

A_∞ -CODERIVATIONS AND THE GERSTENHABER BRACKET ON HOCHSCHILD COHOMOLOGY

C. NEGRON, Y. VOLKOV, AND S. WITHERSPOON

ABSTRACT. We show that Hochschild cohomology of an algebra over a field is a space of infinity coderivations on an arbitrary projective bimodule resolution of the algebra. The Gerstenhaber bracket is the graded commutator of infinity coderivations. We thus generalize, to an arbitrary resolution, Stasheff's realization of the Gerstenhaber bracket on Hochschild cohomology as the graded commutator of coderivations on the tensor coalgebra of the algebra.

1. INTRODUCTION

Stasheff [21] identified the Hochschild cohomology of an algebra A with a space of coderivations on the tensor coalgebra of A (cf. [19, Part II]). In the same article the author also gave an elegant interpretation of the Gerstenhaber bracket as the graded commutator of these coderivations, thus providing a theoretical framework for the explicit formula of the bracket as given by Gerstenhaber [4] on the bar resolution of A .

In this paper we generalize Stasheff's result to an arbitrary bimodule resolution P of A (see Theorem 4.1.1 and Section 4.6). In order to do this, we must consider higher operations on the resolution, starting with a diagonal map for P , that is, a chain map from P to $P \otimes_A P$. If P is the bar resolution, the standard diagonal map is coassociative. In general a diagonal map is only coassociative up to homotopy, and this is where higher operations begin to appear. We show that P may be given the structure of an A_∞ -coalgebra. We show that Hochschild cohomology of A may then be identified with the graded Lie algebra of outer A_∞ -coderivations on P . That is, it may be identified with the quotient of the space of A_∞ -coderivations on P by the space of inner A_∞ -coderivations, and the Gerstenhaber bracket is induced by the graded commutator of A_∞ -coderivations. Thus the present paper contributes to a growing collection of literature concerning alternate interpretations of the Lie structure on Hochschild cohomology [8, 10, 17, 20, 21, 22, 26], and subsequent applications (e.g. [5, 6, 7, 10, 16, 18]). For related constructions one may see also recent works of Herscovich and Tamaroff [9, 23, 24].

We note that even when the higher structure on P vanishes, i.e. when the A_∞ -coalgebra structure on P simply reduces to a dg coalgebra structure, the space of A_∞ -coderivations on P is strictly larger than the space of ordinary coderivations. This occurs, for example, when A is Koszul and P is the Koszul resolution. In that case there is a canonical dg coalgebra structure on P induced by the algebra structure on the Koszul dual $A^!$, and yet it is strictly necessary to consider A_∞ -coderivations of P in order to obtain the identification with Hochschild cohomology.

Many of the proofs in the present work are constructive, and the constructions of A_∞ -coderivations begin with the homotopy liftings introduced by the second author [26] to

generalize work of the first and third authors [17]: Given a Hochschild cocycle f on P , we show how to extend it to an A_∞ -coderivation $\{f_n\}$ for which $f_0 = f$ and f_1 is a homotopy lifting of f as defined in [26]. If one is interested only in expressing the Gerstenhaber bracket of two cocycles on P as another cocycle on P , one needs only consider homotopy liftings, and not the full structure of infinity coderivations. We include some small examples as illustrations; other examples are given in [26].

The order of the main definitions and results is as follows. In Section 2, we give general definitions of A_∞ -coalgebras, A_∞ -coderivations, and a cup product. In Section 3, we show that a projective bimodule resolution P of an algebra A has an A_∞ -coalgebra structure (Theorem 3.1.1). In Section 4 we prove our main theorems: For any Hochschild cocycle f on P , there exists an A_∞ -coderivation on P whose 0-component is f (Theorem 4.4.6) and whose 1-component is a homotopy lifting for f . As a consequence, Hochschild cohomology of A is isomorphic to the Gerstenhaber algebra of A_∞ -coderivations on P modulo inner A_∞ -coderivations (Theorem 4.1.1). In Section 5 we consider connected graded algebras and in particular look closely at Koszul algebras as one large class of examples. Their Koszul resolutions have rich arrays of A_∞ -coderivations for which we give explicit formulas.

For all of our definitions and results, we choose grading and sign conventions to simplify computations. In particular, we adopt the Koszul sign convention: If $f : M_1 \rightarrow N_1$ and $g : M_2 \rightarrow N_2$ are homogeneous morphisms of \mathbb{Z} -graded A^e -modules, then $f \otimes_A g : M_1 \otimes_A M_2 \rightarrow N_1 \otimes_A N_2$ denotes the morphism defined by the equality

$$(f \otimes_A g)(x \otimes_A y) = (-1)^{|g||x|} f(x) \otimes_A g(y)$$

for all homogeneous $x \in M_1, y \in M_2$. As a consequence, $(1 \otimes_A g)(f \otimes_A 1) = (-1)^{|f||g|} (f \otimes_A g)$, where 1 always denotes an appropriate identity map. For complexes of bimodules X and Y , we let ∂ denote the differential on the Hom complex $\text{Hom}_{A^e}(X, Y)$, $\partial(f) = d_Y f - (-1)^{|f|} f d_X$.

Throughout, \mathbf{k} will be a field and A will be a unital \mathbf{k} -algebra. We write $\otimes = \otimes_{\mathbf{k}}$. As usual, $A^e = A \otimes A^{op}$ denotes the enveloping algebra of A . We identify A -bimodules with (left) A^e -modules.

