HOCHSCHILD COHOMOLOGY OF SMASH PRODUCTS AND RANK ONE HOPF ALGEBRAS

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Abstract. We give some general results on the ring structure of Hochschild cohomology of smash products of algebras with Hopf algebras. We compute this ring structure explicitly for a large class of finite dimensional Hopf algebras of rank one.

1. Introduction

Cibils and Solotar [6] gave the ring structure of the Hochschild cohomology of a group algebra kG of a finite abelian group G over a commutative ring k, and Cibils [5] conjectured a formula for this ring structure for a general finite group G. Siegel and the second author [11] proved the conjecture. This Hochschild cohomology is a direct sum of graded vector spaces indexed by the conjugacy classes, and cup products may be described in terms of this decomposition via known maps and products in group cohomology. This vector space decomposition generalizes to a result for the Hochschild cohomology of a Hopf algebra H, where the direct sum is indexed by summands of the adjoint representation of H on itself [7, Prop. 5.6], but in general there is no known formula for the cup product in terms of this decomposition. In the special case that H is commutative, Linckelmann [10] generalized the original result of Cibils and Solotar.

There is a generalization in another direction, to a crossed product of an algebra A with a group algebra kG; again Hochschild cohomology is a direct sum of graded vector spaces indexed by conjugacy classes, and there is a formula for the cup product in terms of this decomposition [15, Thm. 3.16]. How much of this theory generalizes to smash (or crossed) products with Hopf algebras, or to Hopf Galois extensions? In this note, we begin a program to answer this question by (1) computing the ring structure of Hochschild cohomology for a large class of examples, namely some finite dimensional Hopf algebras of rank one defined by Krop and Radford [9], by (2) giving a vector space decomposition of the Hochschild cohomology of a smash product $A\#H$, simultaneously generalizing the known cases $A = k$ and $H = kG$, and by (3) giving some consequences of this decomposition in special cases.

A rank one Hopf algebra is a generalization of a Taft algebra whose grouplike elements may form a nonabelian group. As an algebra, it is a smash product $B = A\#kG$, with $A = k[x]/(x^n)$ and G a finite group acting on A. We compute the graded vector

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space structure of its Hochschild cohomology $\mathrm{HH}^*(B) := \mathrm{Ext}^*_{B\otimes B^{op}}(B,B)$ (where B^{op} is the algebra opposite to B) in Section 2 (see Theorem 2.4). In Section 3 we use explicit chain maps first defined by the Buenos Aires Cyclic Homology Group [3] to compute the ring structure of $\mathrm{HH}^*(B)$, showing that the ring is generated by the G-invariant subring $\mathrm{HH}^*(A)^G$ of $\mathrm{HH}^*(A)$ and by $\mathrm{HH}^0(B) \cong Z(B)$ (see Theorem 3.4).

We give our general result in Section 4 for a smash product $B = A\#H$, where H is a Hopf algebra with bijective antipode and A is an H-module algebra. We introduce a subalgebra $\mathcal{D}$ of $B \otimes B^{op}$ in (4.1) for which $\mathrm{HH}^*(B,M) := \mathrm{Ext}^*_{B\otimes B^{op}}(B,M) \cong \mathrm{Ext}^*_{\mathcal{D}}(A,M)$ for any B-bimodule M (see Theorem 4.3). If $A = k$ and $M = H$, this yields the decomposition of $\mathrm{HH}^*(H)$ in terms of the adjoint representation of H on itself. If $H = kG$ and $M = B$, this yields the decomposition indexed by conjugacy classes. In general if $M = B$, it will give a decomposition in terms of $\mathcal{D}$-submodules of B, and we translate the cup product on $\mathrm{HH}^*(B)$ to one on $\mathrm{Ext}^*_{\mathcal{D}}(A,B)$ described explicitly at the cochain level in (4.6). This is potentially a first step towards understanding the cup product more directly in terms of summands arising from the adjoint representation. It seems difficult to generalize the next step from the special case $H = kG$, as in this case, certain $\mathcal{D}$-submodules of B are coinduced from centralizer subgroups (see the proof of [15, Lem. 3.5]). We also do not know if there is a more general version of our Theorem 4.3 that applies to crossed products or to Hopf Galois extensions.

In the remainder of this note, we give some consequences of Theorem 4.3. We first return to the rank one Hopf algebras of Sections 2 and 3 and take another look at the structure of their Hochschild cohomology, this time in relation to the adjoint representation. Next, in the special case that H is semisimple, we show that Theorem 4.3 directly implies $\mathrm{HH}^*(B) \cong \mathrm{HH}^*(A,B)^H$, where the superscript H denotes invariants (cf. [13, Thm. 3.3]), and we give some resulting formulas for explicit cocycles (see Theorem 4.11). Finally, when H is semisimple, another consequence of Theorem 4.3 is that the Hopf-Hochschild cohomology of A introduced by Kaygun [8] is isomorphic to the Hochschild cohomology of B when coefficients are taken in a B-bimodule (Theorem 5.2). This follows from the observation that Kaygun's "crossed product" $A^e \rtimes H$ is isomorphic to our subalgebra $\mathcal{D}$ of B^e.

We work over a field k. For the explicit computations we require the characteristic of k to be relatively prime to the order of G, however for the general results k could equally well be a commutative ring provided all algebras are projective as k-modules. Let $\otimes = \otimes_k$. We use modified Sweedler notation for the coproduct $\Delta : H \to H \otimes H$ of a Hopf algebra H, symbolically writing $\Delta(h) = h_1 \otimes h_2$ $(h \in H)$.

2. Hochschild cohomology of rank one Hopf algebras

Let G be a finite group whose order is relatively prime to the characteristic of k. Let $\chi : G \to k^\times$ be a character, that is a group homomorphism from G to the multiplicative group of k. Let $n \geq 2$ be a positive integer and $A = k[x]/(x^n)$. Then

G acts by automorphisms on A via

$$^{g}x = \chi(g)x$$

for all $g \in G$. Let $B = A\#kG$, the corresponding *skew group algebra* (or *smash product* of A and kG): As a vector space, $B = A \otimes kG$, and the multiplication is

$$(a \otimes g)(b \otimes h) = a(^{g}b) \otimes gh$$

for all $a, b \in A$ and $g, h \in G$. We abbreviate $a \otimes g$ by ag.

Assume there is a central element $g_1 \in G$ such that $\chi(g_1)$ is a primitive nth root of 1. Then B is a Hopf algebra with coproduct Δ defined by

$$\Delta(x) = x \otimes 1 + g_1 \otimes x \quad \text{and} \quad \Delta(g) = g \otimes g, \tag{2.1}$$

counit ε by $\varepsilon(x) = 0$ and $\varepsilon(g) = 1$, and antipode S by $S(x) = -g_1^{-1}x$ and $S(g) = g^{-1}$, for all $g \in G$. This generalization of a Taft algebra is defined in [1] for abelian groups G, and generalized further in [9] (but with the opposite coproduct). Note that we do not use the coalgebra structure of B until Section 4.

