi_y = \phi^*_{0,2,0}, \qquad \xi_g = \phi^*_{0,0,2}.$$

Using the projectivity of the resolution $Y_\bullet$, one can show that these functions may be extended to chain maps on $Y_\bullet$ as follows:

$$\xi_x(\phi_{i,j,k}) = \phi_{i-2,j,k}, \qquad \xi_y(\phi_{i,j,k}) = \phi_{i,j-2,k}, \qquad \xi_g(\phi_{i,j,k}) = \phi_{i,j,k-2},$$

for all i, j, k, where we set $\phi_{i',j',k'} = 0$ if any one of i', j', k' is negative. Consequently, ξ_x, ξ_y, ξ_g are generators of a polynomial subalgebra $\Bbbk[\xi_x, \xi_y, \xi_g]$ of $\mathrm{H}^*(R\#\,\Bbbk G, \Bbbk)$ in even degrees. For example, the above formulas can be used to show that

$$(\phi^*_{2,0,0})^2 = \phi^*_{4,0,0} \qquad \text{and} \qquad \phi^*_{2,0,0}\smile\phi^*_{0,2,0} = \phi^*_{2,2,0} = \phi^*_{0,2,0}\smile\phi^*_{2,0,0}$$

and generally if i, j, k, i', j', k' are all even, then

$$\phi^*_{i,j,k}\smile\phi^*_{i',j',k'} = \phi^*_{i+i',j+j',k+k'}.$$

We will also need the following lemma, which is [15, Lemma 2.5] as adapted in [9, Lemma 1.6]. A permanent cocycle of degree n in a spectral sequence $E^{*,*}$ is an element of E_r^n for some r that survives to E_∞^n in the following sense: Let d_i denote the differential on E_i and $\pi^i_{i+1} : \mathrm{Ker}(d_i) \to E_{i+1}$ denote the canonical projection, an element α of E_r^n is a *permanent cocycle* if $d_i\pi_i^r\alpha = 0$ for all $i \geq r$, where $\pi_i^s = \pi_i^{i-1}\pi_{i-1}^{i-2}\cdots\pi_{s+1}^s$ for $i > s$ and π_r^r is the identity map. An element of $E_r^{*,*}$ is a permanent cocycle if it is a sum, over n, of permanent cocycles of degree n.

Lemma 5.1.1. *Let $E_1^{p,q} \Longrightarrow E_\infty^{p+q}$ be a multiplicative spectral sequence of $\Bbbk$-algebras concentrated in the half plane $p+q \geq 0$, and let $B^{*,*}$ be a bigraded commutative $\Bbbk$-algebra concentrated in even (total) degrees. Assume that there exists a bigraded map of algebras from $B^{*,*}$ to $E_1^{*,*}$ such that the image of $B^{*,*}$ consists of permanent cocycles, and $E_1^{*,*}$ is a noetherian module over the image of $B^{*,*}$. Then E_∞^* is a noetherian module over* $\mathrm{Tot}(B^{*,*})$.

We are now ready to prove our first main theorem.

Theorem 5.1.2. *Let $R := \Bbbk\langle x, y\rangle/(x^p,\ y^p,\ yx - xy - \frac{1}{2}x^2)$ be the Nichols algebra defined in Section 2.2, and $G := \langle g\rangle$ be a cyclic group of order q divisible by p, acting on R by automorphisms with ${}^g x = x$ and ${}^g y = x + y$. Then the cohomology ring of the bosonization, $\mathrm{H}^*(R\#\,\Bbbk\mathrm{G}, \Bbbk)$, is finitely generated.*

Proof. Without loss of generality we may assume that $q = p^a$ for some a. To see this, note that $G \cong \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ for some ℓ coprime to p and some $a \geq 1$. Elements of the subgroup of G that is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ act trivially on R since their orders are coprime to p, and so $R\#\Bbbk G \cong (R\#\Bbbk\mathbb{Z}/p^a\mathbb{Z}) \otimes (\Bbbk\mathbb{Z}/\ell\mathbb{Z})$ as an algebra. Thus the cohomology of $R\#\Bbbk G$ is the graded tensor product of the cohomology of $R\#\Bbbk\mathbb{Z}/p^a\mathbb{Z}$ and of $\Bbbk\mathbb{Z}/\ell\mathbb{Z}$. The cohomology of $\Bbbk\mathbb{Z}/\ell\mathbb{Z}$ is concentrated in degree 0, where it is simply $\Bbbk$, since ℓ is coprime to p.