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2. A_∞ -COALGEBRAS AND A_∞ -CODERIVATIONS

In this section, we give many definitions that we will need in the paper. In particular, we define A_∞ -coalgebras, A_∞ -coderivations, and a cup product and a graded Lie bracket on A_∞ -coderivations.

2.1. A_∞ -coalgebras and the cobar construction. Let \mathcal{Q} be the category of dg A -bimodules, that is complexes of A -bimodules with morphisms the hom complex $\text{Hom}_{\mathcal{Q}}(X, Y)$ of homogeneous maps between X and Y of arbitrary degrees. Then \mathcal{Q} is a monoidal category under the product \otimes_A and unit A .

For an object C in \mathcal{Q} , we let $T(C)$ denote the free augmented dg algebra in \mathcal{Q} , which is explicitly the sum of the tensor product complexes

$$T(C) = \bigoplus_{n \geq 0} C^{\otimes_A n}$$

with the standard product given by concatenation and differential $d_{T(C)} = \sum_{n=1}^{\infty} d_{C^{\otimes_A n}}$. (Recall that $d_{X \otimes_A Y} = d_X \otimes 1 + 1 \otimes d_Y$, interpreted via the Koszul sign convention when we apply the differential to elements.) The dg algebra $T(C)$ is A -augmented with augmentation $\epsilon_T : T(C) \rightarrow A$ given by projection onto $T^0(C) = A$.

We consider the \mathfrak{m} -adic completion $\widehat{T}(C)$ at the augmentation ideal $\mathfrak{m} = \ker(\epsilon_T)$,

$$\widehat{T}(C) = \varprojlim_n T(C)/(C^{\otimes_A n}) = \prod_{n \geq 0} C^{\otimes_A n},$$

a linear topological, augmented, dg algebra in \mathcal{Q} .

Definition 2.1.1. An A_∞ -coalgebra in \mathcal{Q} is an object C in \mathcal{Q} with the additional information of a homogenous degree 1, A^e -linear, continuous, algebra derivation $\delta : \widehat{T}(C) \rightarrow \widehat{T}(C)$ such that:

- (i) δ preserves the augmentation, that is the composite $\epsilon_T \delta : \widehat{T}(C) \rightarrow \widehat{T}(C) \rightarrow A$ is 0;
- (ii) δ solves the Maurer-Cartan equation in the endomorphism complex $\text{End}_{\mathcal{Q}}(\widehat{T}(C))$:

$$d_{\text{End}}(\delta) + \delta \circ \delta = 0.$$

There is a notion of a ‘‘weakly counital’’ A_∞ -coalgebra, and we will suppose that all of our A_∞ -coalgebras are weakly counital. This definition is easier to give with a componentwise description of an A_∞ -structure on a complex C , and is given in Section 2.3.

The above definition of an A_∞ -coalgebra is constructed following the standard principles of A_∞ -(co)algebras outlined in, say, Lefevre-Hasegawa’s thesis [14]. (See also [11, 12, 15].) Note that the continuous derivation δ is determined by a sequence of degree 1 maps $\delta_n : C \rightarrow C^{\otimes_A n}$ (see Section 2.3 below for explicit conditions on δ_n). Compatibility with the augmentation implies that the term δ_0 vanishes. We make this (standard) assumption to avoid dealing with ‘‘curved’’ structures. Note also that if $((C, d_C), \delta)$ is an A_∞ -coalgebra, then $((C, 0), \delta + d_C)$ and $((C, d_C + \delta_1), \delta - \delta_1)$ are A_∞ -coalgebras too.

The familiar reader will recognize that an A_∞ -coalgebra can be described equivalently as a graded object in the category of A -bimodules with a sequence of maps $\Delta_n : C[1] \rightarrow C[1]^{\otimes_A n}$ of degree $2 - n$ satisfying certain compatibilities. This definition is more natural from a classical perspective, where dg coalgebras are central. Indeed, in the case that Δ_n vanishes for all $n > 2$, $C = (C, \Delta_1, \Delta_2)$ will simply be a dg coalgebra in \mathcal{Q} .

Let us recall a standard construction, which in some sense appeared already in the work of Adams in the 1950’s [1].

Definition 2.1.2. For an A_∞ -coalgebra $C = (C, \delta)$ in \mathcal{Q} we let ΩC denote the dg algebra in \mathcal{Q} which is $\widehat{T}(C)$ as an augmented, topological, algebra, with differential $d_{\Omega C} = d_{\widehat{T}(C)} + \delta$. We call ΩC the *cobar construction* of C .

As before, $d_{\widehat{T}(C)}$ denotes the differential on the (completed) free dg algebra generated by C in the above definition. We work with the completed algebra $\widehat{T}(C)$ in order to allow the δ_n to be nonvanishing for arbitrarily large n . Similarly, for morphisms, we would like to allow arbitrary f_n to be nonvanishing. One otherwise might define the cobar construction

via the tensor algebra $T(C)$ directly. See Section 5 on connected graded algebras where this is sometimes sufficient for our connection between Hochschild cohomology and A_∞ -coderivations, that is our A_∞ -coalgebras there will be locally finite in the following sense:

Definition 2.1.3. An A_∞ -coalgebra C in \mathcal{Q} is called *locally finite* if for each $c \in C$ there exists an integer N_c such that $\delta_N(c) = 0$ for all $N > N_c$.