In order to compute the Hochschild cohomology of B, we use the following subalgebra of $B^e = B \otimes B^{op}$:

$$\mathcal{D} := A^e\#kG \cong \bigoplus_{g\in G}(Ag \otimes Ag^{-1}) \subset B^e, \tag{2.2}$$

where the action of G on A^e is diagonal, that is $^{g}(a \otimes b) = {}^{g}a \otimes {}^{g}b$. The indicated isomorphism is given by $(a \otimes b)g \mapsto ag \otimes (^{g^{-1}}b)g^{-1}$ for all $a, b \in A$ and $g \in G$. Note that A is a $\mathcal{D}$-module under left and right multiplication. The algebra $\mathcal{D}$ is sometimes denoted Δ in the literature on group-graded algebras.

It is known that the Hochschild cohomology $\mathrm{HH}^*(B) := \mathrm{Ext}^*_{B^e}(B, B)$ satisfies

$$\mathrm{HH}^*(B) \cong \mathrm{Ext}^*_{\mathcal{D}}(A, B) \tag{2.3}$$

as graded algebras. This is a consequence of the Eckmann-Shapiro Lemma and the isomorphism of B^e-modules, $B \xrightarrow{\sim} A\uparrow_{\mathcal{D}}^{B^e} := B^e \otimes_{\mathcal{D}} A$ given by $b \mapsto (b \otimes 1) \otimes 1$ with inverse $(b \otimes c) \otimes a \mapsto bac$. (See for example [15, Lemma 3.5], valid more generally for some crossed products.) Alternatively (2.3) follows from our generalization to smash products with Hopf algebras, Theorem 4.3 below. As the characteristic of k is relatively prime to $|G|$, there is a further isomorphism $\mathrm{Ext}^*_{\mathcal{D}}(A, B) \cong \mathrm{Ext}^*_{A^e}(A, B)^G = \mathrm{HH}^*(A, B)^G$ where the latter consists of invariants under the action induced from that of $G \subset \mathcal{D}$ on $\mathcal{D}$-modules. (The resulting isomorphism $\mathrm{HH}^*(B) \cong \mathrm{Ext}^*_{A^e}(A, B)^G$ also follows from [13, Cor. 3.4] or from (4.9) below.) Again as the characteristic of k is relatively prime to $|G|$, G-invariants may be taken in a complex prior to taking cohomology. This will be our approach in proving the following theorem for a rank one Hopf algebra $B = A\#kG$.

Theorem 2.4. *Let $N = \ker \chi \subset G$. For all $i \geq 0$, the dimensions of $\mathrm{HH}^{2i}(B)$ and of $\mathrm{HH}^{2i+1}(B)$ are the same, and equal the number of representatives g of conjugacy classes in G such that $g \in N$ and $\chi^{in}|_{C(g)} = 1$.*

Proof. The following is an A^e-free resolution of A [14, Exer. 9.1.4]:

$$(2.5) \qquad \cdots \xrightarrow{\cdot v} A^e \xrightarrow{\cdot u} A^e \xrightarrow{\cdot v} A^e \xrightarrow{\cdot u} A^e \xrightarrow{\mu} A \to 0,$$

where $u = x \otimes 1 - 1 \otimes x$, $v = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \cdots + 1 \otimes x^{n-1}$, and μ is multiplication. This becomes a $\mathcal{D}$-projective resolution of A as follows. The action of $G \subset \mathcal{D}$ in degree 0 is diagonal on A^e, $g \cdot (a \otimes b) = {}^g a \otimes {}^g b$ for all $a, b \in A$ and $g \in G$. In all other degrees the action must be modified in order that the maps $\cdot u$ and $\cdot v$ be maps of $\mathcal{D}$-modules. In degree $2i$, $g \cdot (a \otimes b) = \chi(g)^{in}({}^g a) \otimes {}^g b$, and in degree $2i+1$, $g \cdot (a \otimes b) = \chi(g)^{in+1}({}^g a) \otimes {}^g b$, for all $a, b \in A$ and $g \in G$. With these actions, (2.5) is indeed a $\mathcal{D}$-projective resolution of A: Since the characteristic of k does not divide the order of G, a $\mathcal{D}$-module is projective if and only if its restriction to A^e is projective. (An A^e-splitting map of $\mathcal{D}$-modules may be "averaged" by applying $\frac{1}{|G|}\sum_{g \in G} g$ to obtain a $\mathcal{D}$-splitting map.)

According to the isomorphism (2.3), we must now apply $\mathrm{Hom}_{\mathcal{D}}(-, B)$ to (2.5). Now $\mathrm{Hom}_{\mathcal{D}}(A^e, B) \cong \mathrm{Hom}_{A^e}(A^e, B)^G$ where G acts on $\mathrm{Hom}_{A^e}(A^e, B)$ by $(g \cdot f)(a \otimes b) = gf(g^{-1} \cdot (a \otimes b))g^{-1}$ for all $f \in \mathrm{Hom}_{A^e}(A^e, B), g \in G$. Such a homomorphism is determined by its value on $1 \otimes 1$. We identify B with $\mathrm{Hom}_k(k, B) \cong \mathrm{Hom}_{A^e}(A^e, B)$, under the correspondence $b \mapsto f_b$ where $f_b(1 \otimes 1) = b$. Thus applying $\mathrm{Hom}_{\mathcal{D}}(-, B)$ to (2.5) yields the complex

$$(2.6) \qquad \cdots \xleftarrow{(\cdot v)^*} B^G \xleftarrow{(\cdot u)^*} B^G \xleftarrow{(\cdot v)^*} B^G \xleftarrow{(\cdot u)^*} B^G \leftarrow 0,$$

the action of G on B depending on the degree as stated above. In degree $2i$, this action is

$$(g \cdot f_b)(1 \otimes 1) = gf_b(g^{-1} \cdot (1 \otimes 1))g^{-1} = gf_b(\chi(g)^{-in} \otimes 1)g^{-1} = \chi(g)^{-in} gbg^{-1},$$

so that $g \cdot b = \chi(g)^{-in} gbg^{-1}$ for all $g \in G$ and $b \in B$. Similarly, in degree $2i+1$, $g \cdot b = \chi(g)^{-in-1} gbg^{-1}$. Thus in degree $2i$, as $\chi(g_1)$ is a primitive nth root of 1, if x^j is invariant then $x^j = g_1 \cdot x^j = \chi(g_1)^{-in+j} x^j = \chi(g_1)^j x^j$, implying $j = 0$. It follows that $B^G \subset kG$. Applying the formula for the action in degree $2i$, we find $B^G = Z(kG)$ (the center of the group algebra kG) in case $\chi^{in} = 1$. If $\chi^{in} \neq 1$, let g be a representative of a conjugacy class of G for which $\chi^{in}|_{C(g)} = 1$. Then B^G is spanned by all $\sum_{h \in G} \chi(h)^{-in} hgh^{-1}$ for such elements g. The coefficients $\chi(h)^{-in}$ are determined only by the conjugates hgh^{-1} of g. Similarly, in degree $2i+1$, the invariants are spanned by elements of the form $\sum_{h \in G} xhgh^{-1}$ in case $\chi^{in} = 1$, and otherwise by $\sum_{h \in G} \chi(h)^{-in} xhgh^{-1}$ for those representatives g of conjugacy classes for which $\chi^{in}|_{C(g)} = 1$.

