

# QUANTUM GROUP ACTIONS, TWISTING ELEMENTS, AND DEFORMATIONS OF ALGEBRAS

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ABSTRACT. We construct twisting elements for module algebras of restricted two-parameter quantum groups from factors of their  $R$ -matrices. We generalize the theory of Giaquinto and Zhang to universal deformation formulas for categories of module algebras and give examples arising from  $R$ -matrices of two-parameter quantum groups.

## 1. INTRODUCTION

The algebraic deformation theory of Giaquinto and Zhang [GZ] laid out a framework for certain types of deformations of associative algebras, namely those arising from actions of bialgebras on algebras. This work generalized known examples of deformations coming from the action of a Lie algebra (equivalently of its universal enveloping algebra). Giaquinto and Zhang illustrated their ideas by constructing a new deformation formula from the action of a noncommutative bialgebra, namely, the universal enveloping algebra of a certain nonabelian Lie algebra. This Lie algebra is spanned by the  $n \times n$  matrix units  $E_{1,p}$ ,  $E_{p,n}$  for  $p = 1, \dots, n$ , along with the  $n \times n$  diagonal matrices of trace 0, and so it is an abelian extension of a Heisenberg Lie algebra when  $n \geq 3$ , and it is a two-dimensional nonabelian Lie algebra when  $n = 2$ . In recent work, Grunspan [G] applied the deformation formula for the  $n = 2$  case to solve the open problem of giving an explicit deformation of the Witt algebra.

In their study of deformations and orbifolds, Căldăraru, Giaquinto, and the second author [CGW] constructed a deformation from a noncocommutative (and noncommutative) bialgebra – the Drinfeld double of a Taft algebra. This was related to unpublished work of Giaquinto and Zhang and was the first known example of an explicit formula for a formal deformation of an algebra arising from a noncocommutative bialgebra.

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*Date:* December 6, 2005.

The authors gratefully acknowledge support from the following grants: NSF grants #DMS-0245082 and #DMS-0443476, and NSA grants MSPF-02G-082 and MDA904-03-1-0068. The second author thanks the Mathematical Sciences Research Institute for its support during the writing of this paper.

The present paper began as an attempt to put the example of [CGW] into a general context. The algebra in [CGW] to undergo deformation was a crossed product of a polynomial algebra and the group algebra of a finite group. We expected that generalizations of the Drinfeld double of a Taft algebra, such as the two-parameter restricted quantum groups  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  of our paper [BW3], would act on similar crossed products, potentially leading to deformations. Indeed,  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  *does* act on such a crossed product, but we were unable to generalize the deformation formula of [CGW] directly to deform the multiplication in the crossed product via the  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ -action. Instead, a generalization in a different direction proved more successful in deforming these particular types of algebras (see [Wi]). However, as we discuss in Sections 3 through 5 in this paper, certain related non-commutative algebras and their crossed products with group algebras also carry a  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ -action. Since  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  is a quotient of the infinite-dimensional two-parameter quantum group  $U_{r,s}(\mathfrak{sl}_n)$ , such algebras also have an action of  $U_{r,s}(\mathfrak{sl}_n)$ , and in some cases they admit formal deformations arising from this action.

In order to obtain these examples, we found it necessary to generalize the definitions of twisting elements and universal deformation formulas given in [GZ], from elements in bialgebras to operators on particular categories of modules. This we do in Section 2. There are known connections between twisting elements and  $R$ -matrices of quasitriangular Hopf algebras (see Section 2 for the details). In Section 3, we factor the  $R$ -matrix of the quasitriangular Hopf algebra  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ , where  $r$  and  $s$  are roots of unity (under a mild numerical constraint), and obtain a twisting element from one of the factors. In the special case of  $\mathfrak{u}_{1,-1}(\mathfrak{sl}_2)$ , we show that this twisting element leads to a universal deformation formula, and via this alternate approach, essentially recover the example of [CGW].

In Section 4, we continue under the assumption that  $r$  and  $s$  are roots of unity, but consider the infinite-dimensional quantum group  $U_{r,s}(\mathfrak{sl}_n)$ . We show that an analogue of an  $R$ -matrix for  $U_{r,s}(\mathfrak{sl}_n)$  gives rise to a universal deformation formula for certain types of algebras. In such formulas, it appears to be necessary to work with the quantum group  $U_{r,s}(\mathfrak{sl}_n)$ , or more particularly with  $U_{q,q^{-1}}(\mathfrak{sl}_n)$ , rather than with its one-parameter quotient  $U_q(\mathfrak{sl}_n)$ , whose relations do not permit the types of actions we describe in Section 5.

For the definitions and general theory in Section 2, we assume only that our algebras are defined over a commutative ring  $\mathbb{K}$ . For the quantum groups in Sections 3 through 5, we assume  $\mathbb{K}$  is a field of characteristic 0 containing appropriate roots of unity.

## 2. TWISTING ELEMENTS AND DEFORMATION FORMULAS

Let  $B$  be an associative bialgebra over a commutative ring  $\mathbb{K}$  with coproduct  $\Delta : B \rightarrow B \otimes B$  and counit  $\epsilon : B \rightarrow \mathbb{K}$ . An associative  $\mathbb{K}$ -algebra  $A$  is called a *left*

$B$ -module algebra if it is a left  $B$ -module such that

$$(2.1) \quad b.1 = \epsilon(b)1, \quad \text{and}$$

$$(2.2) \quad b.(aa') = \sum_{(b)} (b_{(1)}.a)(b_{(2)}.a')$$

for all  $b \in B$  and  $a, a' \in A$ . Here we have adopted the Heyneman-Sweedler convention for the coproduct,  $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ .

Let  $\mathcal{C}$  be a category of  $B$ -module algebras. An element  $F \in B \otimes B$  is a *twisting element* for  $\mathcal{C}$  (based on  $B$ ) if

$$(2.3) \quad (\epsilon \otimes \text{id})(F) = \text{id} \otimes \text{id} = (\text{id} \otimes \epsilon)(F), \quad \text{and}$$

$$(2.4) \quad [(\Delta \otimes \text{id})(F)](F \otimes \text{id}) = [(\text{id} \otimes \Delta)(F)](\text{id} \otimes F)$$

as operators on the  $\mathbb{K}$ -modules  $A \otimes A$  and  $A \otimes A \otimes A$ , respectively, for all objects  $A$  in  $\mathcal{C}$ , where  $\text{id}$  denotes the identity operator. More generally, we allow  $F$  to be a formal infinite sum of elements of  $B \otimes B$ , provided that  $F$  and the operators on each side of equations (2.3) and (2.4) are well-defined operators on the appropriate objects.

The following theorem is essentially the same as [GZ, Thm. 1.3], but stated in terms of the category  $\mathcal{C}$ . We include a proof for completeness.

**Theorem 2.5.** *Let  $A$  be an associative algebra in a category  $\mathcal{C}$  of  $B$ -module algebras having multiplication map  $\mu = \mu_A$ , and assume  $F$  is a twisting element for  $\mathcal{C}$ . Let  $A_F$  denote the  $\mathbb{K}$ -module  $A$  with multiplication map  $\mu \circ F$ . Then  $A_F$  is an associative algebra with multiplicative identity  $1 = 1_A$ .*

*Proof.* Equations (2.1) and (2.3) imply that  $1_A$  is the multiplicative identity in  $A_F$ . Associativity in  $A_F$  is equivalent to the identity

$$\mu \circ F \circ (\mu \otimes \text{id}) \circ (F \otimes \text{id}) = \mu \circ F \circ (\text{id} \otimes \mu) \circ (\text{id} \otimes F)$$

of functions from  $A \otimes A \otimes A$  to  $A$ . We prove this identity by applying the associativity of  $\mu$ , (2.4), and (2.2) twice:

$$\begin{aligned} \mu \circ F \circ (\mu \otimes \text{id}) \circ (F \otimes \text{id}) &= \mu \circ (\mu \otimes \text{id}) \circ [(\Delta \otimes \text{id})(F)] \circ (F \otimes \text{id}) \\ &= \mu \circ (\text{id} \otimes \mu) \circ [(\text{id} \otimes \Delta)(F)] \circ (\text{id} \otimes F) \\ &= \mu \circ F \circ (\text{id} \otimes \mu) \circ (\text{id} \otimes F). \end{aligned}$$

(This identity may be viewed as a commutative diagram as in [GZ, Thm. 1.3].)  $\square$

Known examples of twisting elements include those arising from quasitriangular Hopf algebras, and we recall these ideas next, with a few details for clarity. A Hopf algebra  $H$  is *quasitriangular* if the antipode  $S$  is bijective, and there is an invertible element  $R \in H \otimes H$  such that

$$(2.6) \quad \tau(\Delta(h)) = R\Delta(h)R^{-1}$$

for all  $h \in H$ , where  $\tau(a \otimes b) = b \otimes a$ ,

$$(2.7) \quad (\Delta \otimes \text{id})(R) = R^{13}R^{23}, \quad \text{and}$$

$$(2.8) \quad (\text{id} \otimes \Delta)(R) = R^{13}R^{12}.$$

The notation is standard; for example, if  $R = \sum_i R_i^1 \otimes R_i^2$ , then  $R^{13} = \sum_i R_i^1 \otimes 1 \otimes R_i^2$ .

It can be shown that

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

(see for example [M, Prop. 10.1.8]). Thus left multiplication of (2.7) by  $R^{12} = R \otimes 1$  and left multiplication of (2.8) by  $R^{23} = 1 \otimes R$  yields

$$(R \otimes 1)[(\Delta \otimes \text{id})(R)] = (1 \otimes R)[(\text{id} \otimes \Delta)(R)],$$

which is very similar to (2.4), but applies to *right*  $H$ -module algebras. For *left*  $H$ -module algebras,  $R^{-1}$  (which equals  $(S \otimes \text{id})(R)$  [M, (10.1.10)]) satisfies (2.4) (as noted in [GZ, pp. 139–40]). The element  $R_{21} := \tau(R)$  can also be shown to satisfy (2.4) by the following argument:

Interchange the factors in (2.7) and (2.8) to obtain

$$(2.9) \quad (\text{id} \otimes \Delta)(R_{21}) = R^{21}R^{31}, \quad \text{and}$$

$$(2.10) \quad (\Delta \otimes \text{id})(R_{21}) = R^{32}R^{31}.$$

Using the fact that  $R_{21}$  is itself an  $R$ -matrix for  $H^{\text{coop}}$  (that is, for  $H$  with the opposite coproduct), we see that

$$R^{32}R^{31}R^{21} = R^{21}R^{31}R^{32}.$$

(This may also be shown directly.) Thus, multiplying (2.9) on the right by  $R^{32}$  and multiplying (2.10) on the right by  $R^{21}$  yields

$$(2.11) \quad [(\Delta \otimes \text{id})(R_{21})](R_{21} \otimes 1) = [(\text{id} \otimes \Delta)(R_{21})](1 \otimes R_{21}).$$

Again, as  $R_{21}$  is an  $R$ -matrix for  $H^{\text{coop}}$ , it satisfies (2.3) [M, (10.1.11)]. Therefore we have shown the following.

**Proposition 2.12.** *Let  $H$  be a quasitriangular Hopf algebra with  $R$ -matrix  $R$ . Then  $R_{21}$  is a twisting element for left  $H$ -module algebras.*

Now assume  $A$  is an associative  $\mathbb{K}$ -algebra with multiplication map  $\mu = \mu_A$ . Let  $t$  be an indeterminate and  $A[[t]]$  be the algebra of formal power series with coefficients in  $A$ . A map  $f: A[[t]] \rightarrow A[[t]]$  is said to have  $\deg(f) \geq i$  if  $f$  takes the ideal  $A[[t]]t^j$  to  $A[[t]]t^{i+j}$  for each  $j$ .

A *formal deformation* of  $A$  is an associative  $\mathbb{K}[[t]]$ -algebra structure on the  $\mathbb{K}[[t]]$ -module  $A[[t]]$  for which multiplication takes the form

$$\mu_A + t\mu_1 + t^2\mu_2 + \cdots.$$

Each  $\mu_i : A \otimes_{\mathbb{K}} A \rightarrow A$  is assumed to be a  $\mathbb{K}$ -linear map, extended to be  $\mathbb{K}[[t]]$ -linear. Associativity imposes certain constraints on the maps  $\mu_i$ ; for example,  $\mu_1$  must be a *Hochschild two-cocycle*:

$$\mu_1(a \otimes b)c + \mu_1(ab \otimes c) = a\mu_1(b \otimes c) + \mu_1(a \otimes bc)$$

for all  $a, b, c \in A$ . For further discussion of the conditions on the maps  $\mu_i$ , see [GZ, p. 141].