In the case of a locally finite A_∞ -coalgebra C the derivation $d_\Omega : \Omega C \rightarrow \Omega C$ is such that the restriction to the topological generators C has image in the dense algebra $T(C) \subset \Omega C$. Consequently, d_Ω restricts to a derivation on $T(C)$, and the cobar construction ΩC is the \mathfrak{m} -adic completion of the dg algebra $(T(C), \delta_\Omega|_{T(C)}) = (T(C), d_{T(C)} + \delta|_{T(C)})$.

Definition 2.1.4. For a locally finite A_∞ -coalgebra C in \mathcal{Q} we define the *discrete cobar construction* as the dg algebra $\Omega^{\text{disc}}C = (T(C), d_{T(C)} + \delta|_{T(C)})$.

Although in general it is appropriate to employ the completed algebra ΩC , even when the A_∞ -structure on C is locally finite, we will see in Section 5 that under certain finiteness assumptions one can effectively employ the discrete algebra $\Omega^{\text{disc}}C$.

2.2. DG Lie algebras of A_∞ -coderivations. Note that for any topological dg algebra Ω in \mathcal{Q} , the collection $\text{Der}_\mathcal{Q}(\Omega)$ of continuous, graded, derivations of Ω in \mathcal{Q} forms a subcomplex in the endomorphism complex $\text{End}_\mathcal{Q}(\Omega)$. Furthermore, since the (graded) commutator of derivations is another derivation, the commutator operation

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$$

of graded derivations $f, g : \Omega \rightarrow \Omega$ makes $\text{Der}_\mathcal{Q}(\Omega)$ naturally a dg Lie algebra. To be clear, the differential on $\text{Der}_\mathcal{Q}(\Omega)$ is the adjoint operator

$$d_{\text{Der}} = [d_\Omega, -].$$

This operator d_{Der} is a graded Lie algebra derivation as a consequence of the Jacobi identity. It satisfies the equation $d_{\text{Der}} \circ d_{\text{Der}} = 0$ as a consequence of the identity $[f, [f, g]] = [f \circ f, g]$, which is valid for $f \in \text{End}_\mathcal{Q}(\Omega)$ of odd degree and any $g \in \text{End}_\mathcal{Q}(\Omega)$. We apply these observations in particular to the case $\Omega = \Omega C$ for an A_∞ -coalgebra C in \mathcal{Q} to produce a dg Lie algebra

$$\text{Der}_\mathcal{Q}(\Omega C).$$

By our previous discussions, $\text{Der}_\mathcal{Q}(\Omega C)$ is identified with $\text{Hom}_\mathcal{Q}(C, \Omega C)$ as a vector space.

We view our dg Lie algebra $\text{Der}_\mathcal{Q}(\Omega C)$ as analogous to the complex $\text{Coder}(T(A))$ in [21]. Continuing this analogy, we are interested in the cohomology of this complex. The space of cocycles $\text{Coder}_\mathcal{Q}^\infty(C)$ is referred to as the *A_∞ -coderivations*, and the space of coboundaries $\text{Inn}_\mathcal{Q}^\infty(C)$ is the space of *inner A_∞ -coderivations*.

As with any dg Lie algebra, the space of cocycles in $\text{Der}_\mathcal{Q}(\Omega C)$ forms a Lie subalgebra, the space of coboundaries forms a Lie ideal in the Lie subalgebra of cocycles, and the cohomology

$$H^\bullet(\text{Der}_\mathcal{Q}(\Omega C)) = \text{Coder}_\mathcal{Q}^\infty(C)/\text{Inn}_\mathcal{Q}^\infty(C)$$

is a graded Lie algebra. In Theorem 4.1.1, we will make a connection to Hochschild cohomology of an algebra A when C is a bimodule resolution of A .

2.3. Componentwise definitions. In this section we give componentwise definitions of A_∞ -coalgebras and A_∞ -coderivations that will be needed in calculations later.

Let $C = (C, \delta)$ be an A_∞ -coalgebra. The A_∞ -structure is specified by a collection of maps

$$\delta_n : C \rightarrow C^{\otimes A^n}, \quad \delta_n = p_n d_{\Omega C}|_C,$$

where p_n is the linear projection $\Omega C \rightarrow C^{\otimes A^n}$. The equation $d_{\Omega C}^2 = 0$ can now be written in the form

$$(2.3.1) \quad 0 = \sum_{r+s+t=N} (1^{\otimes Ar} \otimes_A \delta_s \otimes_A 1^{\otimes At}) \delta_{N-s+1}$$

for each $N \geq 0$. We can also write this equation as

$$(2.3.2) \quad \partial(\delta_N) = - \sum_{\substack{1 < s < N \\ r+s+t=N}} (1^{\otimes Ar} \otimes_A \delta_s \otimes_A 1^{\otimes At}) \delta_{N-s+1},$$

where ∂ denotes the differential on the Hom complex $\text{Hom}_{\mathcal{Q}}(C, C^{\otimes A^N})$ for each N ; the differential on the complex C is δ_1 .

Thus, an A_∞ -coalgebra in \mathcal{Q} is a graded A -bimodule C equipped with A^e -linear operators $\delta_n : C \rightarrow C^{\otimes A^n}$ of degree 1, for $n \geq 1$, that solve the equation (2.3.1) for all nonnegative N .

Definition 2.3.3. An A_∞ -coalgebra C is called *weakly counital* if it comes equipped with a degree 1 bimodule map $\mu : C \rightarrow A$ such that $(\mu \otimes_A \mu) \delta_2 = \mu$ and $(\mu^{\otimes A^n}) \delta_n = 0$ for all $n > 2$.