The maps $(\cdot u)^*$ and $(\cdot v)^*$ are:

$$\begin{aligned} (\cdot u)^*(b) &= (\cdot u)^* f_b(1 \otimes 1) = f_b(u) = xb - bx, \\ (\cdot v)^*(b) &= x^{n-1} b + x^{n-2} bx + \cdots + bx^{n-1}, \end{aligned}$$

for all $b \in B^G$. In particular $(\cdot v)^*$ is the 0-map: $(\cdot v)^*(\sum_{h \in G/C(g)} xhgh^{-1}) = 0$ since $x^n = 0$. Thus $\ker(\cdot v)^* = B^G$ and $\mathrm{im}(\cdot v)^* = 0$. In degree $2i$, if $B^G \neq 0$, then $\ker(\cdot u)^*$ is spanned by those sums $\sum_h \chi^{-in}(h) hgh^{-1}$ for which $g \in \ker \chi = N$. In degree $2i+1$, $\mathrm{im}(\cdot u)^*$ is spanned by those elements $\sum_{h\in G}\chi^{-in}(h)xhgh^{-1}$ for which $g\notin N$. Therefore the dimensions of $\mathrm{HH}^{2i}(B)=\ker(\cdot u)^*/\,\mathrm{im}(\cdot v)^*$ and of $\mathrm{HH}^{2i+1}(B)=\ker(\cdot v)^*/\,\mathrm{im}(\cdot u)^*$ are the same, and are as claimed in the theorem. □

3. The ring structure

We next compute the ring structure of $\mathrm{HH}^*(B)$, where $B=A\#kG$ is the rank one Hopf algebra defined in Section 2, under the hypothesis $\chi^n=1$. The computation of cup products in the general case is no more difficult, but the ring structure is harder to describe. We compare the resolution (2.5) with the bar resolution of A,

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{\mu} A \to 0, \tag{3.1}$$

where $\delta_i(a_0\otimes a_1\otimes\cdots\otimes a_{i+1})=\sum_{j=0}^{i}(-1)^j a_0\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_{i+1}$ for $a_0,\dots,a_{i+1}\in A$. The cup product on $\mathrm{HH}^*(B)$ is defined at the chain level, with respect to the bar resolution (3.1) with A replaced by B, as follows: Let $f\in \mathrm{Hom}_{B^e}(B^{\otimes(l+2)},B)\cong \mathrm{Hom}_k(B^{\otimes l},B)$ and $f'\in \mathrm{Hom}_{B^e}(B^{\otimes(m+2)},B)\cong \mathrm{Hom}_k(B^{\otimes m},B)$. Then

$$(f\smile f')(b_1\otimes\cdots\otimes b_{l+m})=f(b_1\otimes\cdots\otimes b_l)f'(b_{l+1}\otimes\cdots\otimes b_m) \tag{3.2}$$

for all $b_1,\dots,b_{l+m}\in B$. It is convenient to consider an l-cochain sometimes to be an element of $\mathrm{Hom}_{B^e}(B^{\otimes(l+2)},B)$ and other times to be an element of $\mathrm{Hom}_k(B^{\otimes l},B)$. This should cause no confusion.

Consider the cup product on $\mathrm{HH}^*(A,B)$ given by (3.2), where $B^{\otimes l}$, $B^{\otimes m}$ are replaced by $A^{\otimes l}$, $A^{\otimes m}$. The isomorphism $\mathrm{HH}^*(B)\cong \mathrm{HH}^*(A,B)^G$ described in Section 2 preserves cup products, that is $\mathrm{HH}^*(B)$ is isomorphic to the G-invariant subalgebra of $\mathrm{HH}^*(A,B)$. This is known, and also follows from our more general results at the end of Section 4 as kG is semisimple, but we outline a direct proof using the algebra $\mathcal{D}$ in this case. Note that the bar resolution for B (as B^e-module) is induced from the $\mathcal{D}$-projective resolution of A:

$$\cdots \xrightarrow{\delta_3} \mathcal{D}_2 \xrightarrow{\delta_2} \mathcal{D}_1 \xrightarrow{\delta_1} \mathcal{D}_0 \xrightarrow{\mu} A \to 0 \tag{3.3}$$

where $\mathcal{D}_0=\mathcal{D}$ and

$$\mathcal{D}_m=\mathrm{Span}_k\{a_0g_0\otimes\cdots\otimes a_{m+1}g_{m+1}\mid a_i\in A, g_i\in G, g_0\cdots g_{m+1}=1\}$$

is a $\mathcal{D}$-submodule of $B^{\otimes(m+2)}$. An isomorphism $\mathcal{D}_m\uparrow_{\mathcal{D}}^{B^e}:=B^e\otimes_{\mathcal{D}}\mathcal{D}_m\xrightarrow{\sim}B^{\otimes(m+2)}$ is given by

$$(b_{-1}\otimes b_{m+2})\otimes(b_0\otimes\cdots\otimes b_{m+1})\mapsto b_{-1}b_0\otimes b_1\otimes\cdots\otimes b_{m+1}b_{m+2},$$

and its inverse by

$$a_0g_0\otimes a_1g_1\otimes\cdots\otimes a_{m+1}g_{m+1}\mapsto(1\otimes g_0\cdots g_{m+1})\otimes(a_0g_0\otimes\cdots\otimes a_{m+1}g_m^{-1}\cdots g_0^{-1}).$$

The bar resolution (3.1) for A is compatible with the action of $\mathcal{D}=A^e\#kG$ given by the usual action of A^e and the diagonal action of G on tensor products $A^{\otimes m}$. Thus

(3.1) is in fact a $\mathcal{D}$-projective resolution of A. There is a $\mathcal{D}$-map from (3.3) to (3.1) given by

$$a_0g_0 \otimes \cdots \otimes a_{m+1}g_{m+1} \mapsto a_0 \otimes {}^{g_0}a_1 \otimes {}^{g_0g_1}a_2 \otimes \cdots \otimes {}^{g_0\cdots g_m}a_{m+1}$$

for all $a_i \in A$ and $g_i \in G$ (see [4, (5.2)]). Under this map and the identification $B^{\otimes(m+2)} \cong \mathcal{D}_m \uparrow_{\mathcal{D}}^{B^e}$, it can be seen that the cup product (3.2) on $\mathrm{HH}^*(B)$ indeed corresponds to that on $\mathrm{HH}^*(A,B)^G \subset \mathrm{HH}^*(A,B)$ induced by multiplication on B.