Let  $B$  be a  $\mathbb{K}$ -bialgebra,  $\mathcal{C}$  be a category of  $B[[t]]$ -module algebras of the form  $A[[t]]$  for  $\mathbb{K}$ -algebras  $A$ , and  $F$  be a twisting element for  $\mathcal{C}$ . We say that  $F$  is a *universal deformation formula* for  $\mathcal{C}$  (based on  $B$ ) if

$$(2.13) \quad F = 1 \otimes 1 + \sum_{j \geq 1} F_j^1 \otimes F_j^2$$

for some elements  $F_j^1, F_j^2$  of  $B[[t]]$  such that each  $F_j^1 \otimes F_j^2$  (when viewed as a transformation on  $A[[t]] \otimes_{\mathbb{K}[[t]]} A[[t]]$  for any  $A[[t]]$  in  $\mathcal{C}$ ) satisfies  $\deg(F_j^1 \otimes F_j^2) \geq 1$ . We assume for each  $i \geq 1$  that there are only finitely many summands  $F_j^1 \otimes F_j^2$  for which  $\deg(F_j^1 \otimes F_j^2) \geq i$  but  $\deg(F_j^1 \otimes F_j^2) \not\geq i + 1$ . In this case,  $A[[t]]$  has a (new) associative algebra structure with multiplication  $\mu_{A[[t]]} \circ F$  and multiplicative identity  $1_{A[[t]]} = 1_A$  by Theorem 2.5. In addition, this multiplication has the following property: for all  $a, b \in A$ ,

$$\mu_{A[[t]]} \circ F(a \otimes b) \equiv \mu_A(a \otimes b) \pmod{(t)}.$$

Note that our definition of a universal deformation formula generalizes that of [GZ, Defn. 1.13], where  $F$  is required to have an expression  $1 \otimes 1 + tF_1 + t^2F_2 + \dots$  with each  $F_i \in B \otimes B$ , and where  $\mathcal{C}$  is the category of  $B[[t]]$ -module algebras arising from  $B$ -module algebras by extension of scalars. The differences are that here  $F$  is not required to satisfy (2.3) and (2.4) on module algebras outside of category  $\mathcal{C}$ , and  $B$  is not required to take each algebra  $A$  in  $\mathcal{C}$  to itself, but only to  $A[[t]]$ . We illustrate such actions in Section 5.

The next example is well-known.

**Example 2.14.** Let  $\mathbb{K}$  be a field of characteristic 0,  $B$  be a commutative  $\mathbb{K}$ -bialgebra, and  $\mathcal{C}$  be any category of  $B[[t]]$ -module algebras of the form  $A[[t]]$  where  $A$  is a  $B$ -module algebra. Let  $P$  be the space of primitive elements of  $B$  and  $p \in P \otimes P$ . Then

$$\exp(tp) = \sum_{i=0}^{\infty} \frac{t^i}{i!} p^i$$

is a universal deformation formula for  $\mathcal{C}$  (see [GZ, Thm. 2.1] for a proof).

The following theorem on universal deformation formulas generalizes a consequence of [GZ, Thm. 1.3 and Defn. 1.13].

**Theorem 2.15.** *Let  $B$  be a  $\mathbb{K}$ -bialgebra and  $\mathcal{C}$  be a category of  $B[[t]]$ -module algebras of the form  $A[[t]]$ , where  $A$  is a  $\mathbb{K}$ -algebra. Let  $F$  be a universal deformation formula for  $\mathcal{C}$  based on  $B$ . Then for each  $A[[t]]$  in  $\mathcal{C}$ ,  $\mu_{A[[t]]} \circ F$  defines a formal deformation of  $A$ .*

*Proof.* Let  $A[[t]] \in \mathcal{C}$ . By Theorem 2.5,  $\mu_{A[[t]]} \circ F$  provides an associative algebra structure on  $A[[t]]$ . For  $a, a' \in A$ , we may write

$$F_j^1.a = \sum_{k \geq 0} \phi_j^k(a)t^k \quad \text{and} \quad F_j^2.a' = \sum_{l \geq 0} \psi_j^l(a')t^l,$$

where  $\phi_j^k : A \rightarrow A$ ,  $\psi_j^l : A \rightarrow A$  are  $\mathbb{K}$ -linear functions. The coefficient of  $t^i$  in  $\mu_{A[[t]]} \circ (F_j^1 \otimes F_j^2)(a \otimes a')$  is  $\sum_{k+l=i} \phi_j^k(a)\psi_j^l(a')$ . As  $\deg(F_j^1 \otimes F_j^2) \geq 1$ , the coefficient of  $t^0$  in  $\mu_{A[[t]]} \circ (F_j^1 \otimes F_j^2)(a \otimes a')$  is 0 for each  $j \geq 1$ . Therefore we have  $\mu_{A[[t]]} \circ F(a \otimes a') = \mu_A(a \otimes a') + t\mu_1(a \otimes a') + t^2\mu_2(a \otimes a') + \cdots$  where

$$\mu_i(a \otimes a') = \sum_{j \geq 1} \sum_{k+l=i} \phi_j^k(a)\psi_j^l(a').$$

This is necessarily well-defined for each  $i$ , since  $F$  is assumed to be a well-defined operator on  $A[[t]] \otimes_{\mathbb{K}[[t]]} A[[t]]$ . As  $F_j^1, F_j^2$  act linearly on  $A$  for each  $j$ , the same is true of the functions  $\phi_j^k, \psi_j^l$ , and thus  $\mu_i$  is bilinear.  $\square$

### 3. TWISTING ELEMENTS FROM FINITE QUANTUM GROUPS

Drinfeld doubles of finite-dimensional Hopf algebras provide a wealth of examples of quasitriangular Hopf algebras. In particular, the two-parameter restricted quantum groups  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  of [BW3] are Drinfeld doubles under some mild assumptions on  $r$  and  $s$ . We discuss implications of this for twisting and deforming algebras later on. The restricted quantum group  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  is a quotient of the unital associative  $\mathbb{K}$ -algebra  $U_{r,s}(\mathfrak{sl}_n)$ , which we introduce next.

Let  $\epsilon_1, \dots, \epsilon_n$  denote an orthonormal basis of a Euclidean space  $E = \mathbb{R}^n$  with an inner product  $\langle \cdot, \cdot \rangle$ . Set  $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \dots, n-1\}$  and  $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$ . Then  $\Phi$  is a finite root system of type  $A_{n-1}$  with  $\Pi$  a base of simple roots. Let  $\mathbb{K}$  be a field. Choose  $r, s \in \mathbb{K}^\times$  with  $r \neq s$ . The algebra  $U = U_{r,s}(\mathfrak{sl}_n)$  is the unital associative  $\mathbb{K}$ -algebra generated by  $e_j, f_j$  ( $1 \leq j < n$ ), and  $\omega_i^{\pm 1}, (\omega'_i)^{\pm 1}$  ( $1 \leq i < n$ ), subject to the following relations.

$$(R1) \quad \text{The } \omega_i^{\pm 1}, (\omega'_j)^{\pm 1} \text{ all commute with one another and } \omega_i \omega_i^{-1} = \omega'_j (\omega'_j)^{-1} = 1,$$

$$(R2) \quad \omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i \quad \text{and} \quad \omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i,$$

$$(R3) \quad \omega'_i e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega'_i \quad \text{and} \quad \omega'_i f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega'_i,$$

$$(R4) \quad [e_i, f_j] = \frac{\delta_{i,j}}{r-s} (\omega_i - \omega'_i).$$

$$\begin{aligned}
 \text{(R5)} \quad & [e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } |i - j| > 1, \\
 \text{(R6)} \quad & e_i^2 e_{i+1} - (r + s)e_i e_{i+1} e_i + r s e_{i+1} e_i^2 = 0, \\
 & e_i e_{i+1}^2 - (r + s)e_{i+1} e_i e_{i+1} + r s e_{i+1}^2 e_i = 0, \\
 \text{(R7)} \quad & f_i^2 f_{i+1} - (r^{-1} + s^{-1})f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0, \\
 & f_i f_{i+1}^2 - (r^{-1} + s^{-1})f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_{i+1}^2 f_i = 0.
 \end{aligned}$$

The algebra  $U$  becomes a Hopf algebra over  $\mathbb{K}$  with  $\omega_i, \omega'_i$  group-like and

$$\begin{aligned}
 \Delta(e_i) &= e_i \otimes 1 + \omega_i \otimes e_i, & \Delta(f_i) &= 1 \otimes f_i + f_i \otimes \omega'_i, \\
 \epsilon(e_i) &= 0, & \epsilon(f_i) &= 0, \\
 S(e_i) &= -\omega_i^{-1} e_i, & S(f_i) &= -f_i (\omega'_i)^{-1}.
 \end{aligned}$$

Let  $U^0$  be the group algebra generated by all  $\omega_i^{\pm 1}, (\omega'_i)^{\pm 1}$ , and let  $U^+$  (respectively,  $U^-$ ) be the subalgebra of  $U$  generated by all  $e_i$  (respectively,  $f_i$ ).

We define

$$\begin{aligned}
 \mathcal{E}_{j,j} &= e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - r^{-1} \mathcal{E}_{i-1,j} e_i \quad (i > j), \\
 \mathcal{F}_{j,j} &= f_j \quad \text{and} \quad \mathcal{F}_{i,j} = f_i \mathcal{F}_{i-1,j} - s \mathcal{F}_{i-1,j} f_i \quad (i > j).
 \end{aligned}$$

The algebra  $U$  has a triangular decomposition  $U \cong U^- \otimes U^0 \otimes U^+$ , and as discussed in [BKL], the subalgebras  $U^+, U^-$  respectively have monomial PBW (Poincaré-Birkhoff-Witt) bases

$$(3.1) \quad \mathcal{E} := \{\mathcal{E}_{i_1, j_1} \mathcal{E}_{i_2, j_2} \cdots \mathcal{E}_{i_p, j_p} \mid (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_p, j_p) \text{ lexicographically}\},$$

$$(3.2) \quad \mathcal{F} := \{\mathcal{F}_{i_1, j_1} \mathcal{F}_{i_2, j_2} \cdots \mathcal{F}_{i_p, j_p} \mid (i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_p, j_p) \text{ lexicographically}\}.$$

The spaces  $U^{\leq 0} = U^- \otimes U^0$  and  $U^{\geq 0} = U^0 \otimes U^+$  are Hopf subalgebras of  $U$ . We view  $U^{\leq 0}$  as a Hopf algebra with the opposite coproduct  $\Delta^{\text{op}}$ . Then there is a Hopf pairing  $(\mid) : U^{\leq 0} \times U^{\geq 0} \rightarrow \mathbb{K}$  given by

$$(3.3) \quad \begin{aligned}
 (f_i \mid e_j) &= \frac{\delta_{i,j}}{s - r} \\
 (\omega'_i \mid \omega_j) &= r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_{j+1}, \alpha_i \rangle}
 \end{aligned}$$

for  $1 \leq i, j < n$ , and all other pairings between generators are 0. Pairings between more complicated expressions can be computed from those values by applying the coproducts  $\Delta$  of  $U^{\geq 0}$  and  $\Delta^{\text{op}}$  of  $U^{\leq 0}$  according to the following rules:

$$(3.4)$$

- (i)  $(1 | b) = \epsilon(b)$ ,  $(b' | 1) = \epsilon(b')$ ,
- (ii)  $(b' | bc) = (\Delta^{\text{op}}(b') | b \otimes c) = \sum_{(b')} (b'_{(2)} | b)(b'_{(1)} | c)$   
(where  $\Delta(b') = \sum_{(b')} b'_{(1)} \otimes b'_{(2)}$ ),
- (iii)  $(b'c' | b) = (b' \otimes c' | \Delta(b)) = \sum_{(b)} (b' | b_{(1)})(c' | b_{(2)})$ ,

for all  $b, c \in U^{\geq 0}$  and  $b', c' \in U^{\leq 0}$ .

For the rest of this section, we assume that  $r$  and  $s$  are roots of unity. Let  $r$  be a primitive  $d$ th root of unity,  $s$  be a primitive  $d'$ th root of unity, and  $\ell$  be the least common multiple of  $d$  and  $d'$ . We also assume that  $\mathbb{K}$  contains a primitive  $\ell$ th root of unity  $\theta$  and that  $r = \theta^y$ ,  $s = \theta^z$ , where  $y, z$  are nonnegative integers.