An A_∞ -coalgebra which is strictly counital (see e.g. [15]) is weakly counital. All A_∞ -coalgebras here will be assumed to be weakly counital for convenience, although the assumption can be avoided. We next give a componentwise definition of A_∞ -coderivation.

Definition 2.3.4. An A_∞ -coderivation f of degree l on C is a sequence of maps

$$f_n : C \rightarrow C^{\otimes A^n}$$

of degree l , for $n \geq 0$, that satisfy

$$(2.3.5) \quad \sum_{r+s+t=N} (1^{\otimes Ar} \otimes_A \delta_s \otimes_A 1^{\otimes At}) f_{N-s+1} = (-1)^l \sum_{r+s+t=N} (1^{\otimes Ar} \otimes_A f_s \otimes_A 1^{\otimes At}) \delta_{N-s+1}$$

for all N .

We note the possible appearance of the scalar term $f_0 : C \rightarrow A$ in the above notion of an A_∞ -coderivation.

We consider an A_∞ -coderivation to be an element of $\text{Hom}_{\mathcal{Q}}(C, \Omega C)$, and identify it with the corresponding function in $\text{Der}_{\mathcal{Q}}(\Omega C)$ as described earlier. In this way we may compose two A_∞ -coderivations. The space of such maps is not closed under composition, but it is closed under a bracket, as we define next.

For f and g as in Definition 2.3.4, of respective degrees l and p , we deduce a Lie bracket given componentwise as $[f, g] = ([f, g]_N)_{N \geq 0}$ from our earlier description. This is the A_∞ -coderivation defined by

$$[f, g]_N = \sum_{r+s+t=N} (1^{\otimes Ar} \otimes_A f_s \otimes_A 1^{\otimes At}) g_{N-s+1} - (-1)^{lp} \sum_{r+s+t=N} (1^{\otimes Ar} \otimes_A g_s \otimes_A 1^{\otimes At}) f_{N-s+1}.$$

The above operation $[\ , \]$ extends immediately to an operation on the collection of tuples $(b_n)_{n \geq 0}$ of maps b_n in $\text{Hom}_{\mathcal{Q}}(C, C^{\otimes A^n})$ such that all the maps b_n have the same degree, and we understand from our earlier description that for any such b the tuple $[\delta, b]$ is an A_∞ -coderivation. We call such A_∞ -coderivations *inner*, as before.

2.4. The cup product on A_∞ -coderivations. Our main Theorem 4.1.1 below gives an isomorphism of Gerstenhaber algebras. Towards this end, we define a cup product on A_∞ -coderivations. We first reiterate the definition of function composition in $\text{Der}_{\mathcal{Q}}(\Omega C)$, expressed componentwise:

Definition 2.4.1. Let (C, δ) be an A_∞ -coalgebra. For $f \in \text{Der}_{\mathcal{Q}}(\Omega C)_n$ and $g \in \text{Der}_{\mathcal{Q}}(\Omega C)_m$, we define $f \circ g \in \text{Hom}_{\mathcal{Q}}(C, \Omega C)_{n+m}$ by the equality

$$(f \circ g)_i = \sum_{r+s+t=i} (1^{\otimes Ar} \otimes_A f_s \otimes_A 1^{\otimes At}) g_{r+t+1}.$$

In terms of the elements of $\text{Der}_{\mathcal{Q}}(\Omega C)$, this definition can be reformulated in the following way. If f and g are tuples of maps in $\text{Hom}_{\mathcal{Q}}(C, \Omega C)$ that correspond to $\bar{f}, \bar{g} \in \text{Der}_{\mathcal{Q}}(\Omega C)$ respectively, then $f \circ g$ corresponds to the element $\overline{f \circ g} \in \text{Der}_{\mathcal{Q}}(\Omega C)$ such that $(\overline{f \circ g} - \bar{f} \circ \bar{g})|_C = 0$.

Lemma 2.4.2. For $f \in \text{Der}_{\mathcal{Q}}(\Omega C)_n$, $g \in \text{Der}_{\mathcal{Q}}(\Omega C)_m$ and $h \in \text{Der}_{\mathcal{Q}}(\Omega C)_p$ one has

$$(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{mn} ((g \circ f) \circ h - g \circ (f \circ h)).$$

Thus, $(\text{Der}_{\mathcal{Q}}(\Omega C), \circ)$ is a graded pre-Lie algebra.

Proof. Let as before f, g and h correspond to the elements \bar{f}, \bar{g} , and $\bar{h} \in \text{Der}_{\mathcal{Q}}(\Omega C)$ respectively. Since $\bar{f} \circ \bar{g} - (-1)^{mn} \bar{g} \circ \bar{f} \in \text{Der}_{\mathcal{Q}}(\Omega C)$, one sees that $(f \circ g - (-1)^{mn} g \circ f) \circ h$ corresponds to the element $H \in \text{Der}_{\mathcal{Q}}(\Omega C)$ such that $H|_C = (\bar{f} \circ \bar{g} \circ \bar{h} - (-1)^{mn} \bar{g} \circ \bar{f} \circ \bar{h})|_C$. On the other hand, $f \circ (g \circ h)$ and $g \circ (f \circ h)$ correspond to the elements $F, G \in \text{Der}_{\mathcal{Q}}(\Omega C)$ such that $(F - \bar{f} \circ \bar{g} \circ \bar{h})|_C = (G - \bar{g} \circ \bar{f} \circ \bar{h})|_C = 0$. Thus, $H = F - (-1)^{mn} G$, and the required equality follows. \square