We first need a chain map ϕ_* from (2.5) to the bar complex (3.1) for A. This was found in a more general setting in [3]. We give the maps explicitly in our setting. Define $\phi_m : A^e \to A^{\otimes(m+2)}$ by

$$\phi_{2l}(1\otimes 1) = 1 \otimes \alpha_l \quad \text{and} \quad \phi_{2l+1}(1\otimes 1) = 1\otimes x \otimes \alpha_l$$

where $\alpha_0 = 1$ and if $l \geq 1$,

$$\alpha_l = \sum_{\{i_1+i_2+\cdots+i_{l+1}=ln-l \mid i_1,i_2,\cdots,i_l \geq 1\}} x^{i_1} \otimes x \otimes x^{i_2} \otimes x \otimes \cdots \otimes x \otimes x^{i_{l+1}}.$$

In this formula, we emphasize that i_{l+1} is allowed to be 0 whereas $i_1, i_2, \ldots, i_l$ must be greater than 0. Note that our maps differ from those in [3] by a sign due to our choice $u = x\otimes 1 - 1 \otimes x$.

We next define a chain map ψ_* from (3.1) to (2.5) as in [3]. Define $\psi_{2l} : A^{\otimes(2l+2)} \to A^e$ and $\psi_{2l+1} : A^{\otimes(2l+3)} \to A^e$ by

$$\begin{aligned}
\psi_{2l}(1\otimes x^{i_1}\otimes x^{i_2}\otimes\cdots\otimes x^{i_{2l}}\otimes 1) &= 1\otimes x^{i_1+i_2-n}x^{i_3+i_4-n}\cdots x^{i_{2l-1}+i_{2l}-n},\\
\psi_{2l+1}(1\otimes x^{i_1}\otimes x^{i_2}\otimes\cdots\otimes x^{i_{2l+1}}\otimes 1) &= \sum_{m=0}^{i_1-1} x^m\otimes x^{i_1-m-1}x^{i_2+i_3-n}x^{i_4+i_5-n}\cdots x^{i_{2l}+i_{2l-1}-n},
\end{aligned}$$

where x^j is defined to be 0 if $j < 0$. By [3, Prop. 1.5], ϕ_* and ψ_* are indeed chain maps. Further, both ϕ_* and ψ_* are compatible with the action of G and thus they are $\mathcal{D}$-maps.

Theorem 3.4. *Assume $\chi^n = 1$, and let $N = \ker\chi \subset G$. Then there is an isomorphism of graded algebras*

$$\mathrm{HH}^*(B) \cong (kN)^G \otimes k[y,z]/(z^2),$$

where $\deg y = 2$ and $\deg z = 1$. In particular $\mathrm{HH}^*(B)$ is generated by $\mathrm{HH}^*(A)^G \cong k[y,z]/(z^2)$ and $\mathrm{HH}^0(B) \cong (kN)^G$.

Proof. Computations will be done at the chain level using the complex (2.6). First let $a, b \in B^G$ be two elements of degrees $2l$ and $2m$, respectively, and let f_a and f_b be the corresponding functions from A^e to B^G. Applying the chain maps ϕ_*, ψ_* given above and the definition of cup product on the bar resolution, the cup product $f_a \smile f_b$ is defined by

$$\begin{aligned}
&(\psi_{2l}^* f_a \smile \psi_{2m}^* f_b)\phi_{2l+2m}(1\otimes 1)\\
&\qquad = (\psi_{2l}^* f_a \smile \psi_{2m}^* f_b)\left(\sum 1\otimes x^{i_1}\otimes x\otimes x^{i_2}\otimes x\otimes\cdots\otimes x\otimes x^{i_{l+m+1}}\right)
\end{aligned}$$

where the sum is over all indices i_s such that $i_1 \cdots + i_{l+m+1} = (l+m)(n-1)$ and $i_1, i_2, \cdots, i_{l+m} \geq 1$. Identifying f_a, f_b with a, b, we have

$$\begin{aligned} a \smile b &= \sum \psi_{2l}^* f_a(x^{i_1} \otimes x \otimes \cdots \otimes x^{i_l} \otimes x) \psi_{2m}^* f_b(x^{i_{l+1}} \otimes x \otimes \cdots \otimes x^{i_{l+m}} \otimes x) x^{i_{l+m+1}} \\ &= \sum f_a(1 \otimes x^{i_1+1-n} \cdots x^{i_l+1-n}) f_b(1 \otimes x^{i_{l+1}+1-n} \cdots x^{i_{l+m}+1-n}) x^{i_{l+m+1}} \ = \ ab \end{aligned}$$

since the sum has only one nonzero term, the one where $i_1 = i_2 = \cdots = i_{l+m} = n-1$ and $i_{l+m+1} = 0$. If $a, b \in B^G$ are elements of degrees $2l$ and $2m+1$, respectively, then a similar calculation shows that $a \smile b = ab$.

Finally, let $a, b \in B^G$ be elements of degrees $2l+1$ and $2m+1$, respectively. Then the cup product $f_a \smile f_b$ is given by

$$\begin{aligned} &(\psi_{2l+1}^* f_a \smile \psi_{2m+1}^* f_b)\phi_{2l+2m+2}(1 \otimes 1) \\ &\qquad = (\psi_{2l+1}^* f_a \smile \psi_{2m+1}^* f_b)(\sum 1 \otimes x^{i_1} \otimes x \otimes \cdots \otimes x \otimes x^{i_{l+m+2}}) \end{aligned}$$

where the sum is over all indices i_s such that $i_1 + \cdots + i_{l+m+2} = (l+m+1)(n-1)$ and $i_1, i_2, \cdots, i_{l+m+1} \geq 1$. So $a \smile b$ is

$$\begin{aligned} &= \sum \psi_{2l+1}^* f_a(x^{i_1} \otimes x \otimes \cdots \otimes x^{i_{l+1}}) \psi_{2m+1}^* f_b(x \otimes x^{i_{l+2}} \otimes \cdots \otimes x^{i_{l+m+1}} \otimes x) x^{i_{l+m+2}} \\ &= \sum \sum_{j=0}^{i_1-1} f_a(x^j \otimes x^{i_1-j-1} x^{i_2+1-n} \cdots x^{i_{l+1}+1-n}) f_b(1 \otimes x^{i_{l+2}+1-n} \cdots x^{i_{l+m+1}+1-n}) x^{i_{l+m+2}} \\ &= \sum_{i_1 \geq 1,\ i_1 + i_{l+m+2} = n-1} \ \sum_{j=0}^{i_1-1} f_a(x^j \otimes x^{i_1-j-1}) f_b(1 \otimes 1) x^{i_{l+m+2}} \\ &= \sum \sum x^j a x^{i_1-j-1} b x^{i_{l+m+2}} \ = \ 0 \end{aligned}$$

since $a, b \in kGx$, $i_1 + i_{l+m+2} = n-1$, and $x^n = 0$.