It is shown in [BW3] that all  $\mathcal{E}_{i,j}^\ell$ ,  $\mathcal{F}_{i,j}^\ell$ ,  $\omega_i^\ell - 1$ , and  $(\omega'_i)^\ell - 1$  ( $1 \leq j \leq i < n$ ) are central in  $U_{r,s}(\mathfrak{sl}_n)$ . The ideal  $I_n$  generated by these elements is in fact a Hopf ideal [BW3, Thm. 2.17]. This is evident from the following expressions for the coproduct given in [BW3, (2.24) and the ensuing text]:

$$(3.5) \quad \Delta(\mathcal{E}_{i,j}^\ell) = \mathcal{E}_{i,j}^\ell \otimes 1 + \omega_{i,j}^\ell \otimes \mathcal{E}_{i,j}^\ell + s^{\ell(\ell-1)/2} (1 - r^{-1}s)^\ell \sum_{p=j}^{i-1} \mathcal{E}_{i,p+1}^\ell \omega_{p,j}^\ell \otimes \mathcal{E}_{p,j}^\ell$$

$$(3.6) \quad \Delta(\mathcal{F}_{i,j}^\ell) = 1 \otimes \mathcal{F}_{i,j}^\ell + \mathcal{F}_{i,j}^\ell \otimes (\omega'_{i,j})^\ell + r^{-\ell(\ell-1)/2} (1 - r^{-1}s)^\ell \sum_{p=j}^{i-1} \mathcal{F}_{p,j}^\ell \otimes \mathcal{F}_{i,p+1}^\ell (\omega'_{p,j})^\ell$$

where  $\omega_{p,j} = \omega_p \omega_{p-1} \cdots \omega_j$  and  $\omega'_{p,j} = \omega'_p \omega'_{p-1} \cdots \omega'_j$ . As  $I_n$  is a Hopf ideal, the quotient

$$(3.7) \quad \mathfrak{u} := \mathfrak{u}_{r,s}(\mathfrak{sl}_n) := U_{r,s}(\mathfrak{sl}_n)/I_n$$

is a Hopf algebra, called the *restricted two-parameter quantum group*. Furthermore,  $\mathfrak{u}$  is finite-dimensional, as can be readily seen from the PBW-bases (3.1) and (3.2). The algebra  $U$  is graded by the root lattice  $Q$  of  $\mathfrak{sl}_n$  by assigning

$$\deg(e_i) = \alpha_i, \quad \deg(f_i) = -\alpha_i, \quad \text{and} \quad \deg(\omega_i) = 0 = \deg(\omega'_i).$$

Since the generators of  $I_n$  are homogeneous,  $\mathfrak{u}$  inherits the grading. Note that by (3.3) and (3.4), if  $X \in \mathcal{E}$  and  $Y \in \mathcal{F}$  with  $(Y | X) \neq 0$ , then  $\deg(XY) = 0$ .

Let  $\mathcal{E}_\ell$  denote the set of monomials in  $\mathcal{E}$  having each  $\mathcal{E}_{i,j}$  appear with exponent at most  $\ell - 1$ . Identifying cosets in  $\mathfrak{u}$  with their representatives, we may assume  $\mathcal{E}_\ell$  is a basis for the subalgebra of  $\mathfrak{u}$  generated by the elements  $e_i$ .

The following proposition will allow us to use the pairing (3.3) on the quotient algebra  $\mathfrak{u}$ .

**Proposition 3.8.** *The ideal  $I_n$  is contained in the radical of the pairing  $(|)$  of  $U_{r,s}(\mathfrak{sl}_n)$ . Thus there is an induced pairing on the quotient  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ .*



*Proof.* For the group-like elements, note that for each pair  $i, j$ ,

$$\begin{aligned} (\omega'_i | \omega_j^\ell - 1) &= (\omega'_i | \omega_j^\ell) - (\omega'_i | 1) \\ &= (r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_{j+1}, \alpha_i \rangle})^\ell - 1 = 0, \end{aligned}$$

as  $r$  and  $s$  are  $\ell$ th roots of 1. Thus  $\omega_j^\ell - 1$  is in the radical of  $(|)$ , and similarly for  $(\omega'_i)^\ell - 1$ .

Now consider  $\mathcal{E}_{i,j}^\ell$  and let  $X$  be any monomial in  $f_1, \dots, f_{n-1}$  of degree  $-\ell\alpha_i - \dots - \ell\alpha_j$ . Such a monomial is the only type that potentially has a nonzero pairing with  $\mathcal{E}_{i,j}^\ell$ . As  $i \geq j$  and  $\ell \geq 2$ , we may write  $X = Y f_k$  for some  $k$ . By (3.4)(iii) and (3.5),

$$(X | \mathcal{E}_{i,j}^\ell) = (Y \otimes f_k | \mathcal{E}_{i,j}^\ell \otimes 1 + \omega_{i,j}^\ell \otimes \mathcal{E}_{i,j}^\ell + s^{\ell(\ell-1)/2} (1 - r^{-1}s)^\ell \sum_{p=j}^{i-1} \mathcal{E}_{i,p+1}^\ell \omega_{p,j}^\ell \otimes \mathcal{E}_{p,j}^\ell).$$

Again as  $\ell \geq 2$  and  $\deg f_k = -\alpha_k$ , each term above is 0.  $\square$

We will need to apply the following results from [BW3], where the corresponding pairing differs from (3.3) by nonzero scalar multiples. This difference does not affect the results. (See the proof of [BW3, Thm. 4.8] where a relevant adjustment is made.)

**Proposition 3.9.** [BW3, (5.8) and Lem. 4.1] *Let  $\theta$  be a primitive  $\ell$ th root of unity, and suppose  $r = \theta^y$  and  $s = \theta^z$ . Let  $\mathfrak{b}$  be the subalgebra of  $\mathfrak{u}$  generated by  $\omega_i, e_i$  ( $1 \leq i < n$ ) and  $\mathfrak{b}'$  be the subalgebra of  $\mathfrak{u}$  generated by  $\omega'_i, f_i$  ( $1 \leq i < n$ ). Then the Hopf pairing  $(|)$  on  $\mathfrak{b}' \times \mathfrak{b}$  satisfying (3.3) and (3.4) is nondegenerate if*

$$(3.10) \quad (y^{n-1} - y^{n-2}z + \dots + (-1)^{n-1}z^{n-1}, \ell) = 1,$$

(that is, the first expression in the parentheses is relatively prime to  $\ell$ , the least common multiple of the orders of  $r$  and  $s$  as roots of 1).

We will use the definition of the Drinfeld double  $D(\mathfrak{b})$  given in [BW3]:  $D(\mathfrak{b}) = \mathfrak{b} \otimes (\mathfrak{b}^*)^{\text{coop}}$  as a coalgebra, where  $(\mathfrak{b}^*)^{\text{coop}}$  denotes the dual Hopf algebra with the opposite coproduct. As an algebra,  $\mathfrak{b}$  and  $(\mathfrak{b}^*)^{\text{coop}}$  become subalgebras of  $D(\mathfrak{b})$  under identifications with  $\mathfrak{b} \otimes 1$  and  $1 \otimes \mathfrak{b}^*$ , respectively, and

$$(1 \otimes b)(a \otimes 1) = \sum_{(a),(b)} (b_{(1)} | S^{-1}(a_{(1)}))(b_{(3)} | a_{(3)}) a_{(2)} \otimes b_{(2)}.$$

**Proposition 3.11.** [BW3, Thm. 4.8] *Assume  $r = \theta^y$  and  $s = \theta^z$ , where  $\theta$  is a primitive  $\ell$ th root of unity, and suppose that (3.10) holds. Then there is an isomorphism of Hopf algebras  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n) \cong D(\mathfrak{b})$ , where  $D(\mathfrak{b})$  is the Drinfeld double of the Hopf subalgebra  $\mathfrak{b}$  of  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  generated by  $\omega_i, e_i$  ( $1 \leq i < n$ ).*

In general, whenever  $\mathfrak{u} \cong \mathfrak{u}_{r,s}(\mathfrak{sl}_n) \cong D(\mathfrak{b})$ , then  $\mathfrak{u}$  is quasitriangular with  $R$ -matrix,

$$R = \sum b \otimes b^*,$$

where  $b$  runs over a basis of  $\mathfrak{b}$  and  $b^*$  runs over the dual basis of the dual space  $\mathfrak{b}^*$ , which can be identified with the Hopf subalgebra  $\mathfrak{b}'$  of  $\mathfrak{u}$  generated by  $\omega'_i, f_i$  ( $1 \leq i < n$ ) but with the opposite coproduct  $\Delta^{\text{op}}$  [BW3, Lem. 4.1].

To illustrate this result in a very special case, take  $r = q$ , a primitive  $\ell$ th root of unity, and  $s = q^{-1}$ . Then  $\mathfrak{u} = \mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_n)$  is isomorphic to the Drinfeld double of  $\mathfrak{b}$  when  $n$  and  $\ell$  are relatively prime. The quotient of  $\mathfrak{u}$  by the ideal generated by the elements  $\omega'_i - \omega_i^{-1}$  for  $1 \leq i < n$  is a one-parameter restricted quantum group related to those which have played a significant role in the study of algebraic groups in the case  $\ell = p$ , a prime (see for example, [AJS]).

**Lemma 3.12.** *Assume  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  is a Drinfeld double as in Proposition 3.11. Then its  $R$ -matrix factors,*

$$(3.13) \quad R = R_{e,f} R_{\omega,\omega'},$$

with  $R_{e,f} = \sum \varepsilon \otimes \varepsilon^*$ , where  $\varepsilon$  runs over the basis  $\mathcal{E}_\ell$  of  $\ell$ -power truncated monomials, and  $R_{\omega,\omega'} = \sum w \otimes w^*$ , where  $w$  runs over the basis  $\Omega := \{\omega^\mathbf{c} = \omega_1^{c_1} \cdots \omega_{n-1}^{c_{n-1}} \mid 0 \leq c_i < \ell \text{ for all } i\}$  of the group algebra generated by the  $\omega_i$  ( $1 \leq i < n$ ).

*Proof.* Let  $\{\varepsilon^* \mid \varepsilon \in \mathcal{E}_\ell\}$  denote elements of  $\mathfrak{b}'$  dual to the vectors of  $\mathcal{E}_\ell$  with respect to this pairing so that  $((\varepsilon')^* \mid \varepsilon) = \delta_{\varepsilon',\varepsilon}$  for all  $\varepsilon', \varepsilon \in \mathcal{E}_\ell$ . For example, if  $\varepsilon = \mathcal{E}_{1,1}\mathcal{E}_{2,2} = e_1e_2$ , then  $\varepsilon^*$  is a linear combination of  $\mathcal{F}_{1,1}\mathcal{F}_{2,2} = f_1f_2$  and  $\mathcal{F}_{2,1} = f_2f_1 - sf_1f_2$ . Similarly, let  $\{w^* \mid w \in \Omega\}$  denote the basis of  $\mathbb{K}G'$  dual to  $\Omega$ , where  $G'$  is the group generated by the  $\omega'_i$ . By the triangular decomposition of  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ , the elements  $\{\varepsilon w \mid \varepsilon \in \mathcal{E}_\ell, w \in \Omega\}$  form a basis of  $\mathfrak{b}$  (see for example, [BW3, (2.16)]). Moreover using  $\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i$  and  $\Delta^{\text{op}}(f_i) = f_i \otimes 1 + \omega'_i \otimes f_i$ , the properties of the pairing in (3.4), and degree considerations, we have

$$\begin{aligned} ((\varepsilon')^*(w')^* \mid \varepsilon w) &= ((\varepsilon')^* \otimes (w')^* \mid \Delta(\varepsilon w)) \\ &= ((\varepsilon')^* \otimes (w')^* \mid \Delta(\varepsilon)(w \otimes w)) \\ &= ((\varepsilon')^* \mid \varepsilon w)((w')^* \mid w) \\ &= (\Delta^{\text{op}}((\varepsilon')^*) \mid \varepsilon \otimes w) \delta_{w',w} \\ &= ((\varepsilon')^* \mid \varepsilon)(1 \mid w) \delta_{w',w} \\ &= \delta_{\varepsilon',\varepsilon} \delta_{w',w}. \end{aligned}$$

This calculation shows that  $\varepsilon^*w^*$  is the dual basis element of  $\varepsilon w$  relative to the pairing. Thus,

$$\begin{aligned} R &= \sum_{\varepsilon \in \mathcal{E}_\ell, w \in \Omega} \varepsilon w \otimes \varepsilon^* w^* \\ &= \left( \sum_{\varepsilon \in \mathcal{E}_\ell} \varepsilon \otimes \varepsilon^* \right) \left( \sum_{w \in \Omega} w \otimes w^* \right). \end{aligned}$$

□

It follows from Lemma 3.12 that  $R_{21} = R_{f,e}R_{\omega',\omega}$ , where  $R_{f,e} = \sum_{\varepsilon \in \mathcal{E}_\ell} \varepsilon^* \otimes \varepsilon = (R_{e,f})_{21}$  and  $R_{\omega',\omega} = \sum_{w \in \Omega} w^* \otimes w = (R_{\omega,\omega'})_{21}$ . Equation (2.11) holds for  $R_{21}$ , that is

$$(3.14) \quad [(\Delta \otimes \text{id})(R_{f,e}R_{\omega',\omega})](R_{f,e}R_{\omega',\omega} \otimes 1) = [(\text{id} \otimes \Delta)(R_{f,e}R_{\omega',\omega})](1 \otimes R_{f,e}R_{\omega',\omega}).$$

We will show that the factor  $R_{f,e}$  alone satisfies this equation and so is a twisting element.