Now assume that $q = p^a$. Let $w = g - 1$ and note that since the order of g is q,

$$R\#\,\Bbbk\mathrm{G} \cong \Bbbk\langle w, x, y\rangle/(w^q,\ x^p,\ y^p,\ yx - xy - \frac{1}{2}x^2,\ xw - wx,\ yw - wy - wx - x).$$

Assign the degree lexicographic order on monomials in w, x, y, with $w < x < y$. This gives rise to an $\mathbb{N}$-filtration on $R\#\Bbbk G$ for which $\mathrm{gr}(R\#\,\Bbbk\mathrm{G}) \cong \Bbbk[w, x, y]/(w^q, x^p, y^p)$. (See, e.g., [2, Theorem 4.6.5].)

We will apply the May spectral sequence [17] in our context, for which:

$$E_1^{*,*} \cong \mathrm{H}^*(\mathrm{gr}(R\#\,\Bbbk\mathrm{G}), \Bbbk) \Longrightarrow E_\infty^{*,*} \cong \mathrm{gr}\,\mathrm{H}^*(R\#\,\Bbbk\mathrm{G}, \Bbbk).$$

The algebra $\mathrm{gr}(R\#\,\Bbbk\mathrm{G}) \cong \Bbbk[w, x, y]/(w^q, x^p, y^p)$ has a resolution given by a tensor product as in Lemma 4.2.3, equivalently by repeating the twisted tensor product construction in Section 3.3 but with trivial twisting. We find that there are elements in degree 2 of $\mathrm{H}^*(\mathrm{gr}(R\#\,\Bbbk\mathrm{G}), \Bbbk)$ corresponding to $\xi_w, \xi_x, \xi_y \in \mathrm{H}^2(R\#\,\Bbbk\mathrm{G}, \Bbbk)$ (here, we identify $\xi_w = \xi_g$), and we use the same notation for them, by abuse of notation. These elements are permanent cocycles in the May spectral sequence: We have already seen that these cocycles exist for the filtered algebra $R\#\,\Bbbk\mathrm{G}$, as constructed in Section 3.3. They are permanent cocycles as we may identify their images with the corresponding elements of $\mathrm{H}^*(R\#\,\Bbbk\mathrm{G}, \Bbbk)$.

Specifically, let $B^{*,*} = \Bbbk[\xi_w, \xi_x, \xi_y]$. By identifying $\mathrm{H}^*(\mathrm{gr}(R\#\,\Bbbk\mathrm{G}), \Bbbk)$ with group cohomology, or by arguments in [15, Section 4], we see that $E_1^{*,*} \cong \mathrm{H}^*(\mathrm{gr}(R\#\,\Bbbk\mathrm{G}), \Bbbk)$ is a noetherian $B^{*,*}$-module (it is generated over $B^{*,*}$ by some elements η_w, η_x, η_y in degree 1). By Lemma 5.1.1, $E_\infty^{*,*} \cong \mathrm{gr}\,\mathrm{H}^*(R\#\,\Bbbk\mathrm{G}, \Bbbk)$ is a noetherian module over $\Bbbk[\xi_w, \xi_x, \xi_y]$. By an appropriate Zariskian filtration [14, Chapter 2], one can lift information from the associated graded ring to the filtered ring; thus, $\mathrm{H}^*(R\#\,\Bbbk\mathrm{G}, \Bbbk)$ is noetherian over $\Bbbk[\xi_w, \xi_x, \xi_y]$. Therefore, by [7, Proposition 2.4], $\mathrm{H}^*(R\#\,\Bbbk\mathrm{G}, \Bbbk)$ is finitely generated as an algebra. □

Remark 5.1.3. There is a different proof of Theorem 5.1.2 that is closer to Evens' original proof of finite generation of group cohomology, given in [20, Section 5 and Erratum]. That proof uses [20, Theorem 3.1] which gives some sufficient conditions for $\mathrm{H}^*(R\#\Bbbk G, \Bbbk)$ to be noetherian, where R is any finite dimensional augmented algebra with action of finite group G preserving the augmentation map. These conditions arise from a Lyndon-Hochschild-Serre spectral sequence relating $\mathrm{H}^*(R\#\Bbbk G, \Bbbk)$ to $\mathrm{H}^*(G, \mathrm{H}^*(R, \Bbbk))$, instead of the May spectral sequence associated to an algebra filtration that we use here.