Now let $z = x$ in degree 1 and $y = 1$ in degree 2. Comparing to the proof of Theorem 2.4, we see that y and z together with $\mathrm{HH}^0(B) \cong (kN)^G$ generate $\mathrm{HH}^*(B)$, and the ring structure is as claimed. $\square$

A consequence of the theorem is that $\mathrm{HH}^*(B)$ is finitely generated. It has been conjectured that the Hochschild cohomology ring, modulo nilpotent elements, of any finite dimensional algebra is finitely generated [12].

We remark that the cup products could equally well have been computed using Yoneda composition. The resolution (2.5) is a $\mathcal{D}$-projective resolution of A, and may be induced to B^e to obtain a B^e-projective resolution of $A \uparrow_{\mathcal{D}}^{B^e} \cong B$. The technique for computing Yoneda compositions from a projective resolution given in [2, §2.6] applies to this resolution to yield an alternative proof of Theorem 3.4.

It would be interesting to determine the Hochschild cohomology more generally for *all* finite dimensional rank one Hopf algebras, including those for which the relation $x^n = 0$ is replaced by $x^n = g_1^n - 1$ (see [9]). This would require a different approach. A generalization in another direction would be to allow the characteristic of k to divide $|G|$ while remaining relatively prime to n.

4. Hochschild cohomology of smash products

In this section, we let A be any k-algebra and H any Hopf algebra over k, with bijective antipode S, for which A is an H-module algebra. That is, A is an H-module for which ${}^h(ab) = ({}^{h_1}a)({}^{h_2}b)$ and ${}^h1 = \varepsilon(h)$ for all $a, b \in A$ and $h \in H$. Let $B = A\#H$ be the *smash product* of A with H: As a vector space, $B = A \otimes H$, and multiplication is given by

$$(a \otimes h)(b \otimes l) = a({}^{h_1}b) \otimes h_2 l$$

for all $a, b \in A$ and $h, l \in H$. We abbreviate $a \otimes h$ by ah.

We now generalize the algebra $\mathcal{D}$ defined in (2.2) in the case $H = kG$. Let $\delta : H \to H \otimes H^{op}$ be the map given by $\delta(h) = h_1 \otimes S(h_2)$ for all $h \in H$. Note that δ is injective as its composition with $\mathrm{id} \otimes \varepsilon$ is injective, so that $H \cong \delta(H)$. Let

$$\mathcal{D} := (A \otimes A^{op})\delta(H), \tag{4.1}$$

a subalgebra of B^e: To see that $\mathcal{D}$ is closed under multiplication, use the relation $aS(h) = S(h_1)({}^{h_2}a)$ for all $h \in H, a \in A$:

$$\begin{aligned}
(a \otimes b)(h_1 \otimes S(h_2))(c \otimes d)(l_1 \otimes S(l_2)) &= (a \otimes b)(h_1 c \otimes dS(h_2))(l_1 \otimes S(l_2)) \\
&= (a \otimes b)(({}^{h_1}c)h_2 \otimes S(h_3)({}^{h_4}d))(l_1 \otimes S(l_2)) \\
&= (a \otimes b)({}^{h_1}c \otimes {}^{h_4}d)(h_2 \otimes S(h_3))(l_1 \otimes S(l_2))
\end{aligned}$$

for all $a, b, c, d \in A$ and $h, l \in H$. Unlike the case $H = kG$, the algebra $\mathcal{D}$ appears not to be a smash product in general.

Note that A is a $\mathcal{D}$-module under left and right multiplication since $h_1 a S(h_2) = ({}^{h_1}a)h_2 S(h_3) = {}^h a \in A$ for all $h \in H$ and $a \in A$. Let $A \uparrow_{\mathcal{D}}^{B^e} = B^e \otimes_{\mathcal{D}} A$ denote the induced left B^e-module.

Lemma 4.2. *There is an isomorphism of left B^e-modules, $B \cong A \uparrow_{\mathcal{D}}^{B^e}$.*

Proof. First note that $H^e = (H \otimes 1)\delta(H)$ as sets: If $a, b \in H$, we have

$$a \otimes S(b) = a\varepsilon(b_1) \otimes S(b_2) = aS(b_1)b_2 \otimes S(b_3) = (aS(b_1) \otimes 1)(b_2 \otimes S(b_3)),$$

an element of $(H \otimes 1)\delta(H)$. This suffices since S is bijective.

Now define B^e-maps ϕ and ψ:

$$\begin{array}{ccccccccc}
B^e \otimes_{\mathcal{D}} A & \xrightarrow{\phi} & B & , & & B & \xrightarrow{\psi} & B^e \otimes_{\mathcal{D}} A & . \\
(b \otimes c) \otimes a & \mapsto & bac & & & b & \mapsto & (b \otimes 1) \otimes 1 &
\end{array}$$

That ψ is a B^e-map uses the relation from the first paragraph. Clearly ϕ is well-defined. We next check that ϕ and ψ are inverses. By the above arguments, $B^e = H^e A^e = (H \otimes 1)\delta(H)A^e$. Note that $\mathcal{D} = A^e\delta(H) = \delta(H)A^e$: If $h \in H$ and $a, b \in A$, then

$$(h_1 \otimes S(h_2))(a \otimes b) = h_1 a \otimes bS(h_2) = (h_1.a)h_2 \otimes S(h_3)h_4.b = (h_1.a \otimes h_4.b)(h_2 \otimes S(h_3)),$$

so that $\delta(H)A^e \subseteq A^e\delta(H)$. The other containment may be shown similarly.

We claim that B^e is a free right $\mathcal{D}$-module, with free $\mathcal{D}$-basis given by any k-basis of $H \otimes 1$. In case $A = k$, this follows from the fact that a tensor product of a free

H-module with another module is free [2, Prop. 3.1.5]. In general, note that B^e is free over A^e with free basis any k-basis of H^e, by construction, and we may take as k-basis of H^e the product of a k-basis of $H \otimes 1$ with a k-basis of $\delta(H)$, again by [2, Prop. 3.1.5]. This shows that a k-basis of $H \otimes 1$ forms a free $\mathcal{D}$-basis of B^e. Therefore we may write elements of $B^e \otimes_{\mathcal{D}} A$ as linear combinations of elements $(h \otimes 1) \otimes a$ $(h \in H, a \in A)$. Then

$$(\psi \circ \phi)((h \otimes 1) \otimes a) = \psi(ha) = (ha \otimes 1) \otimes 1 = (h \otimes 1) \otimes a$$

for all $h \in H, a \in A$, since $a \otimes 1 \in \mathcal{D}$, and $(\phi \circ \psi)(b) = \phi((b \otimes 1) \otimes 1) = b$ for all $b \in B$. □

The following theorem generalizes part of [15, Lemma 3.5].

Theorem 4.3. *Let M be a B-bimodule, and $\mathcal{D}$ the subalgebra of B^e defined in (4.1). Then*

$$\mathrm{HH}^*(B, M) \cong \mathrm{Ext}^*_{\mathcal{D}}(A, M)$$

as graded vector spaces.