Let  $\mathcal{C}$  be the category of left  $\mathfrak{u}$ -module algebras. Let  $G$  be the group generated by the  $\omega_i$  and  $G'$  the group generated by the  $\omega'_i$  ( $1 \leq i < n$ ). Since  $\omega_i, \omega'_i$  ( $1 \leq i < n$ ) generate the finite abelian group  $G \times G'$  of order a power of  $\ell$ , and  $\mathbb{K}$  contains a primitive  $\ell$ th root of 1, each  $A \in \mathcal{C}$  decomposes into a direct sum of common eigenspaces (or weight spaces) for the  $\omega_i, \omega'_i$ . That is,  $A = \bigoplus_{\chi} A_{\chi}$  for some group homomorphisms  $\chi : G \times G' \rightarrow \mathbb{K}^{\times}$ , where  $\omega_i \cdot a = \chi(\omega_i)a$  and  $\omega'_i \cdot a = \chi(\omega'_i)a$  for all  $a \in A_{\chi}$ ,  $i = 1, \dots, n-1$ . We will use this decomposition to show that  $R_{f,e}$  is itself a twisting element for  $\mathcal{C}$ .

**Theorem 3.15.** *Let  $\mathfrak{u} := \mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ . The factor  $R_{f,e}$  of the opposite  $R_{21}$  of the  $R$ -matrix for  $\mathfrak{u}$  is a twisting element for the category of left  $\mathfrak{u}$ -module algebras.*

*Proof.* Let  $A$  be a left  $\mathfrak{u}$ -module algebra, and apply both sides of (3.14) to  $a \otimes a' \otimes a'' \in A_{\chi} \otimes A_{\psi} \otimes A_{\phi}$ , for weights  $\chi, \psi, \phi$  of  $A$ . Note that  $R_{\omega',\omega}$  acts as the scalar  $\xi(\chi, \psi) := \sum_{w \in \Omega} \chi(w^*)\psi(w)$  on  $a \otimes a'$ . Applying the left side of (3.14) to  $a \otimes a' \otimes a''$ , we have

$$\begin{aligned} & [(\Delta \otimes \text{id})(R_{f,e}R_{\omega',\omega})](R_{f,e} \otimes 1)\xi(\chi, \psi)a \otimes a' \otimes a'' \\ &= [(\Delta \otimes \text{id})(R_{f,e})](R_{f,e} \otimes 1)\xi(\chi \cdot \psi, \phi)\xi(\chi, \psi)a \otimes a' \otimes a'', \end{aligned}$$

since each term of  $R_{f,e}$  when applied to  $a \otimes a'$  changes the weight of  $a$  and the weight of  $a'$  by multiplication by functions which are inverses of one another, with a net effect of no change at all in the weight. Similarly, applying the right side of (3.14) to  $a \otimes a' \otimes a''$ , we have

$$\begin{aligned} & [(\text{id} \otimes \Delta)(R_{f,e}R_{\omega',\omega})](1 \otimes R_{f,e})\xi(\psi, \phi)a \otimes a' \otimes a'' \\ &= [(\text{id} \otimes \Delta)(R_{f,e})](1 \otimes R_{f,e})\xi(\chi, \psi \cdot \phi)\xi(\psi, \phi)a \otimes a' \otimes a''. \end{aligned}$$

Now  $R_{\omega',\omega}$  is itself an  $R$ -matrix for the group algebra  $\mathbb{K}[G \times G']$ , because our constructions show that this group algebra is the Drinfeld double of  $\mathbb{K}[G]$ . Similar calculations to those above prove that

$$\xi(\chi \cdot \psi, \phi)\xi(\chi, \psi) = \xi(\chi, \psi \cdot \phi)\xi(\psi, \phi)$$

for all weights  $\chi, \psi, \phi$ . (Note this implies that  $\xi$  is a 2-cocycle for the dual group to  $G \times G'$ .) Moreover, as  $R_{\omega',\omega}$  is invertible, none of these values is zero. Thus we may cancel the factors  $\xi(\chi \cdot \psi, \phi)\xi(\chi, \psi)$  and  $\xi(\chi, \psi \cdot \phi)\xi(\psi, \phi)$  from the above expressions and apply (3.14) to obtain

$$[(\Delta \otimes \text{id})(R_{f,e})](R_{f,e} \otimes 1)(a \otimes a' \otimes a'') = [(\text{id} \otimes \Delta)(R_{f,e})](1 \otimes R_{f,e})(a \otimes a' \otimes a''),$$

as desired. We also have  $(\epsilon \otimes 1)(R_{f,e}) = 1 \otimes 1 = (1 \otimes \epsilon)(R_{f,e})$  by the definition of  $R_{f,e}$ .  $\square$

**Remark 3.16.** Theorem 4.15 in the next section provides an alternate approach to proving Theorem 3.15.

**Examples 3.17.** We will give some examples of  $\mathfrak{u}$ -module algebras, which may be twisted by  $R_{f,e}$  according to Theorems 2.5 and 3.15. First let  $V$  be the natural  $n$ -dimensional module for  $U = U_{r,s}(\mathfrak{sl}_n)$ ; that is,  $V$  has a basis  $v_1, \dots, v_n$  and

$$\begin{aligned} e_i.v_j &= \delta_{i,j-1}v_{j-1} \\ f_i.v_j &= \delta_{i,j}v_{j+1} \\ \omega_i.v_j &= r^{\delta_{i,j}}s^{\delta_{i,j-1}}v_j, \\ \omega'_i.v_j &= r^{\delta_{i,j-1}}s^{\delta_{i,j}}v_j, \end{aligned}$$

where  $v_0 = 0 = v_{n+1}$ . Because each  $\mathcal{E}_{i,j}^2$  acts as 0,  $V$  becomes a  $\mathfrak{u}$ -module via the induced action. This action extends to a representation of  $\mathfrak{u}$  on the tensor algebra  $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ , where the action on  $V^{\otimes k}$  is by  $\Delta^{k-1}$  for  $k \geq 1$ . When  $k = 0$ , then  $V^{\otimes k} = \mathbb{K}$ , and the action is given by the counit. By definition then,  $T(V)$  is a  $\mathfrak{u}$ -module algebra.

Let  $J$  be the (two-sided) ideal of  $T(V)$  generated by all elements of the form

$$v_i \otimes v_j - rv_j \otimes v_i \quad (j > i).$$

Then  $J$  is homogeneous,  $J = \bigoplus_{k=2}^{\infty} J_k$ , where  $J_k = J \cap V^{\otimes k}$ . In fact,  $J$  is a  $\mathfrak{u}$ -submodule of  $T(V)$ : A computation such as in [BW2, Prop. 5.3] shows that  $J_2 := J \cap V^{\otimes 2}$  is a  $\mathfrak{u}$ -submodule of  $V^{\otimes 2}$ , and as  $J = T(V) \otimes J_2 \otimes T(V)$ , the result follows. Thus, the quantum plane  $\mathbb{K}_r[x_1, \dots, x_n]$ , with  $x_i x_j = rx_j x_i$  if  $j > i$ , is isomorphic to the  $\mathfrak{u}$ -module algebra  $T(V)/J$ , under the identification  $x_i = v_i + J$ . Let

$$x(\underline{d}) = x(d_1, \dots, d_n) = x_1^{d_1} \cdots x_n^{d_n}.$$

Then the  $\mathfrak{u}$ -module action is given by

$$\begin{aligned} e_i.x(\underline{d}) &= r^{d_i-d_{i+1}+1}[d_{i+1}]x(d_1, \dots, d_i+1, d_{i+1}-1, \dots, d_n), \\ f_i.x(\underline{d}) &= r^{d_{i+1}-d_i+1}[d_i]x(d_1, \dots, d_i-1, d_{i+1}+1, \dots, d_n), \\ \omega_i.x(\underline{d}) &= r^{d_i}s^{d_{i+1}}x(\underline{d}), \\ \omega'_i.x(\underline{d}) &= r^{d_{i+1}}s^{d_i}x(\underline{d}), \end{aligned}$$

where

$$[d] := \frac{r^d - s^d}{r - s}.$$

Other quotients of  $T(V)$  provide examples as well: Choose a positive integer  $p$ , and let  $J$  be the ideal of  $T(V)$  generated by the  $\mathfrak{u}$ -submodule of  $T(V)$  having as generators all  $v_i^{\otimes p}$ . As an abbreviation, we will write  $v_i^p = v_i^{\otimes p}$ , and similarly leave out the tensor product symbol in the notation for words in  $v_1, \dots, v_n$ . For example, if  $n = 2$  and  $p = 2$ , the  $\mathfrak{u}$ -submodule of  $V^{\otimes 2}$  generated by  $v_1^2$  and  $v_2^2$  has basis  $v_1^2, v_2^2, v_1v_2 + sv_2v_1$ . If  $n = 2$  and  $p = 3$ , the  $\mathfrak{u}$ -submodule of  $V^{\otimes 3}$  generated by  $v_1^3$  and  $v_2^3$  has basis  $v_1^3, v_2^3, v_1v_2^2 + sv_2v_1v_2 + s^2v_2^2v_1, v_1^2v_2 + sv_1v_2v_1 + s^2v_2v_1^2$ . If  $q$  is a primitive third root of 1, and  $r = q, s = q^{-1}$ , then  $\mathfrak{u} = \mathfrak{u}_{q,q^{-1}}(\mathfrak{sl}_2)$  acts on the quotient of  $T(V)$  by the ideal generated by  $v_1^2v_2 + sv_1v_2v_1 + s^2v_2v_1^2, v_1v_2^2 + sv_2v_1v_2 + s^2v_2^2v_1$ , which is the down-up algebra  $A(1 + s^2, -s^2, 0) = A(1 + r, -r, 0)$  in the notation of [BR]. This example is related to one appearing in the work of Montgomery and Schneider [MS] on skew derivations and actions of the double of the Taft algebra.

The family of down-up algebras  $A(1+r, -r, 0)$  (where  $r$  is an arbitrary primitive  $\ell$ th root of unity) is especially interesting, as the finite-dimensional modules for these algebras are completely reducible [CM]. This family includes the universal enveloping algebra  $U(\mathfrak{sl}_2)$  for  $r = 1$ , the universal enveloping algebra  $U(\mathfrak{osp}_{1,2})$  of the Lie superalgebra  $\mathfrak{osp}_{1,2}$  for  $r = -1$ , and the algebras appearing in this example for  $r$  a primitive third root of 1.

**Example 3.18.** Here we generalize the last example by allowing  $n$  to be arbitrary. Again we suppose  $r = q, s = q^{-1}$  for  $q$  a primitive third root of 1, but take  $\mathfrak{u} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  for any  $n \geq 3$ . Consider the subspace  $Y$  of  $V^{\otimes 3}$  spanned by the following elements:

$$\begin{aligned} y(i, j, k) &= v_i v_j v_k + s v_j v_i v_k + s v_i v_k v_j + s^2 v_j v_k v_i + s^2 v_k v_i v_j + v_k v_j v_i \\ &\hspace{15em} (1 \leq i < j < k \leq n), \\ y(i, i, k) &= v_i^2 v_k + s v_i v_k v_i + s^2 v_k v_i^2 \hspace{10em} (1 \leq i < k \leq n), \\ y(i, k, k) &= v_i v_k^2 + s v_k v_i v_k + s^2 v_k^2 v_i \hspace{10em} (1 \leq i < k \leq n), \\ y(i, i, i) &= 0 \hspace{15em} (1 \leq i \leq n). \end{aligned}$$

From the formulas below it is easy to see that  $Y$  is a  $\mathfrak{u}$ -submodule of  $T(V)$ :

$$\begin{aligned}\omega_t.y(i, j, k) &= r^{(\epsilon_t, \epsilon_i + \epsilon_j + \epsilon_k)} s^{(\epsilon_{t+1}, \epsilon_i + \epsilon_j + \epsilon_k)} y(i, j, k), \\ \omega'_t.y(i, j, k) &= r^{(\epsilon_{t+1}, \epsilon_i + \epsilon_j + \epsilon_k)} s^{(\epsilon_t, \epsilon_i + \epsilon_j + \epsilon_k)} y(i, j, k),\end{aligned}$$

for all  $1 \leq i \leq j \leq k \leq n$ ; and using the fact that  $r + s = -1$ , we have

$$\begin{aligned}e_t.y(t+1, j, k) &= y(t, j, k) && (t+1 \leq j < k) \text{ or } (t+1 < j = k), \\ e_t.y(i, t+1, k) &= (-1)^{\delta_{i,t}} y(i, t, k) && (i < t+1 \leq k), \\ e_t.y(i, j, t+1) &= (-1)^{\delta_{j,t}} y(i, j, t) && (i \leq j \leq t), \\ e_t.y(i, j, k) &= 0 && \text{otherwise;} \\ \\ f_t.y(t, j, k) &= (-1)^{\delta_{j,t+1}} y(t+1, j, k) && (t < j \leq k), \\ f_t.y(i, t, k) &= (-1)^{\delta_{k,t+1}} y(i, t+1, k) && (i \leq t < k), \\ f_t.y(i, j, t) &= y(i, j, t+1) && (i \leq j < t) \text{ or } (i < j = t), \\ f_t.y(i, j, k) &= 0 && \text{otherwise.}\end{aligned}$$

The restricted quantum group  $\mathfrak{u}$  acts on the quotient algebra  $A = T(V)/\langle Y \rangle$  obtained by factoring out the ideal generated by  $Y$ . In the quotient, each pair  $v_i, v_k$  ( $1 \leq i < k \leq n$ ) generates a down-up algebra  $A(1 + s^2, -s^2, 0) = A(1 + r, -r, 0)$ .