We next define an A_∞ -algebra structure on $\text{Der}_{\mathcal{Q}}(\Omega C)$ that is induced by convolution of the A_∞ -coalgebra structure and the algebra structure on ΩC :

Definition 2.4.3. Let (C, δ) be an A_∞ -coalgebra. Let $f_i \in \text{Der}_{\mathcal{Q}}(\Omega C)_{p_i}$ for $1 \leq i \leq l$, $l \geq 2$. Then we define $m_l(f_1, \dots, f_l) \in \text{Der}_{\mathcal{Q}}(\Omega C)_{\sum_{i=1}^l p_i + 1}$ by the equality

$$m_l(f_1, \dots, f_l)_n = (-1)^{\sum_{i=1}^l (i-1)p_i} \sum_{\substack{\sum_{k=0}^l i_k + \sum_{k=1}^l j_k = n}} (1^{\otimes A i_0} \otimes_A (f_1)_{j_1} \otimes_A \dots \otimes_A (f_l)_{j_l} \otimes_A 1^{\otimes A i_l}) \delta_{\sum_{k=0}^l i_k + l}.$$

Since we will be mainly interested in the case $l = 2$, let us write down that, for $f \in \text{Der}_{\mathcal{Q}}(\Omega C)_n$ and $g \in \text{Der}_{\mathcal{Q}}(\Omega C)_p$, the element $f \smile g = m_2(f, g) \in \text{Der}_{\mathcal{Q}}(\Omega C)_{n+m+1}$ is defined by the equality

$$(f \smile g)_i = (-1)^p \sum_{r+s+t+u+v=i} (1^{\otimes Ar} \otimes_A f_s \otimes_A 1^{\otimes At} \otimes_A g_u \otimes_A 1^{\otimes Av}) \delta_{r+t+v+2}.$$

Lemma 2.4.4. For any l -tuple $(f_1, \dots, f_l) \in \text{Der}_{\mathcal{Q}}(\Omega C)_{p_1} \times \dots \times \text{Der}_{\mathcal{Q}}(\Omega C)_{p_l}$, one has

$$\begin{aligned} [\delta, m_l(f_1, \dots, f_l)] &= \sum_{i=1}^l (-1)^{\sum_{k=1}^{i-1} (p_k+1)} m_l(f_1, \dots, f_{i-1}, [\delta, f_i], f_{i+1}, \dots, f_l) \\ &\quad - (-1)^{\sum_{k=1}^l p_k} \sum_{j=2}^{l-1} \sum_{i=0}^{l-j} (-1)^{i+j} \sum_{k=i+j+1}^l p_k m_{l-j+1}(f_1, \dots, f_i, m_j(f_{i+1}, \dots, f_{i+j}), f_{i+j+1}, \dots, f_l). \end{aligned}$$

Proof. Let us introduce

$$\begin{aligned} \mathbf{f} &= m_l(f_1, \dots, f_l), \quad \mathbf{f}_i^\delta = m_{l+1}(f_1, \dots, f_i, \delta, f_{i+1}, \dots, f_l), \\ \mathbf{f}_i^{\delta \circ} &= m_l(f_1, \dots, f_{i-1}, \delta \circ f_i, f_{i+1}, \dots, f_l), \quad \text{and } \mathbf{f}_i^{\circ \delta} = m_l(f_1, \dots, f_{i-1}, f_i \circ \delta, f_{i+1}, \dots, f_l). \end{aligned}$$

Let us look at the left hand side $[\delta, \mathbf{f}] = \delta \circ \mathbf{f} + (-1)^{\sum_{k=1}^l p_k} \mathbf{f} \circ \delta$ of the required equality. Direct calculations show that

$$\begin{aligned} \delta \circ \mathbf{f} &= \sum_{i=0}^l (-1)^{\sum_{k=1}^l p_k + i} \mathbf{f}_i^\delta + \sum_{i=1}^l (-1)^{\sum_{k=1}^{i-1} (p_k+1)} \mathbf{f}_i^{\delta \circ} = \sum_{i=0}^l (-1)^{\sum_{k=1}^l p_k + i} \mathbf{f}_i^\delta \\ &\quad - \sum_{i=1}^l (-1)^{\sum_{k=1}^i (p_k+1)} \mathbf{f}_i^{\circ \delta} + \sum_{i=1}^l (-1)^{\sum_{k=1}^{i-1} (p_k+1)} m_l(f_1, \dots, f_{i-1}, [\delta, f_i], f_{i+1}, \dots, f_l). \end{aligned}$$

Let us introduce the element $H \in \text{Der}_{\mathcal{Q}}(\Omega C)_{\sum_{i=1}^l p_i + 2}$ by the equality

$$H_n = (-1)^{\sum_{k=1}^l k p_k} \sum_{\substack{\sum_{k=0}^l i_k + \sum_{k=1}^l j_k = n}} (1^{\otimes A^{i_0}} \otimes_A (f_1)_{j_1} \otimes_A \dots \otimes_A (f_l)_{j_l} \otimes_A 1^{\otimes A^{i_l}}) (\delta \circ \delta)_{\sum_{k=0}^l i_k + l}.$$