We remark that any decomposition of M into a direct sum of $\mathcal{D}$-submodules now leads to a similar decomposition of $\mathrm{HH}^*(B, M)$.

Proof. Since B^e is a free right $\mathcal{D}$-module we may apply the Eckmann-Shapiro Lemma and Lemma 4.2 to obtain the claimed isomorphism. □

As a consequence, $\mathrm{HH}^*(B) \cong \mathrm{Ext}^*_{\mathcal{D}}(A, B)$ has a graded vector space decomposition indexed by $\mathcal{D}$-summands of B. In case $B = kG$, there are $\mathcal{D}$-summands that are indexed by the conjugacy classes of the group, and this leads to a useful description of cup products on $\mathrm{HH}^*(kG)$ that was used to compute several examples [11]. In case $B = H$ is a commutative Hopf algebra, H is trivial as a $\mathcal{D}$-module. Combined with an explicit description of the cup product given in (4.6) below, this yields an isomorphism of graded algebras $\mathrm{HH}^*(H) \cong H \otimes \mathrm{Ext}^*_H(k, k)$, an alternative proof of [10, Thm. 1]. In case $B = A\#H$ with $H = kG$, the cup product on $\mathrm{Ext}^*_{\mathcal{D}}(A, B)$ is described in [15, Thm. 3.16] in terms of summands indexed by conjugacy classes.

We give an explicit formula for the product on $\mathrm{Ext}^*_{\mathcal{D}}(A, B)$ induced by the cup product on $\mathrm{HH}^*(B)$, by expressing the bar resolution of B as induced from a $\mathcal{D}$-resolution of A. Define $\delta^m : H^{\otimes(m+1)} \to H^{\otimes(m+2)}$ by

$$\delta^m(h^0 \otimes \cdots \otimes h^m) = h^0_1 \otimes \cdots \otimes h^m_1 \otimes S(h^0_2 \cdots h^m_2)$$

for $h^i \in H$. Let

$$\mathcal{D}_m := (A^{\otimes(m+2)})\delta^m(H^{\otimes(m+1)}),$$

where indicated products occur in B. Note that $\delta^0 = \delta$ and $\mathcal{D}_0 = \mathcal{D}$ since

$$ah_1 \otimes bS(h_2) = ah_1 \otimes S(h_2)({}^{h_3}b) = (a \otimes {}^{h_3}b)(h_1 \otimes S(h_2)) \in \mathcal{D} \subset B^e.$$

By Lemma 4.2 we have an isomorphism of B^e-modules, $A\uparrow_{\mathcal{D}}^{B^e}\cong B$. By construction, $\mathcal{D}_0\uparrow_{\mathcal{D}}^{B^e}\cong B^e$. A calculation shows that for all $m\geq 0$, $\mathcal{D}_m\uparrow_{\mathcal{D}}^{B^e}\xrightarrow{\sim} B^{\otimes(m+2)}$ as B^e-modules, via the map

$$(b_{-1}\otimes b_{m+2})\otimes(b_0\otimes\cdots\otimes b_{m+1})\mapsto b_{-1}b_0\otimes b_1\otimes\cdots\otimes b_m\otimes b_{m+1}b_{m+2}, \tag{4.4}$$

whose inverse is

$$a_0h^0\otimes a_1h^1\otimes\cdots\otimes a_{m+1}h^{m+1}\mapsto(1\otimes h_3^0\cdots h_3^m h^{m+1})\otimes(a_0h_1^0\otimes\cdots\otimes a_mh_1^m\otimes a_{m+1}S(h_2^0\cdots h_2^m)). \tag{4.5}$$

We claim that $\mathcal{D}_m$ is $\mathcal{D}$-projective: First note that

$$\mathcal{D}_m=\mathcal{D}\{1\otimes a_1h_1^1\otimes\cdots\otimes a_mh_1^m\otimes S(h_2^1\cdots h_2^m)\}$$

as sets. We use this to define an isomorphism $\mathcal{D}_m\xrightarrow{\sim}(A\otimes B^{\otimes m}\otimes A)\uparrow_{A^e}^{\mathcal{D}}$ via the map

$$\begin{aligned}(a_0l_1\otimes a_{m+1}S(l_2))(1\otimes a_1h_1^1\otimes\cdots\otimes a_mh_1^m\otimes S(h_2^1\cdots h_2^m))&\\ \mapsto(a_0l_1\otimes a_{m+1}S(l_2))\otimes(1\otimes a_1h^1\otimes\cdots\otimes a_mh^m\otimes 1)&\end{aligned}$$

whose inverse is

$$\begin{aligned}(a_{-1}l_1\otimes a_{m+2}S(l_2))\otimes a_0\otimes a_1h^1\otimes\cdots\otimes a_mh^m\otimes a_{m+1}&\\ \mapsto a_{-1}l_1a_0\otimes a_1h_1^1\otimes\cdots\otimes a_mh_1^m\otimes S(h_2^1\cdots h_2^m)a_{m+1}a_{m+2}S(l_2).&\end{aligned}$$

Since $A\otimes B^{\otimes m}\otimes A$ is clearly A^e-free and $\mathcal{D}$ is free as a right A-module, the induced $\mathcal{D}$-module $\mathcal{D}_m$ is $\mathcal{D}$-free. The differentials for the bar resolution of B preserve $\mathcal{D}_*$, and it may be checked that $\mathcal{D}_*$ is a resolution under the restriction of these differentials. The bar resolution of B is thus induced from $\mathcal{D}_*$. If $f:\mathcal{D}_l\to B$ and $f':\mathcal{D}_m\to B$ are two cocycles, define $f\smile f':\mathcal{D}_{l+m}\to B$ by

$$\begin{aligned}(f\smile f')(a_0h_1^0\otimes\cdots\otimes a_{l+m}h_1^{l+m}\otimes a_{l+m+1}S(h_2^0\cdots h_2^{l+m}))=f(a_0h_1^0\otimes\cdots\otimes a_lh_1^l\otimes S(h_2^0\cdots h_2^l))\cdot&\\ f'(h_3^0\cdots h_3^l\otimes a_{l+1}h_1^{l+1}\otimes\cdots\otimes a_{l+m}h_1^{l+m}\otimes a_{l+m+1}S(h_4^0\cdots h_4^lh_2^{l+1}\cdots h_2^{l+m}))&\end{aligned} \tag{4.6}$$

for all $a_i\in A$ and $h^i\in H$. This agrees with the cup product on $\mathrm{Ext}^*_{B^e}(B,B)$ under the given isomorphism.

In the special case $A=k$, Theorem 4.3 implies that

$$\mathrm{HH}^*(H)\cong\mathrm{H}^*(H,H^{ad}), \tag{4.7}$$

where H^{ad} is the H-module H under the left adjoint action defined by $(\mathrm{ad}\,h)(l)=h_1lS(h_2)$ for $h,l\in H$, and $\mathrm{H}^*(H,H^{ad}):=\mathrm{Ext}^*_H(k,H^{ad})$. This isomorphism appears in [7] as Prop. 5.6.