5.2. **Cohomology of some pointed Hopf algebras of dimension 27.** In this section, we let $\Bbbk$ be a field of characteristic $p = 3$ and consider the Hopf algebras $H(\epsilon, \mu, \tau)$ defined in Section 2.3. Consider $\Bbbk$ to be the $H(\epsilon, \mu, \tau)$-module on which w, x, y each act as 0.

Theorem 5.2.1. *Let $H(\epsilon, \mu, \tau)$ be the Hopf algebra of dimension 27 defined in Section 2.3. Then the cohomology ring $\mathrm{H}^*(H(\epsilon, \mu, \tau), \Bbbk)$ is finitely generated.*

Proof. Choose the ordering $w < x < y$ as before, and the corresponding degree lexicographic ordering on monomials. Due to the form of the relations, this gives rise to an $\mathbb{N}$-filtration on $H(\epsilon, \mu, \tau)$

for which the associated graded algebra is $\operatorname{gr} H(\epsilon,\mu,\tau)\cong \Bbbk[w,x,y]/(w^3,x^3,y^3)$. (See, e.g., [2, Theorem 4.6.5].) We consider the Anick resolution $P_\bullet$ of $\Bbbk$ over $H(\epsilon,\mu,\tau)$, filtered correspondingly, and the resulting May spectral sequence of the complex $\operatorname{Hom}_{H(\epsilon,\mu,\tau)}(P_\bullet,\Bbbk)$. This is a multiplicative spectral sequence under the product induced by a diagonal map $P_\bullet\to P_\bullet\otimes P_\bullet$ lifting the identity map on $\Bbbk$. Denote the associated graded resolution to $P_\bullet$ by $\operatorname{gr} P_\bullet$, which we may identify with the Anick resolution of $\Bbbk$ over $\operatorname{gr} H(\epsilon,\mu,\tau)$, described in Lemma 4.2.3.

The Anick resolution of $\Bbbk$ over $A = H(\epsilon,\mu,\tau)$ has the same free basis sets C_n as that for $\operatorname{gr} H(\epsilon,\mu,\tau)$ described in Section 4.2. Direct calculations show that it has the following differentials in degrees 2 and 3 (recall that the parameter ϵ only takes the values 0 or 1): By the proof of Theorem 4.1.2, values of d_2 on tips correspond to the relations, specifically,

$$\begin{aligned}
d_2(1\otimes w^3) &= w^2\otimes w,\\
d_2(1\otimes x^3) &= x^2\otimes x-\epsilon\otimes x,\\
d_2(1\otimes y^3) &= y^2\otimes y+\epsilon y\otimes y+(\mu\epsilon-\tau-\mu^2)\otimes y,\\
d_2(1\otimes xw) &= x\otimes w-w\otimes x-\epsilon w\otimes w-\epsilon\otimes w,\\
d_2(1\otimes yw) &= y\otimes w-w\otimes y-w\otimes x-1\otimes x+(\mu-\epsilon)w\otimes w+(\mu-\epsilon)\otimes w,\\
d_2(1\otimes yx) &= y\otimes x-x\otimes y+x\otimes x-(\mu+\epsilon)\otimes x-\epsilon\otimes y+\tau w\otimes w-\tau\otimes w.
\end{aligned}$$