It follows from $\delta \circ \delta = 0$ that $H = 0$. On the other hand, it is not difficult to see that

$$\begin{aligned} H &= \sum_{i=0}^l (-1)^{\sum_{k=1}^l p_k + i} \mathbf{f}_i^\delta - \sum_{i=1}^l (-1)^{\sum_{k=1}^i (p_k+1)} \mathbf{f}_i^{\delta \circ} + (-1)^{\sum_{k=1}^l p_k} \mathbf{f} \circ \delta \\ &\quad + (-1)^{\sum_{k=1}^l p_k} \sum_{j=2}^{l-1} \sum_{i=0}^{l-j} (-1)^{i+j} \sum_{k=i+j+1}^l p_k m_{l-j+1}(f_1, \dots, f_i, m_j(f_{i+1}, \dots, f_{i+j}), f_{i+j+1}, \dots, f_l). \end{aligned}$$

Subtracting the obtained expression from the formula for $\delta \circ \mathbf{f}$ obtained above, we get the required equality. \square

Lemma 2.4.4 says that $\text{Der}_{\mathcal{Q}}(\Omega C)$ is an A_∞ -algebra under the convolution operations $\{m_n\}_n$. As with A_∞ -algebras more generally, the subspace of cocycles is not an A_∞ -subalgebra immediately, but the cocycles are closed under the (possibly non-associative) multiplication m_2 .

Corollary 2.4.5. The operation $\smile = m_2$ induces a degree preserving operation on $\text{Coder}_{\mathcal{Q}}^\infty(C)[1]$. Moreover, $\text{Inn}_{\mathcal{Q}}^\infty(C)[1]$ is an ideal in $\text{Coder}_{\mathcal{Q}}^\infty(C)[1]$ with respect to the operation \smile , and the algebra $\left((\text{Coder}_{\mathcal{Q}}^\infty(C)/\text{Inn}_{\mathcal{Q}}^\infty(C))[1], \smile \right)$ is associative.

Proof. Everything except the associativity follows directly from Lemma 2.4.4 with $l = 2$. The stated associativity follows from Lemma 2.4.4 with $l = 3$. \square

Lemma 2.4.6. *Let (C, δ) be an A_∞ -coalgebra. Then*

$$(-1)^p(f \circ (g \circ \delta) - (f \circ g) \circ \delta) = f \smile g - (-1)^{(p+1)(n+1)}g \smile f$$

for any $f \in \text{End}_{\mathcal{Q}}(C)_n$ and $g \in \text{End}_{\mathcal{Q}}(C)_p$. In particular, if f and g are A_∞ -coderivations, then $f \smile g + (-1)^{(p+1)(n+1)}g \smile f = (-1)^n[\delta, f \circ g] \in \text{Inn}_{\mathcal{Q}}^\infty(C)$.

Proof. The first equality easily follows from the definitions of the operations \circ and \smile . The second equality follows from the first one and Lemma 2.4.2. \square

Theorem 2.4.7. *If (C, δ) is an A_∞ -coalgebra, then $\text{Inn}_{\mathcal{Q}}^\infty(C)$ is an ideal in $\text{Coder}_{\mathcal{Q}}^\infty(C)$ with respect to the operations \smile and $[\ , \]$. Moreover, $\left((\text{Coder}_{\mathcal{Q}}^\infty(C)/\text{Inn}_{\mathcal{Q}}^\infty(C))[1], \smile, [\ , \] \right)$ is a Gerstenhaber algebra (in general, nonunital).*

Proof. It was shown above that \smile and $[\ , \]$ have the required degrees, and $[\ , \]$ satisfies the conditions of a graded Lie algebra.

The required associativity and graded commutativity of \smile follow from Lemma 2.4.6 and Corollary 2.4.5. Thus, it remains to prove the Poisson identity.

For an l -tuple $(f_1, \dots, f_l) \in \text{Der}_{\mathcal{Q}}(\Omega C)_{p_1} \times \dots \times \text{Der}_{\mathcal{Q}}(\Omega C)_{p_l}$ and $h \in \text{Der}_{\mathcal{Q}}(\Omega C)_q$, we define $m_l^h(f_1, \dots, f_l) \in \text{Der}_{\mathcal{Q}}(\Omega C)_{\sum_{i=1}^l p_i + q}$ by the equality

$$m_l^h(f_1, \dots, f_l)_n = \sum_{\substack{\sum_{k=0}^l i_k + \sum_{k=1}^l j_k = n}} (1^{\otimes A^{i_0}} \otimes_A (f_1)_{j_1} \otimes_A \dots \otimes_A (f_l)_{j_l} \otimes_A 1^{\otimes A^{i_l}}) h_{\sum_{k=0}^l i_k + l}.$$

Let us take $f \in \text{Der}_{\mathcal{Q}}(\Omega C)_n$, $g \in \text{Der}_{\mathcal{Q}}(\Omega C)_r$ and $h \in \text{Der}_{\mathcal{Q}}(\Omega C)_q$. Direct calculations show that

$$\begin{aligned} [f \smile g, h] &= (f \smile g) \circ h + (-1)^{(r+n+1)(q+1)}m_3(h, f, g) - (-1)^{(r+n+1)q}(h \circ f) \smile g \\ &\quad - (-1)^{r(q+1)}m_3(f, h, g) - (-1)^{r^q}f \smile (h \circ g) - (-1)^q m_3(f, g, h) = (f \smile g) \circ h \\ &\quad + (-1)^{(r+n+1)(q+1)}m_3(h, f, g) - (-1)^{r(q+1)}m_3(f, h, g) - (-1)^q m_3(f, g, h) \\ &\quad - (-1)^{(r+1)q}(f \circ h) \smile g - f \smile (g \circ h) + (-1)^{(r+1)q}[f, h] \smile g + f \smile [g, h], \end{aligned}$$