**Example 3.19.** As a special instance of Theorem 3.15, consider the algebra  $\mathfrak{u}_{1,-1}(\mathfrak{sl}_2)$  for which

$$R_{f,e} = 1 \otimes 1 - 2f \otimes e.$$

(Note we omit subscripts when discussing  $\mathfrak{sl}_2$ , as there is only one of each type of generator.) We know from Theorem 3.15 that this is a twisting element, but this also may be verified directly quite easily. This quantum group has as a quotient

$$\mathfrak{u}_{-1}(\mathfrak{sl}_2) := \mathfrak{u}_{1,-1}(\mathfrak{sl}_2)/(\omega - \omega'),$$

and in  $\mathfrak{u}_{-1}(\mathfrak{sl}_2)$ , the images of  $e$  and  $f$  commute by relation (R4). Therefore, the defining relations of  $\mathfrak{u}_{-1}(\mathfrak{sl}_2)$  are all homogeneous with respect to powers of  $e$ . Consequently, the following is a twisting element based on  $\mathfrak{u}_{-1}(\mathfrak{sl}_2)[[t]]$ , as may be checked directly:

$$F = 1 \otimes 1 - 2tf \otimes e.$$

Thus,  $F$  is a universal deformation formula for the category of  $\mathfrak{u}_{-1}(\mathfrak{sl}_2)[[t]]$ -module algebras arising from  $\mathfrak{u}_{-1}(\mathfrak{sl}_2)$ -module algebras by extension of scalars. In fact, there is a one-parameter family of such deformation formulas, given by  $F_c = 1 \otimes 1 + ctf \otimes e$ , for any  $c \in \mathbb{K}$ . The choice  $c = 1$  yields precisely the universal deformation formula of [CGW, Lem. 6.2] that was applied to a certain  $\mathfrak{u}_{-1}(\mathfrak{sl}_2)$ -module algebra  $A$  (given by a crossed product of a polynomial ring with a finite group) to obtain a formal deformation of  $A$ .

However, it is an accident due to the choices  $\ell = 2$ ,  $n = 2$  for  $\mathfrak{u}_{1,-1}(\mathfrak{sl}_2)$  that this technique produces a universal deformation formula directly, as relation (R4) of  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  is *not* homogeneous with respect to the powers of the  $e_i$  (nor of the  $f_i$ ), and in general, there is no reasonable quotient in which it becomes so. In the next section we will remedy this situation by returning to the infinite-dimensional Hopf algebra  $U = U_{r,s}(\mathfrak{sl}_n)$  and choosing an *action* of  $U$  that itself incorporates the indeterminate  $t$ . Using this infinite-dimensional Hopf algebra necessitates a more complicated (but related) construction of a twisting element.

#### 4. TWISTING ELEMENTS FROM INFINITE QUANTUM GROUPS

We consider the infinite-dimensional quantum group  $U = U_{r,s}(\mathfrak{sl}_n)$ , defined in the previous section, where  $r$  and  $s$  are roots of unity. We adopt the same notation as before, so that  $\ell$  is the least common multiple of the orders of  $r$  and  $s$  as roots of 1, and  $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$  is the root lattice of  $\mathfrak{sl}_n$ . We will also assume (3.10) holds, so that we may use Proposition 3.9. Let  $\mathfrak{K}$  be a commutative  $\mathbb{K}$ -algebra, and extend  $U$  to a  $\mathfrak{K}$ -algebra  $U_{\mathfrak{K}}$  that is free as a  $\mathfrak{K}$ -module. (For the applications in the next section,  $\mathfrak{K}$  will be the  $\mathbb{K}$ -algebra of Laurent polynomials in  $t$  with finitely many negative powers of  $t$ , and  $U_{\mathfrak{K}}$  will be a similar extension.)

As in [BW1], we will be interested in weight modules, this time for  $U_{\mathfrak{K}}$ . We will consider only  $U_{\mathfrak{K}}$ -modules that are free as  $\mathfrak{K}$ -modules. For every  $\lambda \in Q$ , define the corresponding algebra homomorphism  $\hat{\lambda} : U^0 \rightarrow \mathbb{K}$  by

$$(4.1) \quad \hat{\lambda}(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle} \quad \text{and} \quad \hat{\lambda}(\omega'_j) = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle},$$

which may be extended to yield an algebra homomorphism from  $U_{\mathfrak{K}}^0$  to  $\mathfrak{K}$ . As  $r$  and  $s$  are roots of unity, the set  $\widehat{Q} = \{\hat{\lambda} \mid \lambda \in Q\}$  is finite.

Let  $M$  be a  $U_{\mathfrak{K}}$ -module, assumed to be free as a  $\mathfrak{K}$ -module. If as a  $U_{\mathfrak{K}}^0$ -module,  $M$  decomposes into the direct sum of eigenspaces

$$M_{\chi} = \{m \in M \mid (\omega_i - \chi(\omega_i))m = 0 = (\omega'_i - \chi(\omega'_i))m \text{ for all } i\}$$

for algebra homomorphisms  $\chi : U_{\mathfrak{K}}^0 \rightarrow \mathfrak{K}$ , we say  $U_{\mathfrak{K}}^0$  acts *semisimply* on  $M$ . The homomorphisms  $\chi$  such that  $M_{\chi} \neq 0$  are called the *weights* of  $M$ . Note that

$$(4.2) \quad e_j \cdot M_{\chi} \subseteq M_{\chi \cdot \widehat{\alpha}_j} \quad \text{and} \quad f_j \cdot M_{\chi} \subseteq M_{\chi \cdot (-\widehat{\alpha}_j)}$$

by (R2), (R3), and (4.1). It follows that if  $M$  is a simple  $U_{\mathfrak{K}}$ -module with a nonzero weight space  $M_{\chi}$ , then  $U_{\mathfrak{K}}^0$  acts semisimply on  $M$ , since the (necessarily direct) sum of the weight spaces is a submodule. In fact, if  $M = U_{\mathfrak{K}} \cdot m$ , any cyclic  $U_{\mathfrak{K}}$ -module generated by a weight vector  $m \in M_{\chi}$ , then  $U_{\mathfrak{K}}^0$  acts semisimply on  $M$ , and all the weights of  $M$  are of the form  $\chi \cdot \hat{\zeta}$  ( $\zeta \in Q$ ).

Let  $\mathcal{N}$  be the category of unital  $U_{\mathfrak{K}}$ -modules  $M$  that are free left  $\mathfrak{K}$ -modules and satisfy the following conditions:

- (N1)  $U_{\mathfrak{R}}^0$  acts semisimply on  $M$ ;  
(N2) For each  $i, j$  ( $1 \leq j \leq i < n$ ), both  $\mathcal{E}_{i,j}^\ell$  and  $\mathcal{F}_{i,j}^\ell$  annihilate  $M$ .

Note that  $\mathcal{N}$  is closed under direct sums and quotients, and it follows from (3.5) and (3.6) that  $\mathcal{N}$  is closed under tensor products.

Examples of modules  $M$  satisfying (N2) are the  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ -modules (see the previous section) with scalars extended to  $\mathfrak{R}$ , which can be viewed as  $U_{\mathfrak{R}}$ -modules. However we will also be interested in some modules in category  $\mathcal{N}$  that do not correspond to  $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ -modules, where neither  $\omega_i^\ell$  nor  $(\omega'_i)^\ell$  acts as the identity.

We remark that (N2) has been included so that certain operators will be well-defined. It is not necessarily true that  $e_i, f_i$  act nilpotently on all finite-dimensional modules as happens in the non-root of unity case [BW1, Cor. 3.14]. The argument used there fails, as  $\widehat{Q}$  is finite.

For any two modules  $M$  and  $M'$  in  $\mathcal{N}$ , we will construct a  $U_{\mathfrak{R}}$ -module homomorphism  $R_{M',M} : M' \otimes M \rightarrow M \otimes M'$  by Jantzen's method (see [J, Ch. 7]). The main difference between the two-parameter version of Jantzen's method (see [BW1, Sec. 4]) and what we are about to do here is that now we are assuming both  $r$  and  $s$  are roots of unity, a case excluded from consideration in [J, BW1]. As a consequence, we will not know whether  $R_{M',M}$  is invertible. However we will make the necessary adjustments to show that most of the arguments of [BW1, Secs. 4,5], in particular those needed to obtain a twisting element, apply in this context.

The desired function  $R_{M',M}$  will be the composition of three  $\mathfrak{R}$ -linear functions  $\tau, \tilde{f}$ , and  $F$ , where  $\tau$  is the map that interchanges tensor factors as before, and  $\tilde{f}$  and  $F$  are as follows. (Ultimately we will show that  $F$  is a twisting element, accounting for the choice of notation.)

As  $M$  satisfies (N1), by (4.2) there are algebra homomorphisms  $\chi : U_{\mathfrak{R}}^0 \rightarrow \mathfrak{R}$  such that  $M = \bigoplus_{\chi} M(\chi)$ , where  $M(\chi)$  is a  $U_{\mathfrak{R}}$ -submodule having weights contained in  $\chi \cdot \widehat{Q}$ . (For example, we may take the sum over a set of representatives  $\chi$  of cosets of  $\widehat{Q}$  in  $\text{Alg}_{\mathfrak{R}}(U_{\mathfrak{R}}^0, \mathfrak{R})$ .) For the purposes of this section, it will suffice to deal with each summand  $M(\chi)$  separately. Assume  $M = M(\chi)$  and  $M' = M'(\psi)$  are such modules. Let  $\lambda, \mu \in Q$  and  $m \in M_{\chi \cdot \lambda}, m' \in M_{\psi \cdot \mu}$ . Set

$$\tilde{f}(m \otimes m') = f_{\chi,\psi}(\lambda, \mu)(m \otimes m')$$

where  $f_{\chi,\psi} : Q \times Q \rightarrow \mathfrak{R}^\times$  is given by

$$f_{\chi,\psi}(\lambda, \mu) = \psi(\omega_\lambda^{-1})\chi(\omega'_\mu)(\omega'_\mu | \omega_\lambda^{-1})$$

with  $\omega_\lambda = \omega_1^{\lambda_1} \cdots \omega_{n-1}^{\lambda_{n-1}}, \omega'_\mu = (\omega'_1)^{\mu_1} \cdots (\omega'_{n-1})^{\mu_{n-1}}, \lambda = \sum_{i=1}^{n-1} \lambda_i \alpha_i, \mu = \sum_{i=1}^{n-1} \mu_i \alpha_i$ , and the Hopf pairing is as in (3.3), (3.4). Thus  $f_{1,1}$  is the function  $f$  of [BW1, (4.2)] as restricted to the root lattice  $Q$ , and it can be shown that  $\tilde{f}$  generalizes the function  $\tilde{f}$  there by looking at cosets of the weight lattice modulo  $Q$ . In particular,



the following hold:

$$(4.3) \quad \begin{aligned} f_{\chi,\psi}(\lambda + \mu, \nu) &= f_{\chi,\psi}(\lambda, \nu)\psi(\omega_\mu^{-1})(\omega'_\nu | \omega_\mu^{-1}) \\ f_{\chi,\psi}(\lambda, \mu + \nu) &= f_{\chi,\psi}(\lambda, \mu)\chi(\omega'_\nu)(\omega'_\nu | \omega_\lambda^{-1}) \\ (\omega'_\mu | \omega_\lambda^{-1}) &= \hat{\mu}(\omega_\lambda^{-1}) = \hat{\lambda}(\omega'_\mu). \end{aligned}$$

The definition of  $F$  is similar to that of  $\Theta$  in [BW1, Sec. 4]. We will construct  $F$  as a sum of elements of  $U \otimes U$ , which then may also be considered elements of  $U_{\mathfrak{R}} \otimes_{\mathfrak{R}} U_{\mathfrak{R}}$ . The subalgebra  $U^+$  of  $U$  generated by 1 and  $e_i$  ( $1 \leq i < n$ ) may be decomposed as

$$U^+ = \bigoplus_{\zeta \in Q^+} U_\zeta^+$$

where  $U_\zeta^+ = \{x \in U^+ \mid x \text{ is homogeneous of degree } \zeta\}$ , and  $Q^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}^{\geq 0} \alpha_i$ . For each  $\zeta \in Q^+$ , let  $\bar{U}_\zeta^+$  be the linear span of all PBW basis elements in  $U_\zeta^+$  (see (3.1)) in which the power of each  $\mathcal{E}_{i,j}$  is less than  $\ell$ . Similarly define  $\bar{U}_{-\zeta}^-$ . By Propositions 3.8 and 3.9, the spaces  $\bar{U}_\zeta^+$ ,  $\bar{U}_{-\zeta}^-$  are nondegenerately paired under the assumption that (3.10) holds. We define

$$(4.4) \quad F = \sum_{\zeta \in Q^+} F_\zeta$$

where  $F_\zeta = \sum_{k=1}^{\bar{d}_\zeta} v_k^\zeta \otimes u_k^\zeta$ ,  $\bar{d}_\zeta = \dim_{\mathbb{K}} \bar{U}_\zeta^+$ ,  $\{u_k^\zeta\}_{k=1}^{\bar{d}_\zeta}$  is a basis for  $\bar{U}_\zeta^+$ , and  $\{v_k^\zeta\}_{k=1}^{\bar{d}_\zeta}$  the dual basis for  $\bar{U}_{-\zeta}^-$ . Note that if  $\bar{U}_\zeta^+ = 0$ , then  $F_\zeta = 0$ , and if  $\zeta \in Q \setminus Q^+$ , we will also set  $F_\zeta = 0$  for convenience.