Values of d_3 require some computation, using the algorithm outlined as part of the proof of Theorem 4.1.2, and on free basis elements they are:

$$\begin{aligned}
d_3(1\otimes w^4) &= w\otimes w^3,\quad d_3(1\otimes x^4) = x\otimes x^3,\quad d_3(1\otimes y^4) = y\otimes y^3,\\
d_3(1\otimes xw^3) &= x\otimes w^3-w^2\otimes xw,\\
d_3(1\otimes x^3w) &= x^2\otimes xw+w\otimes x^3+\epsilon wx\otimes xw+\epsilon x\otimes xw+\epsilon w\otimes xw,\\
d_3(1\otimes yw^3) &= y\otimes w^3-w^2\otimes yw+w^2\otimes xw+w\otimes xw,\\
d_3(1\otimes yxw) &= y\otimes xw-x\otimes yw+w\otimes yx+\epsilon w\otimes yw+x\otimes xw+(\mu+\epsilon)w\otimes xw,\\
d_3(1\otimes y^3w) &= y^2\otimes yw+w\otimes y^3+wy\otimes yx+wx\otimes yx+(\epsilon-\mu)wy\otimes yw\\
&\quad+(\mu-\epsilon)wx\otimes yw-\tau w^2\otimes yw+y\otimes yx-(\epsilon+\mu)y\otimes yw\\
&\quad+\tau w^2\otimes xw+x\otimes yx+(\mu-\epsilon)x\otimes yw+(\mu^2-\epsilon\mu)w\otimes yw+\tau w\otimes xw,\\
d_3(1\otimes yx^3) &= y\otimes x^3-x^2\otimes yx+\tau wx\otimes xw+\epsilon x\otimes yx-\tau x\otimes xw+\epsilon\tau w\otimes xw,\\
d_3(1\otimes y^3x) &= y^2\otimes yx+x\otimes y^3-xy\otimes yx-\tau wx\otimes yw-\tau wy\otimes yw\\
&\quad+\tau w^2\otimes yx+\tau wx\otimes xw+\epsilon\tau w^2\otimes yw+(\epsilon\tau+\mu\tau)w^2\otimes xw\\
&\quad+\mu y\otimes yx+\tau y\otimes yw-\mu x\otimes yx+\tau x\otimes xw\\
&\quad+\tau w\otimes yx+(\epsilon\tau+\mu\tau)w\otimes yw+\epsilon\tau w\otimes xw.
\end{aligned}$$

For example, to find $d_3(1\otimes yw^3)$, we first compute

$$d_3(1\otimes yw^3) = y\otimes w^3 - s_1d_2(y\otimes w^3) = y\otimes w^3 - s_1(yw^2\otimes w).$$

Using the relations, rewrite yw^2 as $w^2y-w^2x-wx+(\mu+\epsilon)w^2+\epsilon w$, so the above expression is

$$= y\otimes w^3 - s_1(w^2y\otimes w - w^2x\otimes w - wx\otimes w + (\mu+\epsilon)w^2\otimes w+\epsilon w\otimes w).$$

Now $d_2(w^2\otimes yw) = w^2y\otimes w - w^2\otimes x+(\mu-\epsilon)w^2\otimes w$ and, adding and subtracting the expression $-w^2\otimes x+(\mu-\epsilon)w^2\otimes w$, the above may be rewritten as

$$\begin{aligned}
&= y\otimes w^3 - s_1(w^2y\otimes w - w^2\otimes x+(\mu-\epsilon)w^2\otimes w + w^2\otimes x-(\mu-\epsilon)w^2\otimes w\\
&\quad - w^2x\otimes w - wx\otimes w+(\mu+\epsilon)w^2\otimes w+\epsilon w\otimes w)\\
&= y\otimes w^3 - w^2\otimes yw - s_1(w^2\otimes x - w^2x\otimes w - wx\otimes w - \epsilon w^2\otimes w+\epsilon w\otimes w)).
\end{aligned}$$

For the next two steps, we note that $d_2(w^2 \otimes xw) = w^2x \otimes w - \epsilon w^2 \otimes w$ and $d_2(w \otimes xw) = wx \otimes w - w^2 \otimes x - \epsilon w^2 \otimes w - \epsilon w \otimes w$, and so the above may be rewritten as

$$= y \otimes w^3 - w^2 \otimes yw - s_1(w^2 \otimes x - w^2x \otimes w + \epsilon w^2 \otimes w + \epsilon w^2 \otimes w - wx \otimes w + \epsilon w \otimes w)$$
$$= y \otimes w^3 - w^2 \otimes yw + w^2 \otimes xw - s_1(w^2 \otimes x - wx \otimes w + \epsilon w^2 \otimes w + \epsilon w \otimes w).$$

We recognize the argument of s_1 above as $d_2(w \otimes xw)$ and so we obtain the value of $d_3(1 \otimes yw^3)$ as claimed.