$$\begin{aligned} [\delta, m_2^h(f, g)] &= m_3^h(\delta, f, g) + m_2^h(\delta \circ f, g) + (-1)^n m_3^h(f, \delta, g) + (-1)^n m_2^h(f, \delta \circ g) \\ &\quad + (-1)^{r+n} m_3^h(f, g, \delta) - (-1)^{r+n+q} m_2^h(f, g) \circ \delta, \end{aligned}$$

$$\begin{aligned} m_2^{\delta \circ h}(f, g) &= (-1)^{r+n} m_3^h(\delta, f, g) + (-1)^r m_2^h(f \circ \delta, g) + (-1)^r m_3^h(f, \delta, g) \\ &\quad + m_2^h(f, g \circ \delta) + m_3^h(f, g, \delta) + (-1)^r (f \smile g) \circ h, \end{aligned}$$

$$\begin{aligned} m_2^{h \circ \delta}(f, g) &= (-1)^{(r+n)q+n} m_3(h, f, g) + (-1)^{r(q+1)}(f \circ h) \smile g \\ &\quad + (-1)^{(r+1)q} m_3(f, h, g) + (-1)^{r+q} f \smile (g \circ h) + (-1)^r m_3(f, g, h) + m_2^h(f, g) \circ \delta. \end{aligned}$$

Hence,

$$[f \smile g, h] - f \smile [g, h] - (-1)^{(r+1)l}[f, h] \smile g = (-1)^r m_2^{[\delta, h]}(f, g) \\ + (-1)^n m_2^h([\delta, f], g) + m_2^h(f, [\delta, g]) - (-1)^n [\delta, m_2^h(f, g)].$$

In particular, if $f, g, h \in \text{Coder}_{\mathcal{Q}}^{\infty}(C)$, then

$$[f \smile g, h] - f \smile [g, h] - (-1)^{(r+1)l}[f, h] \smile g = (-1)^{n+1} [\delta, m_2^h(f, g)] \in \text{Inn}_{\mathcal{Q}}^{\infty}(C).$$

□

Remark 2.4.8. One should note the similarity between our definition of the higher multiplications m_l on $\text{Der}_{\mathcal{Q}}(\Omega C)$ and the brace operations which appear on usual Hochschild cochains [25, Section 3]. Indeed, the m_l are essentially given by certain dualized brace operations.

3. A_{∞} -COALGEBRA STRUCTURE ON A PROJECTIVE RESOLUTION

In this section, we show how to put an A_{∞} -coalgebra structure on a projective A -bimodule resolution of the \mathbf{k} -algebra A . We will consider resolutions of A that are shifted: Let $P = P'[-1]$ where P' is a projective A -bimodule resolution of A , so that $P_1 = P'_0$ and there is an A -bimodule chain map $\mu_P : P \rightarrow A$ of degree -1 that induces an isomorphism $H_*(P) \cong A[-1]$. We will call this complex P a *shifted projective bimodule resolution* of A .

Let X and Y be A -bimodule complexes and $f : X \rightarrow Y$ a homogeneous A -bimodule homomorphism (equivalently an A^e -module homomorphism where $A^e = A \otimes A^{op}$). Denote the differential on $\text{Hom}_{\mathcal{Q}}(X, Y)$ by ∂ , that is

$$\partial(f) = d_Y f - (-1)^{|f|} f d_X.$$

We will write also $f \sim g$ if f is homotopic to g .

3.1. Existence of an A_{∞} -coalgebra structure. We show that any projective resolution of the A^e -module A has an A_{∞} -coalgebra structure of a particular type.

Theorem 3.1.1. *Let (P, μ_P) be a shifted bimodule resolution of A . Then P admits a weakly counital A_{∞} -coalgebra structure δ with $\delta_1 = d_P$.*

Proof. There exists such a δ_2 by uniqueness of projective resolutions, and the fact that $P \otimes_A P$ is a (shift of a) resolution of A . Consider the exact sequence of complexes

$$0 \rightarrow K_n \rightarrow \text{Hom}_{\mathcal{Q}}(P, P^{\otimes A^n}) \xrightarrow{(\mu^{\otimes A^n})_*} \text{Hom}_{\mathcal{Q}}(P, A)[-n] \rightarrow 0,$$

where K_n is simply the kernel of $(\mu^{\otimes A^n})_*$. Note that since $(\mu_P^{\otimes A^n})_*$ is a quasi-isomorphism, the complex K_n is acyclic. One sees this by considering the long exact sequence on cohomology.

Suppose that $n > 2$ and we have already constructed δ_i for $i < n$ in such a way that $(\delta \circ \delta)_i = 0$ for all $i < n$, and $\delta_i \in K_i$ for $i > 2$. Let us prove that there exists δ_n such that $(\delta \circ \delta)_n = 0$. The required equality is

$$(3.1.2) \quad 0 = \sum_{i=1}^n \delta_i \circ \delta_{n-i+1} = \sum_{i=2}^{n-1} \delta_i \circ \delta_{n-i+1} + \partial(\delta_n).$$

One applies $\mu_P^{\otimes A^n}$ to see that $\sum_{i=2}^{n-1} \delta_i \circ \delta_{n-i+1}$ is in K_n . Since K_n is acyclic, the desired δ_n exists if and only if this sum is a cocycle.