As  $\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i$ , for all  $x \in U_\zeta^+$  we have

$$\Delta(x) \in \sum_{0 \leq \nu \leq \zeta} U_{\zeta-\nu}^+ \omega_\nu \otimes U_\nu^+,$$

where  $\nu \leq \zeta$  means  $\zeta - \nu \in Q^+$ . For each  $i$ , there are elements  $p_i(x)$  and  $p'_i(x) \in U_{\zeta-\alpha_i}^+$  such that

$$\begin{aligned} \Delta(x) &= x \otimes 1 + \sum_{i=1}^{n-1} p_i(x) \omega_i \otimes e_i + \pi(x), \\ \Delta(x) &= \omega_\zeta \otimes x + \sum_{i=1}^{n-1} e_i \omega_{\zeta-\alpha_i} \otimes p'_i(x) + \pi'(x), \end{aligned}$$

where  $\pi(x)$  (respectively,  $\pi'(x)$ ) is a sum of terms involving products of more than one  $e_j$  in the second factor (respectively, in the first factor). Similarly, if  $y \in U_{-\zeta}^-$ ,

we define  $p_i(y)$  and  $p'_i(y)$  by

$$\begin{aligned}\Delta(y) &= y \otimes \omega'_\zeta + \sum_{i=1}^{n-1} p_i(y) \otimes f_i \omega'_{\zeta - \alpha_i} + \pi(y), \\ \Delta(y) &= 1 \otimes y + \sum_{i=1}^{n-1} f_i \otimes p'_i(y) \omega'_i + \pi'(y).\end{aligned}$$

The following identities from [BW1] hold in our context, as the proof consists of calculations that do not use properties of the parameters  $r$  and  $s$ .

**Lemma 4.5.** [BW1, Lem. 4.6, Lem. 4.8] *For all  $x \in U_\zeta^+$ ,  $x' \in U_{\zeta'}^+$ , and  $y \in U^-$ , the following hold:*

- (i)  $(f_i y | x) = (f_i | e_i)(y | p'_i(x)) = (s - r)^{-1}(y | p'_i(x))$ .
- (ii)  $(y f_i | x) = (f_i | e_i)(y, p_i(x)) = (s - r)^{-1}(y | p_i(x))$ .
- (iii)  $f_i x - x f_i = (s - r)^{-1}(p_i(x) \omega_i - \omega'_i p'_i(x))$ .

For all  $y \in U_{-\zeta}^-$ ,  $y' \in U_{-\zeta'}^-$ , and  $x \in U^+$ , the following hold:

- (iv)  $(y | e_i x) = (f_i | e_i)(p_i(y) | x) = (s - r)^{-1}(p_i(y) | x)$ .
- (v)  $(y, x e_i) = (f_i | e_i)(p'_i(y) | x) = (s - r)^{-1}(p'_i(y) | x)$ .
- (vi)  $e_i y - y e_i = (s - r)^{-1}(p'_i(y) \omega'_i - \omega_i p_i(y))$ .

Since the spaces  $\overline{U}_\zeta^+$  and  $\overline{U}_{-\zeta}^-$  are nondegenerately paired with dual bases  $\{u_k^\zeta\}_{k=1}^{\overline{d}_\zeta}$  and  $\{v_k^\zeta\}_{k=1}^{\overline{d}_\zeta}$ , for each  $x \in \overline{U}_\zeta^+$  and  $y \in \overline{U}_{-\zeta}^-$ , we have

$$(4.6) \quad x = \sum_{k=1}^{\overline{d}_\zeta} (v_k^\zeta | x) u_k^\zeta \quad \text{and} \quad y = \sum_{k=1}^{\overline{d}_\zeta} (y | u_k^\zeta) v_k^\zeta.$$

The next lemma is a modified version of [BW1, Lem. 4.10] or [J, Lem. 7.1]. The main distinction is that here the identities are as operators on modules rather than as elements of  $U \otimes U$ .

**Lemma 4.7.** *Let  $\zeta \in Q^+$  and  $1 \leq i < n$ . Then the following relations hold for operators on tensor products of pairs of modules in category  $\mathcal{N}$ :*

- (i)  $(\omega_i \otimes \omega_i) F_\zeta = F_\zeta(\omega_i \otimes \omega_i)$  and  $(\omega'_i \otimes \omega'_i) F_\zeta = F_\zeta(\omega'_i \otimes \omega'_i)$ .
- (ii)  $(e_i \otimes 1) F_\zeta + (\omega_i \otimes e_i) F_{\zeta - \alpha_i} = F_\zeta(e_i \otimes 1) + F_{\zeta - \alpha_i}(\omega'_i \otimes e_i)$ .
- (iii)  $(1 \otimes f_i) F_\zeta + (f_i \otimes \omega'_i) F_{\zeta - \alpha_i} = F_\zeta(1 \otimes f_i) + F_{\zeta - \alpha_i}(f_i \otimes \omega_i)$ .

*Proof.* We will verify that (ii) holds. Identity (iii) is similar to (ii), and (i) is immediate from (R2) and (R3). In the calculations below, we use Lemma 4.5(iv)–(vi), (4.6), and the fact that on a module in category  $\mathcal{N}$  any element of  $U_{-\zeta}^-$  acts

as its projection in  $\overline{U}_{-\zeta}^-$ .

$$\begin{aligned}
 (e_i \otimes 1)F_\zeta - F_\zeta(e_i \otimes 1) &= \sum_{k=1}^{\overline{d}_\zeta} (e_i v_k^\zeta - v_k^\zeta e_i) \otimes u_k^\zeta \\
 &= \frac{1}{s-r} \sum_k (p'_i(v_k^\zeta) \omega'_i - \omega_i p_i(v_k^\zeta)) \otimes u_k^\zeta \\
 &= \frac{1}{s-r} \sum_k \left( \sum_{j=1}^{\overline{d}_{\zeta-\alpha_i}} (p'_i(v_k^\zeta) | u_j^{\zeta-\alpha_i}) v_j^{\zeta-\alpha_i} \right) \omega'_i \otimes u_k^\zeta \\
 &\quad - \frac{1}{s-r} \sum_k \omega_i \left( \sum_{j=1}^{\overline{d}_{\zeta-\alpha_i}} (p_i(v_k^\zeta) | u_j^{\zeta-\alpha_i}) v_j^{\zeta-\alpha_i} \right) \otimes u_k^\zeta \\
 &= \sum_k \left( \sum_j (v_k^\zeta | u_j^{\zeta-\alpha_i} e_i) v_j^{\zeta-\alpha_i} \right) \omega'_i \otimes u_k^\zeta \\
 &\quad - \sum_k \omega_i \left( \sum_j (v_k^\zeta | e_i u_j^{\zeta-\alpha_i}) v_j^{\zeta-\alpha_i} \right) \otimes u_k^\zeta \\
 &= \sum_j v_j^{\zeta-\alpha_i} \omega'_i \otimes \left( \sum_k (v_k^\zeta | u_j^{\zeta-\alpha_i} e_i) u_k^\zeta \right) \\
 &\quad - \sum_j \omega_i v_j^{\zeta-\alpha_i} \otimes \left( \sum_k (v_k^\zeta | e_i u_j^{\zeta-\alpha_i}) u_k^\zeta \right) \\
 &= \sum_{j=1}^{\overline{d}_{\zeta-\alpha_i}} (v_j^{\zeta-\alpha_i} \omega'_i \otimes u_j^{\zeta-\alpha_i} e_i - \omega_i v_j^{\zeta-\alpha_i} \otimes e_i u_j^{\zeta-\alpha_i}) \\
 &= F_{\zeta-\alpha_i}(\omega'_i \otimes e_i) - (\omega_i \otimes e_i) F_{\zeta-\alpha_i}.
 \end{aligned}$$

□

Notice that (ii) and (iii) of the above lemma hold even in the cases where  $\overline{U}_\zeta^+ = 0$  but  $\overline{U}_{\zeta-\alpha_i}^+ \neq 0$ , because in these cases both sides of each equation annihilate modules in category  $\mathcal{N}$ .

The following is a modification of [BW1, Thm. 4.11] or [J, Thm. 7.3].

**Theorem 4.8.** *Let  $M$  and  $M'$  be modules in category  $\mathcal{N}$ . Then the map*

$$R_{M',M} = F \circ \tilde{f} \circ \tau : M' \otimes M \rightarrow M \otimes M'$$

*is a homomorphism of  $U$ -modules.*

*Proof.* We must prove that the action of each generator of  $U$  commutes with the map  $R$ . By Lemma 4.7,  $\omega_i$  and  $\omega'_i$  commute with  $R$ . We will check this for  $e_i$  and leave the similar calculation for  $f_i$  as an exercise. We may assume  $M = M(\chi)$  and  $M' = M'(\psi)$  for some algebra homomorphisms  $\chi, \psi : U_{\mathfrak{R}}^0 \rightarrow \mathfrak{K}$ . Let  $m \in M_{\chi, \hat{\lambda}}$ ,  $m' \in M'_{\psi, \hat{\mu}}$ . By (4.3),

$$\begin{aligned} (F \circ \tilde{f} \circ \tau)\Delta(e_i)(m' \otimes m) &= (F \circ \tilde{f})(m \otimes e_i m' + e_i m \otimes \omega_i m') \\ &= f_{\chi, \psi}(\lambda, \mu + \alpha_i)F(m \otimes e_i m') + f_{\chi, \psi}(\lambda + \alpha_i, \mu)F(e_i m \otimes \omega_i m') \\ &= f_{\chi, \psi}(\lambda, \mu)\chi(\omega'_i)\hat{\lambda}(\omega'_i)F(1 \otimes e_i)(m \otimes m') \\ &\quad + f_{\chi, \psi}(\lambda, \mu)\psi(\omega_i^{-1})\hat{\mu}(\omega_i^{-1})F(e_i \otimes \omega_i)(m \otimes m'). \end{aligned}$$

Now we may replace  $F$  by  $\sum_{\zeta \in Q} F_\zeta$  or  $\sum_{\zeta \in Q} F_{\zeta - \alpha_i}$ ,  $\chi(\omega'_i)\hat{\lambda}(\omega'_i)m$  by  $\omega'_i m$ , and  $\psi(\omega_i^{-1})\hat{\mu}(\omega_i^{-1})\omega_i m'$  by  $m'$ . Thus we obtain the following expression to which we apply Lemma 4.7(ii):

$$\begin{aligned} f_{\chi, \psi}(\lambda, \mu) &\left( \left( \sum_{\zeta} F_{\zeta - \alpha_i} \right) (\omega'_i \otimes e_i) + \left( \sum_{\zeta} F_{\zeta} \right) (e_i \otimes 1) \right) (m \otimes m') \\ &= f_{\chi, \psi}(\lambda, \mu) \left( (e_i \otimes 1) \left( \sum_{\zeta} F_{\zeta} \right) + (\omega_i \otimes e_i) \left( \sum_{\zeta} F_{\zeta - \alpha_i} \right) \right) (m \otimes m') \\ &= \Delta(e_i)F\tilde{f}\tau(m' \otimes m). \end{aligned}$$

□

We will need the following relation [BW1, (5.1)]:

$$(4.9) \quad (y\omega'_\zeta | x\omega_\eta) = (y | x)(\omega'_\zeta | \omega_\eta)$$

for all  $x \in U_\gamma^+$  and  $y \in U_{-\gamma}^-$ . A similar identity was proven and used in our Lemma 3.12.