Looking at the values of d_3 given above, note that the coefficients in the factor $A = H(\epsilon, \mu, \tau)$ of $A \otimes \Bbbk C_2$ in the image of each of these free basis elements (under d_3) are in the augmentation ideal. (A similar statement does *not* apply to d_2.) In particular, letting $(w^3)^*, \ldots$ denote elements in the dual basis in $\mathrm{Hom}_{\Bbbk}(\Bbbk C_2, \Bbbk) \cong \mathrm{Hom}_A(A \otimes \Bbbk C_2, \Bbbk)$, to the tips $w^3, \ldots$ of $\Bbbk C_2$, it follows that

$$d_3^*((w^3)^*) = 0, \quad d_3^*((x^3)^*) = 0, \quad d_3^*((y^3)^*) = 0.$$

Setting $\xi_w = (w^3)^*$, $\xi_x = (x^3)^*$, and $\xi_y = (y^3)^*$, we see that these functions are cocycles in $\mathrm{Hom}_{\Bbbk}(\Bbbk C_2, \Bbbk) \cong \mathrm{Hom}_A(A \otimes \Bbbk C_2, \Bbbk)$.

It follows from the above observations that ξ_w, ξ_x, ξ_y are permanent cocycles in the May spectral sequence, and we may use them in an application of Lemma 5.1.1: On the E_1-page, ξ_w, ξ_x, ξ_y correspond to analogous 2-cocycles on $\mathrm{gr}\, H(\epsilon, \mu, \tau)$ that generate a polynomial subalgebra of its cohomology ring by a similar analysis to that in earlier sections. That is, by Lemma 4.2.3, the Anick resolution is essentially the same as the (twisted) tensor product resolution used earlier. Now let $B = \Bbbk[\xi_w, \xi_x, \xi_y]$. Let $\eta_w = (w)^*$, $\eta_x = (x)^*$, $\eta_y = (y)^*$ in $\mathrm{Hom}_{\Bbbk}(\Bbbk C_1, \Bbbk) \cong \mathrm{Hom}_A(A \otimes \Bbbk C_1, \Bbbk)$. The cohomology of $\mathrm{gr}\, H(\epsilon, \mu, \tau)$ is finitely generated as a module over B, by η_w, η_x, η_y and their products (note $\eta_w^2 = 0$, $\eta_x^2 = 0$, $\eta_y^2 = 0$, so these products constitute a finite set). By Lemma 5.1.1, using an appropriate Zariskian filtration [14, Chapter 2], the cohomology $\mathrm{H}^*(H(\epsilon, \mu, \tau), \Bbbk)$ is noetherian over $\Bbbk[\xi_w, \xi_x, \xi_y]$. By [7, Proposition 2.4], it is finitely generated as an algebra. □

Remark 5.2.2. An alternative proof of our earlier Theorem 5.1.2 would proceed just as our above proof of Theorem 5.2.1: One could compute the differentials on the Anick resolution of the algebra to show existence of the needed elements ξ_w, ξ_x, ξ_y. We chose instead to use the twisted tensor product construction there, for which we were able to give formulas for the differentials in all degrees, yielding a more explicit, if not shorter, presentation. Thus our earlier Theorem 5.1.2 has several proofs, of different flavors: One proof is in [20, Section 5], using a Lyndon-Hochschild-Serre spectral sequence for a skew group algebra, one proof is that given in Section 5.1 using a May spectral sequence and a twisted tensor product resolution, and yet one more proof would proceed similarly to the proof of Theorem 5.2.1, using a May spectral sequence and the Anick resolution. By contrast, we offer just this one proof of our Theorem 5.2.1. One may not use a Lyndon-Hochschild-Serre spectral sequence directly since $H(\epsilon, \mu, \tau)$ is not a skew group algebra. One might potentially use a May spectral sequence with a twisted tensor product resolution, but constructing such a resolution may be more difficult in this context and we do not pursue this.

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Department of Mathematics, Hood College, Frederick, MD 21701
E-mail address: nguyen@hood.edu

Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: xingting@temple.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843
E-mail address: sjw@math.tamu.edu