Example 4.8. We outline an alternative approach to the Hochschild cohomology of rank one Hopf algebras that we computed in Section 2, based on (4.7). This allows us to relate the structure of $\mathrm{HH}^*(B)$ to the adjoint representation of $B=H$ for these Hopf algebras. We first find a decomposition of B^{ad}.

Using the coproducts (2.1), we have

$$(\mathrm{ad}\,g)(x^ih)=\chi(g^i)x^ighg^{-1}\quad\text{ and }\quad(\mathrm{ad}\,x)(x^ih)=(1-\chi(g_1^ih))x^{i+1}h$$

for all $g, h \in G$. Let $g_0, \ldots, g_t$ be a set of representatives of conjugacy classes of G, where g_1 is the central element from Section 2. Assume g_1 has order m and $g_0 = 1,\ g_2 = g_1^2,\ \ldots,\ g_{m-1} = g_1^{m-1}$. Then B^{ad} has the following decomposition as a B-module (however the summands are not necessarily indecomposable):

$$B^{ad} = \bigoplus_{k=0}^{t} \mathrm{Span}_k\{x^i h g_k h^{-1} \mid 0 \le i \le n-1,\ h \in G\}.$$

Each summand above potentially splits into the sum of two B-submodules: For each $h \in G$, let j_h $(0 \le j_h \le n-1)$ be the smallest such that $\chi(h) = \chi(g_1)^{-j_h}$ if this exists, and otherwise let $j_h = n-1$. Then the kth summand above becomes

$$\mathrm{Span}_k\{x^i h g_k h^{-1} \mid 0 \le i \le j_{g_k},\ h \in G\} \oplus \mathrm{Span}_k\{x^i h g_k h^{-1} \mid j_{g_k}+1 \le i \le n-1,\ h \in G\},$$

although this is not needed for the computation of cohomology. In order to compute $\mathrm{H}^*(B, B^{ad})$, we use the free A-resolution of k:

$$\cdots \xrightarrow{\cdot x^{n-1}} A \xrightarrow{\cdot x} A \xrightarrow{\cdot x^{n-1}} A \xrightarrow{\cdot x} A \xrightarrow{\varepsilon} k \to 0.$$

This may be extended to a projective B-resolution of k by giving A the following actions of G: In degree $2i$, $g \cdot a = \chi(g)^{in}({}^g a)$, and in degree $2i+1$, $g \cdot a = \chi(g)^{in+1}({}^g a)$ for all $a \in A$ and $g \in G$. This leads to an alternative proof of Theorem 2.4.

We now return to a more general setting. Assume that H is any finite dimensional semisimple Hopf algebra and $B = A\#H$. In this case, a direct consequence of Theorem 4.3 is that

$$\mathrm{HH}^*(B) \cong \mathrm{HH}^*(A, B)^H \tag{4.9}$$

(cf. [13, Thm. 3.3]). To see this, first use the relation $\mathrm{Hom}_{\mathcal{D}}(M, N) \cong \mathrm{Hom}_{A^e}(M, N)^H$ for any two $\mathcal{D}$-modules M, N, where the superscript H denotes invariants under the action

$$(h \cdot f)(m) = h_1 \cdot (f(S(h_2) \cdot m) = h_1 f(S(h_4) m S^2(h_3)) S(h_2)$$

induced by the embedding $\delta : H \to H^e$. Next we must see that taking H-invariants after taking cohomology is equivalent to taking H-invariants before taking cohomology. This follows from the observation that H-invariants are precisely the image of a nonzero integral since H is semisimple. Using this, we now give explicit formulas for cocycles and cup products on $\mathrm{HH}^*(A, B)^H$.

Let

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0 \tag{4.10}$$

be any $\mathcal{D}$-projective resolution of A. We claim that the bar complex for A is itself a $\mathcal{D}$-projective resolution of A where $h \in H \cong \delta(H)$ acts on $A^{\otimes m}$ diagonally. We thank A. Kaygun for explaining to us a proof of this fact, in the context of Hopf-Hochschild cohomology. We summarize the proof here: Let Λ be an integral for H with $\varepsilon(\Lambda) = 1$. Then the $\mathcal{D}$-map $\mathcal{D} \to A^e$ defined by $ah_1 \otimes S(h_2)b \mapsto \varepsilon(h) a \otimes b$ is split by the $\mathcal{D}$-map $A^e \to \mathcal{D}$ defined by $a \otimes b \mapsto a\Lambda_1 \otimes S(\Lambda_2)b$ for all $a, b \in A$ and $h \in H$. Therefore A^e is $\mathcal{D}$-projective. If $m > 0$, we see that $A^{\otimes(m+2)}$ is $\mathcal{D}$-projective as follows. Since H is semisimple, A is H-projective, and so $A^{\otimes m}$ is a direct summand of a sum of copies of $H^{\otimes m}$. A standard argument (see the proof of [2, Prop. 3.1.5]) shows that $H^{\otimes m} \cong H \otimes (H_{tr}^{\otimes(m-1)})$, where H_{tr} is H with the trivial action of H. Now $A \otimes H \otimes A \cong \mathcal{D}$ as $\mathcal{D}$-modules, and k-bases of the remaining factors $H_{tr}^{\otimes(m-1)}$ arising from $A^{\otimes(m+2)}$ provide a $\mathcal{D}$-basis of a free $\mathcal{D}$-module having $A^{\otimes(m+2)}$ as a direct summand. It follows that the bar resolution of A is a $\mathcal{D}$-projective resolution of A in case H is semisimple.

Let $\psi_n : A^{\otimes(m+2)} \to P_m$ be $\mathcal{D}$-homomorphisms giving a map of chain complexes from the bar complex to (4.10). The following theorem generalizes [4, Thm. 5.4], which is useful in case a resolution other than a bar-type resolution is used to compute the cohomology. For example, it was used to find explicit formulas for Hochschild 2-cocycles in [4] when the cohomology was computed via a Koszul resolution.

Theorem 4.11. *Assume H is a finite dimensional semisimple Hopf algebra. Let $f : P_m \to B$ be a function representing an element of* $\mathrm{HH}^m(A,B)^H$ *expressed via the complex (4.10). The corresponding function $\widetilde{f} \in \mathrm{Hom}_k(B^{\otimes m}, B) \cong \mathrm{Hom}_{B^e}(B^{\otimes(m+2)}, B)$ expressed via the bar complex is defined by*

$$\widetilde{f}(a_1h^1 \otimes \cdots \otimes a_mh^m)$$
$$= ((f \circ \psi_m)(1 \otimes a_1 \otimes {}^{h_1^1}a_2 \otimes {}^{h_2^1h_1^2}a_3 \otimes \cdots \otimes {}^{h_{m-1}^1h_{m-2}^2h_1^{m-1}}a_m \otimes 1))h_m^1h_{m-1}^2 \cdots h_2^{m-1}h^m$$

for all $a_1, \ldots, a_m \in A$ and $h^1, \ldots, h^m \in H$.