**Lemma 4.10.** *If  $x \in \overline{U}_\gamma^+$  and  $y \in \overline{U}_{-\gamma}^-$ , then the following hold as relations of operators on a tensor product of two modules from category  $\mathcal{N}$ :*

$$\begin{aligned} \text{(i)} \quad \Delta(x) &= \sum_{0 \leq \zeta \leq \gamma} \sum_{i,j} (v_i^{\gamma-\zeta} v_j^\zeta | x) u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta, \\ \text{(ii)} \quad \Delta(y) &= \sum_{0 \leq \zeta \leq \gamma} \sum_{i,j} (y | u_i^{\gamma-\zeta} u_j^\zeta) v_j^\zeta \otimes v_i^{\gamma-\zeta} \omega'_\zeta. \end{aligned}$$

*Proof.* As in [BW1, Lem. 5.2], we may write  $\Delta(x) = \sum_{\zeta, i, j} c_{i,j}^\zeta u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta$  for some scalars  $c_{i,j}^\zeta$ , this time as an identity of operators on a tensor product of modules

from category  $\mathcal{N}$ . For all  $k, l$ , and  $\nu$ , by (3.4) and (4.9), we have

$$(v_k^{\gamma-\nu} v_l^\nu | x) = \sum_{\zeta, i, j} c_{i, j}^\zeta (v_k^{\gamma-\nu} | u_i^{\gamma-\zeta} \omega_\zeta) (v_l^\nu | u_j^\zeta) = c_{k, l}^\nu.$$

The proof of (ii) is similar.  $\square$

Now let  $F^{\text{op}} = \sum_{\gamma \in Q^+} \sum_{i=1}^{\bar{d}_\gamma} u_i^\gamma \otimes v_i^\gamma$ ,  $F^{12} = \sum_{\gamma \in Q^+} \sum_i v_i^\gamma \otimes u_i^\gamma \otimes 1$ , similarly  $F^{ij}$ , and  $F_f^{ij} = F^{ij} \circ \tilde{f}_{ij}$ , where  $\tilde{f}_{ij}$  denotes  $\tilde{f}$  applied to the  $i, j$  tensor slots. Recall the notation  $R^{ij}$  for  $R$  introduced in Section 2, where the notation  $R_{21}$  was reserved for  $\tau(R)$ .

**Lemma 4.11.** *The following holds as an identity of operators on a tensor product of three modules from category  $\mathcal{N}$ :  $(\Delta \otimes \text{id})(F^{\text{op}}) \circ \tilde{f}_{31} \circ \tilde{f}_{32} = F_f^{31} \circ F_f^{32}$ .*

*Proof.* Let  $M, M', M'' \in \mathcal{N}$  and assume  $M = M(\chi)$ ,  $M' = M'(\psi)$ , and  $M'' = M''(\phi)$ . Let  $m \in M_{\chi \cdot \hat{\lambda}}$ ,  $m' \in M_{\psi \cdot \hat{\mu}}$ ,  $m'' \in M''_{\phi \cdot \hat{\nu}}$ . The left side of (i) applied to  $m \otimes m' \otimes m''$  is

$$\begin{aligned} & (\Delta \otimes \text{id})(F^{\text{op}}) \circ \tilde{f}_{31} \circ \tilde{f}_{32} \\ &= f_{\phi, \psi}(\nu, \mu) f_{\phi, \chi}(\nu, \lambda) (\Delta \otimes \text{id}) \left( \sum_{\gamma, k} u_k^\gamma \otimes v_k^\gamma \right) \\ &= f_{\phi, \psi}(\nu, \mu) f_{\phi, \chi}(\nu, \lambda) \sum_{\gamma, k} \sum_{\zeta, i, j} (v_i^{\gamma-\zeta} v_j^\zeta | u_k^\gamma) u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \otimes v_k^\gamma \\ &= f_{\phi, \psi}(\nu, \mu) f_{\phi, \chi}(\nu, \lambda) \sum_{\gamma, \zeta, i, j} u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \otimes \left( \sum_k (v_i^{\gamma-\zeta} v_j^\zeta | u_k^\gamma) v_k^\gamma \right) \\ &= f_{\phi, \psi}(\nu, \mu) f_{\phi, \chi}(\nu, \lambda) \sum_{\gamma, \zeta, i, j} u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \otimes v_i^{\gamma-\zeta} v_j^\zeta. \end{aligned}$$

On the other hand, by (4.3),

$$\begin{aligned} & F_f^{31} \circ F_f^{32}(m \otimes m' \otimes m'') \\ &= f_{\phi, \psi}(\nu, \mu) \sum_{\eta, \zeta, i, j} f_{\phi, \chi}(\nu - \zeta, \lambda) u_i^\eta m \otimes u_j^\zeta m' \otimes v_i^\eta v_j^\zeta m'' \\ &= f_{\phi, \psi}(\nu, \mu) f_{\phi, \chi}(\nu, \lambda) \sum_{\eta, \zeta, i, j} \chi(\omega_\zeta) \hat{\lambda}(\omega_\zeta) u_i^\eta m \otimes u_j^\zeta m' \otimes v_i^\eta v_j^\zeta m''. \end{aligned}$$

Re-summing over  $\gamma = \zeta + \eta$ , and replacing  $\chi(\omega_\zeta) \hat{\lambda}(\omega_\zeta) m$  by  $\omega_\zeta m$ , we see that (i) holds.  $\square$

The next result is the quantum Yang-Baxter equation for  $R = F \circ \tilde{f} \circ \tau$ .

**Theorem 4.12.** *Let  $M, M', M'' \in \mathcal{N}$ . Then  $R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}$  as maps from  $M \otimes M' \otimes M''$  to  $M'' \otimes M' \otimes M$ .*

*Proof.* If  $\sigma$  is a permutation of  $\{1, 2, 3\}$ , let  $\tau_\sigma$  denote the corresponding permutation of three tensor factors, that is  $\tau_\sigma(m_1 \otimes m_2 \otimes m_3) = m_{\sigma^{-1}(1)} \otimes m_{\sigma^{-1}(2)} \otimes m_{\sigma^{-1}(3)}$ . In particular, if  $\sigma$  equals the transposition  $(ij)$ , we write simply  $\tau_{ij}$ , as  $\tau_{ij}$  is just the same as  $\tau$  applied to tensor slots  $i$  and  $j$  in that case.

Note that  $\tau_\sigma \circ F_f^{ij} = F_f^{\sigma(i)\sigma(j)} \circ \tau_\sigma$  for all  $\sigma$ , and that  $\tilde{f}_{31} \circ \tilde{f}_{32} \circ F^{12} = F^{12} \circ \tilde{f}_{31} \circ \tilde{f}_{32}$  by two applications of (4.3). We apply these identities, Theorem 4.8, and Lemma 4.11 to obtain the following:

$$\begin{aligned}
R^{12} R^{23} R^{12} &= \tau_{12} \tau_{23} F_f^{31} F_f^{32} R^{12} \\
&= \tau_{12} \tau_{23} (\Delta \otimes \text{id})(F^{\text{op}}) \tilde{f}_{31} \tilde{f}_{32} F^{12} \tilde{f}_{12} \tau_{12} \\
&= \tau_{12} \tau_{23} (\Delta \otimes \text{id})(F^{\text{op}}) F^{12} \tilde{f}_{31} \tilde{f}_{32} \tilde{f}_{12} \tau_{12} \\
&= \tau_{12} \tau_{23} (\Delta \otimes \text{id})(F^{\text{op}}) F^{12} \tilde{f}_{12} \tau_{12} \tilde{f}_{32} \tilde{f}_{31} \\
&= \tau_{12} \tau_{23} (\Delta \otimes \text{id})(F^{\text{op}}) R^{12} \tilde{f}_{32} \tilde{f}_{31} \\
&= \tau_{12} \tau_{23} R^{12} (\Delta \otimes \text{id})(F^{\text{op}}) \tilde{f}_{32} \tilde{f}_{31} \\
&= \tau_{12} \tau_{23} F_f^{12} \tau_{12} F_f^{31} F_f^{32} \\
&= F_f^{23} \tau_{12} \tau_{23} \tau_{12} F_f^{31} F_f^{32} \\
&= F_f^{23} \tau_{23} \tau_{12} \tau_{23} F_f^{31} F_f^{32} \\
&= R^{23} R^{12} R^{23}.
\end{aligned}$$

□

We will need one more lemma in order to obtain the hexagon identities, from which it will follow that  $F$  is a twisting element for  $\mathcal{N}$ .

**Lemma 4.13.** *The following are identities of operators on a tensor product of three modules from  $\mathcal{N}$ :*

- (i)  $(\Delta \otimes \text{id})(F_\gamma) = \sum_{0 \leq \zeta \leq \gamma} (F_{\gamma-\zeta})^{23} (F_\zeta)^{13} (1 \otimes \omega'_\zeta \otimes 1)$ .
- (ii)  $(\text{id} \otimes \Delta)(F_\gamma) = \sum_{0 \leq \zeta \leq \gamma} (F_{\gamma-\zeta})^{12} (F_\zeta)^{13} (1 \otimes \omega_\zeta \otimes 1)$ .
- (iii)  $\tilde{f}_{12} \circ (F_\zeta)^{13} = (F_\zeta)^{13} \circ (1 \otimes \omega_\zeta \otimes 1) \circ \tilde{f}_{12}$ .
- (iv)  $\tilde{f}_{23} \circ (F_\zeta)^{13} = (F_\zeta)^{13} \circ (1 \otimes \omega'_\zeta \otimes 1) \circ \tilde{f}_{23}$ .

*Proof.* We will check (ii) and (iv); the proofs of (i) and (iii) are similar. By Lemma 4.10 and (4.6), considering operators on modules we have

$$\begin{aligned}
 (\text{id} \otimes \Delta)(F_\gamma) &= \sum_k \sum_{\zeta, i, j} v_k^\gamma \otimes (v_i^{\gamma-\zeta} v_j^\zeta | u_k^\gamma) u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \\
 &= \sum_{\zeta, i, j} \left( \sum_k (v_i^{\gamma-\zeta} v_j^\zeta | u_k^\gamma) v_k^\gamma \right) \otimes u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \\
 &= \sum_{\zeta, i, j} v_i^{\gamma-\zeta} v_j^\zeta \otimes u_i^{\gamma-\zeta} \omega_\zeta \otimes u_j^\zeta \\
 &= \sum_\zeta (F_{\gamma-\zeta})^{12} (F_\zeta)^{13} (1 \otimes \omega_\zeta \otimes 1),
 \end{aligned}$$

which proves (ii).

Let  $M, M', M'' \in \mathcal{N}$  and  $m \in M_{\chi, \hat{\lambda}}$ ,  $m' \in M'_{\psi, \hat{\mu}}$ ,  $m'' \in M''_{\phi, \hat{\nu}}$ . By (4.3),

$$\begin{aligned}
 \tilde{f}_{23}(F_\zeta)^{13}(m \otimes m' \otimes m'') &= f_{\psi, \phi}(\mu, \nu + \zeta) \sum_i v_i^\zeta m \otimes m' \otimes u_i^\zeta m'' \\
 &= f_{\psi, \phi}(\mu, \nu) \psi(\omega'_\zeta)(\omega'_\zeta | \omega_\mu^{-1}) \sum_i v_i^\zeta m \otimes m' \otimes u_i^\zeta m'' \\
 &= f_{\psi, \phi}(\mu, \nu) \sum_i v_i^\zeta m \otimes \omega'_\zeta m' \otimes u_i^\zeta m'' \\
 &= (F_\zeta)^{13} (1 \otimes \omega'_\zeta \otimes 1) \tilde{f}_{23}(m \otimes m' \otimes m''),
 \end{aligned}$$

which proves (iv). □

Next we will prove the hexagon identities.

**Theorem 4.14.** *Let  $M, M', M'' \in \mathcal{N}$ . Then the following are identities of maps from  $M \otimes M' \otimes M''$  to  $M'' \otimes M \otimes M'$  (respectively,  $M' \otimes M'' \otimes M$ ):*

- (i)  $R^{12} \circ R^{23} = (\text{id} \otimes \Delta)(F) \circ \tilde{f}_{12} \circ \tilde{f}_{13} \circ \tau_{12} \circ \tau_{23}$ .
- (ii)  $R^{23} \circ R^{12} = (\Delta \otimes \text{id})(F) \circ \tilde{f}_{23} \circ \tilde{f}_{13} \circ \tau_{23} \circ \tau_{12}$ .

*Proof.* We will prove (ii). The proof of (i) is similar. Let  $m \otimes m' \otimes m'' \in M_{\chi, \tilde{\lambda}} \otimes M_{\psi, \tilde{\mu}} \otimes M_{\phi, \tilde{\nu}}$ . By Lemma 4.13, we have

$$\begin{aligned}
R^{23}R^{12}(m \otimes m' \otimes m'') &= F^{23}\tilde{f}_{23}\tau_{23}F^{12}\tilde{f}_{12}\tau_{12}(m \otimes m' \otimes m'') \\
&= F^{23}\tilde{f}_{23}F^{13}\tilde{f}_{13}(m' \otimes m'' \otimes m) \\
&= \sum_{\zeta} F^{23}(F_{\zeta})^{13}(1 \otimes \omega'_{\zeta} \otimes 1)\tilde{f}_{23}\tilde{f}_{13}(m' \otimes m'' \otimes m) \\
&= \sum_{\gamma} \sum_{0 \leq \zeta \leq \gamma} (F_{\gamma-\zeta})^{23}(F_{\zeta})^{13}(1 \otimes \omega'_{\zeta} \otimes 1)\tilde{f}_{23}\tilde{f}_{13}(m' \otimes m'' \otimes m) \\
&= \sum_{\gamma} (\Delta \otimes \text{id})(F_{\gamma})\tilde{f}_{23}\tilde{f}_{13}(m' \otimes m'' \otimes m) \\
&= (\Delta \otimes \text{id})(F)\tilde{f}_{23}\tilde{f}_{13}\tau_{23}\tau_{12}(m \otimes m' \otimes m'').
\end{aligned}$$

□

Finally we show that  $F$  is a twisting element for  $\mathcal{N}$ .

**Theorem 4.15.** *Let  $M, M', M''$  be modules in category  $\mathcal{N}$ . Then for  $F$  as defined in (4.4) we have*

$$[(\Delta \otimes \text{id})(F)](F \otimes \text{id}) = [(\text{id} \otimes \Delta)(F)](\text{id} \otimes F)$$

as operators on  $M \otimes M' \otimes M''$ . Thus  $F$  is a twisting element for any subcategory of  $\mathcal{N}$  consisting of  $U$ -module algebras.

*Proof.* In Theorem 4.14, multiply (i) by  $R^{12}$  on the right, multiply (ii) by  $R^{23}$  on the right, and apply Theorem 4.12 to obtain the identity

$$[(\Delta \otimes \text{id})(F)] \circ \tilde{f}_{23} \circ \tilde{f}_{13} \circ \tau_{23} \circ \tau_{12} \circ R^{23} = [(\text{id} \otimes \Delta)(F)] \circ \tilde{f}_{12} \circ \tilde{f}_{13} \circ \tau_{12} \circ \tau_{23} \circ R^{12}$$

as functions from  $M \otimes M' \otimes M''$  to  $M'' \otimes M' \otimes M$ , for any  $M, M', M'' \in \mathcal{N}$ . It may be checked that this is equivalent to the identity stated in the theorem, using (4.3). Thus (2.4) holds, and (2.3) is immediate from the definition of  $F$ . □

We remark that all of the above arguments apply equally well to  $U = U_{r,s}(\mathfrak{sl}_n)$ , if  $rs^{-1}$  is *not* a root of unity and our  $U$ -module algebras are (possibly infinite) direct sums of modules arising from the category  $\mathcal{O}$  modules of [BW1, Sec. 4] by shifting weights by some functions  $\chi : U_{\mathfrak{R}}^0 \rightarrow \mathfrak{K}$ . The function  $\tilde{f}$  there merely needs to be replaced by our more general function  $\tilde{f}$  here. The element  $\Theta$  from that paper provides another example of a twisting element similar to the well-known examples arising from  $R$ -matrices of one-parameter quantum groups.

We also note that we may derive Theorem 3.15 as a special case of Theorem 4.15: When  $\mathfrak{K} = \mathbb{K}$ , the category of modules for  $\mathfrak{u} = \mathfrak{u}_{r,s}(\mathfrak{sl}_n)$  is equivalent to a subcategory of  $\mathcal{N}$ . The image of  $F$  in  $\mathfrak{u} \otimes \mathfrak{u}$  is precisely  $R_{f,e}$  by their definitions.



However, the construction of  $R_{f,e}$  in Section 3 shows that it is invertible, while we do not know if  $F$  is invertible in general.

### 5. DEFORMATION FORMULAS FROM INFINITE QUANTUM GROUPS

Let  $U = U_{r,s}(\mathfrak{sl}_n)$ , and assume throughout this section that  $r = \theta^y$  and  $s = \theta^z$  are both roots of unity with  $y, z$  satisfying condition (3.10). Let  $\mathfrak{K} = \mathbb{K}[[t, t^{-1}]$ , Laurent polynomials in  $t$  with finitely many negative powers of  $t$ . Let  $X$  be any finite-dimensional  $U$ -module such that the extension  $X_{\mathfrak{K}} = X[[t, t^{-1}]$  is a  $U_{\mathfrak{K}}$ -module in category  $\mathcal{N}$ . Let  $T(X_{\mathfrak{K}}) = T_{\mathfrak{K}}(X_{\mathfrak{K}})$  be the tensor algebra of  $X_{\mathfrak{K}}$  over  $\mathfrak{K}$ , considered as a  $U_{\mathfrak{K}}$ -module algebra. As  $\mathcal{N}$  is closed under tensor products and infinite direct sums,  $T(X_{\mathfrak{K}})$  is also in category  $\mathcal{N}$ . Therefore Theorem 4.15 may be applied to twist the multiplication of  $T(X_{\mathfrak{K}})$ .

We wish instead to go further and produce a deformation formula that may be applied to quotients of  $T(X_{\mathfrak{K}})$  to obtain formal deformations. We define a new action of  $U$  on  $T(X_{\mathfrak{K}})$  denoted  $*$ , that is  $\mathfrak{K}$ -linear, by specifying

$$\begin{aligned} e_i * x &= (e_i \cdot x)t \\ f_i * x &= f_i \cdot x \\ \omega_i * x &= (\omega_i \cdot x)t, & \omega_i^{-1} * x &= (\omega_i^{-1} \cdot x)t^{-1} \\ \omega'_i * x &= (\omega'_i \cdot x)t, & (\omega'_i)^{-1} * x &= ((\omega'_i)^{-1} \cdot x)t^{-1} \end{aligned}$$

for all  $x \in X$ , and extending to  $T(X_{\mathfrak{K}})$  via the  $U_{\mathfrak{K}}$ -module algebra conditions (2.1), (2.2). (We are just forcing the generators of  $U_{\mathfrak{K}}$  to act as scalar multiples of their original actions, the scalars being powers of  $t$ .) The powers of  $t$  arising in the new action  $*$  are chosen so that the defining relations of  $U$  are preserved. One way to obtain this module structure is to let  $U$  act on the completion  $\mathbb{K}[[t, t^{-1}]] \widehat{\otimes}_{\mathbb{K}} X$  of the tensor product in which infinite sums are allowed. Here  $\mathbb{K}[[t, t^{-1}]$  is regarded as a weight module for  $U$  with the single weight  $\chi(\omega_i) = \chi(\omega'_i) = t$ , so that  $e_i$  and  $f_i$  annihilate  $\mathbb{K}[[t, t^{-1}]]$  for each  $i$ , and the  $U$ -action on  $\mathbb{K}[[t, t^{-1}]] \widehat{\otimes}_{\mathbb{K}} X$  is via the coproduct. Observe that this new  $U$ -module structure on  $T(X_{\mathfrak{K}})$  yields a module in category  $\mathcal{N}$ , as the only change is in the weights  $\chi$ .

**Theorem 5.1.** *The operator  $F$  defined in (4.4) is a universal deformation formula for any subcategory of  $\mathcal{N}$  (over  $\mathfrak{K} = \mathbb{K}[[t, t^{-1}]$ ) consisting of  $U_{\mathfrak{K}}$ -module algebras for which (2.13) holds.*

*Proof.* This follows immediately from Theorem 4.15 and the definition of a universal deformation formula for a category.  $\square$

Of course, the  $U_{\mathfrak{K}}$ -module algebras  $T(X_{\mathfrak{K}})$  constructed above satisfy the hypotheses of the theorem, so by Theorem 2.15,  $\mu_{T(X_{\mathfrak{K}})} \circ F$  defines a formal deformation of  $T(X)$ .

If  $X$  is *any* finite-dimensional vector space, the Hochschild cohomology groups  $\mathrm{HH}^i(T(X))$  are 0 for  $i \geq 2$  (e.g. see [W, Prop. 9.1.6]). Thus all formal deformations of  $T(X)$  are equivalent to the trivial one, and so we are interested in applying Theorem 5.1 to proper quotients. For example, we may take the truncated tensor algebra obtained by letting  $p$  be a fixed positive integer and taking the quotient by the ideal  $W_p := \bigoplus_{k \geq p} X^{\otimes k}$ . The second Hochschild cohomology group of a truncated tensor algebra is nontrivial. (See for example [C, Lem. 4.1], where we need to identify our  $T(X)/W_p$  with Cibils'  $\mathbb{K}\mathcal{Q}/\mathcal{F}^p$ ,  $\mathcal{Q}$  the quiver with one vertex and  $\dim(X)$  loops, and  $\mathcal{F}$  the ideal of the path algebra  $\mathbb{K}\mathcal{Q}$  generated by the loops.)

**Example 5.2.** Next we consider some examples that are group crossed products. Let  $V$  be the natural  $n$ -dimensional module for  $U = U_{r,s}(\mathfrak{sl}_n)$ , defined in Section 3. We will specify an action of  $U$  on a crossed product  $T(V)\#\mathfrak{A}$ , where  $\mathfrak{A}$  is the abelian group  $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$  on generators  $a_1, \dots, a_{n-1}$ , written multiplicatively. Choose arbitrary  $\ell$ th roots of unity  $\beta_i$  ( $1 \leq i \leq n-1$ ), and let an action of  $\mathfrak{A}$  on  $T(V)$  as automorphisms be defined by

$$a_i.v_j = \beta_i v_j$$

for  $i = 1, \dots, n-1$  and  $j = 1, \dots, n$ . The *smash product*  $T(V)\#\mathfrak{A}$  is the vector space  $T(V) \otimes_{\mathbb{K}} \mathfrak{A}$  with the multiplication

$$(x \otimes a)(y \otimes b) = x(a.y) \otimes ab,$$

for  $x, y \in T(V)$  and  $a, b \in \mathfrak{A}$ . Define the following action of  $U$  on the smash product  $(T(V)\#\mathfrak{A})[[t, t^{-1}]$ :

$$\begin{aligned} e_i.v_j &= \delta_{i,j-1} v_i a_i t, \\ f_i.v_j &= \delta_{i,j} v_{i+1}, \\ \omega_i.v_j &= r^{\delta_{i,j}} s^{\delta_{i,j-1}} v_j a_i t, & \omega_i^{-1}.v_j &= r^{-\delta_{i,j}} s^{-\delta_{i,j-1}} v_j a_i^{-1} t^{-1}, \\ \omega'_i.v_j &= r^{\delta_{i,j-1}} s^{\delta_{i,j}} v_j a_i t, & (\omega'_i)^{-1}.v_j &= r^{-\delta_{i,j-1}} s^{-\delta_{i,j}} v_j a_i^{-1} t^{-1}, \end{aligned}$$

where  $U$  acts trivially on  $\mathfrak{A}$ . (It is possible to modify this example to involve nontrivial actions of the  $\omega_i, \omega'_i$  on  $\mathfrak{A}$ .) It may be checked that  $(T(V)\#\mathfrak{A})[[t, t^{-1}]$  is a  $U$ -module algebra. Note that  $T(V)\#\mathfrak{A}$  is in category  $\mathcal{N}$ , which may be seen by decomposing  $\mathbb{K}\mathfrak{A}$  as an algebra into a direct sum of copies of  $\mathbb{K}$  (as  $\mathfrak{A}$  is a finite abelian group), and partitioning the elements of each  $V^{\otimes i}\#\mathfrak{A}$  accordingly, (or simply by letting  $\mathfrak{K} = \mathbb{K}[[t, t^{-1}]]\mathfrak{A}$  be the ring of coefficients). If elements of  $\mathfrak{A}$  are assigned degree 0, it may be verified that  $F$  satisfies equation (2.13), and so yields a formal deformation of  $T(V)\#\mathfrak{A}$ . This example is similar in some respects to that given in [CGW], although in that example it was possible to take a further quotient of  $T(V)$ , namely a polynomial algebra, because of the special nature of the parameters.

Just as for  $T(V)$ , formal deformations of  $T(V)\#\mathfrak{A}$  are necessarily infinitesimally trivial, as the second Hochschild cohomology group is again trivial. However, we may again truncate the tensor algebra and consider the resulting crossed product with  $\mathfrak{A}$  as a  $U$ -module algebra and corresponding deformations.

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