Proof. This follows by explicitly tracing through the Eckmann-Shapiro Lemma as it applies to $\mathrm{HH}^*(B) \cong \mathrm{Ext}^*_{\mathcal{D}}(A,B)$ in the proof of Theorem 4.3. We use the explicit map from the bar resolution for B to $\mathcal{D}_* \uparrow_{\mathcal{D}}^{B^e}$ given in (4.5). We also need a $\mathcal{D}$-map from $\mathcal{D}_*$ to the bar resolution for A, and this is

$$(4.12) \qquad a_0h_1^0 \otimes \cdots \otimes a_mh_1^m \otimes a_{m+1}S(h_2^0 \cdots h_2^m)$$
$$\mapsto a_0 \otimes {}^{h_1^0}a_1 \otimes {}^{h_2^0h_1^1}a_2 \otimes \cdots \otimes {}^{h_{m+1}^0h_m^1 \cdots h_2^{m-1}h^m}a_{m+1}.$$

(This generalizes [4, (5.2)].)

Applying (4.5) first, $\widetilde{f}(a_1h^1 \otimes \cdots \otimes a_mh^m) = \widetilde{f}(1 \otimes a_1h^1 \otimes \cdots \otimes a_mh^m \otimes 1)$ may be identified with

$$\widetilde{f}((1 \otimes h_3^1 \cdots h_3^m) \otimes (1 \otimes a_1h_1^1 \otimes \cdots \otimes a_mh_1^m \otimes S(h_2^1 \cdots h_2^m))$$
$$= \widetilde{f}(1 \otimes a_1h_1^1 \otimes \cdots \otimes a_mh_1^m \otimes S(h_2^1 \cdots h_2^m))h_3^1 \cdots h_3^m.$$

Now applying (4.12), this is

$$((f \circ \psi_m)(1 \otimes a_1 \otimes {}^{h_1^1}a_2 \otimes {}^{h_2^1h_1^2}a_3 \otimes \cdots \otimes {}^{h_{m-1}^1h_{m-2}^2 \cdots h_1^{m-1}}a_m \otimes 1))h_m^1h_{m-1}^2 \cdots h_2^{m-1}h^m.$$

□

We now describe the cup product on $\mathrm{HH}^*(A,B)^H$. Just as in Section 3, where $H = kG$, the cup product on $\mathrm{HH}^*(A,B)^H \subseteq \mathrm{HH}^*(A,B)$ induced by the algebra structure on B corresponds to the cup product on $\mathrm{HH}^*(B)$, and in particular if $f : A^{\otimes(l+2)} \to B$ and $f' : A^{\otimes(m+2)} \to B$, then

$$(f \smile f')(a_0 \otimes \cdots \otimes a_{l+m+1}) = f(a_0 \otimes \cdots \otimes a_l \otimes 1)f'(1 \otimes a_{l+1} \otimes \cdots \otimes a_{l+m+1})$$

by (4.12) and (4.6).

5. Hopf-Hochschild cohomology is Hochschild cohomology

Let H be a bialgebra and A an H-module algebra. In [8], Kaygun introduces an algebra $\Gamma = A^e \otimes H$ with the following multiplication:

$$(a \otimes b \otimes h)(c \otimes d \otimes l) = (a({}^{h_1}c) \otimes ({}^{h_3}d)b \otimes h_2 l)$$

for all $a, b, c, d \in A$ and $h, l \in H$. By [8, Lem. 3.2], Γ is an associative algebra, denoted $A^e \rtimes H$ there. The bar resolution for A is a differential graded Γ-module under the usual action of A^e and the tensor product action of H (see [8, Lem. 3.5]).

Let M be an *H-equivariant* A-bimodule, that is M is both an H-module and an A-bimodule, and ${}^h(amb) = ({}^{h_1}a)({}^{h_2}m)({}^{h_3}b)$ for all $m \in M$, $a, b \in A$, and $h \in H$. Equivalently, M is a Γ-module where $(a \otimes b \otimes h)m = a({}^h m)b$ by [8, Lem. 3.3]. The *Hopf-Hochschild cohomology* $\mathrm{HH}^*_{\mathrm{Hopf}}(A, M)$ of A with coefficients in M is defined in [8] to be the cohomology of the cochain complex $\mathrm{CH}^*_{\mathrm{Hopf}}(A, M)$ where

$$\mathrm{CH}^m_{\mathrm{Hopf}}(A, M) := \mathrm{Hom}_\Gamma(A^{\otimes(m+2)}, M),$$

and the differentials are induced from those of the bar complex for A.

Lemma 5.1. *Let H be a Hopf algebra, and A an H-module algebra. There is an isomorphism of algebras $\Gamma \cong \mathcal{D}$, where $\mathcal{D}$ is defined in (4.1).*

Proof. Define $\phi : \mathcal{D} \to \Gamma$ by

$$ah_1 \otimes S(h_2)b \to a \otimes b \otimes h$$

for all $a, b \in A$, $h \in H$. Then clearly ϕ has inverse ψ defined by $\psi(a \otimes b \otimes h) = ah_1 \otimes S(h_2)b$. We verify that ϕ is multiplicative:

$$\begin{aligned}\phi((ah_1 \otimes S(h_2)b)(cl_1 \otimes S(l_2)d)) &= \phi(a({}^{h_1}c)h_2 l_1 \otimes S(h_3 l_2)({}^{h_4}d)b) \\ &= a({}^{h_1}c) \otimes ({}^{h_3}d)b \otimes h_2 l \\ &= \phi(ah_1 \otimes S(h_2)b)\phi(cl_1 \otimes S(l_2)d)\end{aligned}$$

for all $a, b, c, d \in A$ and $h, l \in H$. □

If M is an $A\#H$-bimodule, then M is a module for $\Gamma \cong \mathcal{D}$ by restriction to $\mathcal{D} \subset (A\#H)^e$. Therefore M has the structure of an H-equivariant A-bimodule. The next theorem shows that the Hopf-Hochschild cohomology of A with coefficients in M is isomorphic to Hochschild cohomology under the assumption that H is semisimple (cf. [8, Thm. 3.7]).

Theorem 5.2. *Let H be a finite dimensional semisimple Hopf algebra, and A an H-module algebra. Then*

$$\mathrm{HH}^m_{\mathrm{Hopf}}(A, M) \cong \mathrm{HH}^m(A\#H, M)$$

for all m and any $A\#H$-bimodule M.

Proof. The bar resolution of A is a $\mathcal{D}$-projective resolution of A, as explained towards the end of Section 4. This fact, together with Lemma 5.1 and Theorem 4.3 imply

$$\mathrm{HH}^m_{\mathrm{Hopf}}(A, M) \cong \mathrm{Ext}^m_{\mathcal{D}}(A, M) \cong \mathrm{HH}^m(A\#H, M).$$

□

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