Applying ∂ to $\sum_{i=2}^{n-1} \delta_i \circ \delta_{n-i+1}$, we obtain

$$\begin{aligned}
\partial \left(\sum_{i=2}^{n-1} \delta_i \circ \delta_{n-i+1} \right) &= \sum_{i=2}^{n-1} \partial(\delta_i) \circ \delta_{n-i+1} - \sum_{i=2}^{n-1} \delta_i \circ \partial(\delta_{n-i+1}) \\
&= - \sum_{i=2}^{n-1} \left(\sum_{j=2}^{i-1} \delta_j \circ \delta_{i-j+1} \right) \circ \delta_{n-i+1} + \sum_{i=2}^{n-1} \delta_i \circ \left(\sum_{j=2}^{n-i} \delta_j \circ \delta_{n-i-j+2} \right) \\
&= \sum_{\substack{i+j+k=n+2, \\ i,j,k \geq 2}} \left(-(\delta_i \circ \delta_j) \circ \delta_k + \delta_i \circ (\delta_j \circ \delta_k) \right) = 0.
\end{aligned}$$

Hence, $\sum_{i=2}^{n-1} \delta_i \circ \delta_{n-i+1}$ is a chain map from P to $P^{\otimes An}$ of degree 2, and we can find the desired solution δ_n to equation (3.1.2). \square

Of course, the theorem tells us that any shifted resolution $\mu_P : P \rightarrow A$ can be endowed with an A_∞ -coalgebra structure such that μ_P provides a weak counit on P . Throughout this work, when we speak of an A_∞ -coalgebra structure on a shifted resolution, we mean one such that the structure map $\mu_P : P \rightarrow A$ provides a weak counit.

3.2. Examples of A_∞ -coalgebra structures. We next give some examples, beginning with bar resolutions having coassociative coalgebra structures (i.e., A_∞ -coalgebra structures δ with $\delta_m = 0$ for all $m \geq 3$) in Example 3.2.1. Then we give in Example 3.2.2 an A_∞ -coalgebra structure on a projective bimodule resolution that includes some nonzero maps δ_m for $m \geq 3$.

Example 3.2.1. Let P be the shifted bar resolution of the \mathbf{k} -algebra A :

$$P : \quad \dots \xrightarrow{d} A^{\otimes 4} \xrightarrow{d} A^{\otimes 3} \xrightarrow{d} A \otimes A$$

where $\otimes = \otimes_{\mathbf{k}}$ and $d(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}$ for all $a_0, \dots, a_{n+1} \in A$, with augmentation $\mu_P : A \otimes A \rightarrow A$ given by the multiplication on A . That is, $P_1 = A \otimes A$, $P_0 = A^{\otimes 3}$, $P_{-1} = A^{\otimes 4}$, and so on. The standard diagonal map Δ_2 for P may be modified by including some signs to obtain $\delta_2 : P \otimes_A P \rightarrow P$ defined by

$$\delta_2(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i (a_0 \otimes \dots \otimes a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \dots \otimes a_{n+1})$$

for all $a_0, \dots, a_{n+1} \in A$. We may take $\delta_i = 0$ for all $i \geq 3$.

Example 3.2.2. Let $A = \mathbf{k}[x]/(x^n)$, $n > 2$. (The case $n = 2$ is Koszul; see Example 4.7.1 and Section 5.3.) Consider

$$P : \quad \dots \xrightarrow{\cdot v} A \otimes A \xrightarrow{\cdot u} A \otimes A \xrightarrow{\cdot v} A \otimes A \xrightarrow{\cdot u} A \otimes A,$$

where $\otimes = \otimes_{\mathbf{k}}$, $u = x \otimes 1 - 1 \otimes x$ and $v = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \dots + 1 \otimes x^{n-1}$, with augmentation $\mu_P : A \otimes A \rightarrow A$ given by the multiplication on A . For each i , let e_i denote the element $1 \otimes 1$ in $A \otimes A$ in degree $1 - i$. By convention, we set $e_i = 0$ whenever $i < 0$. A

diagonal map Δ_2 is given in [17, Section 5] in case $n = p$ and $\text{char}(\mathbf{k}) = p$. We modify it to define δ_2 here for all $n \geq 3$, independent of characteristic:

$$\begin{aligned}\delta_2(e_{2i}) &= \sum_{j+l=i} e_{2j} \otimes_A e_{2l} - \sum_{\substack{j+l=i \\ a+b+c=n-2}} x^a e_{2j+1} \otimes_A x^b e_{2l-1} x^c, \\ \delta_2(e_{2i+1}) &= \sum_{j+l=i} e_{2j} \otimes_A e_{2l+1} - \sum_{j+l=i} e_{2j+1} \otimes_A e_{2l}.\end{aligned}$$

For $3 \leq m \leq n$, let

$$\begin{aligned}\delta_m(e_{2i}) &= (-1)^{m+1} \sum_{\substack{j_1+\dots+j_m=i-1 \\ a_1+\dots+a_{m+1}=n-m}} x^{a_1} e_{2j_1+1} \otimes_A x^{a_2} e_{2j_2+1} \otimes_A \cdots \otimes_A x^{a_{m-1}} e_{2j_{m-1}+1} \otimes_A x^{a_m} e_{2j_m+1} x^{a_{m+1}}, \\ \delta_m(e_{2i+1}) &= 0 \text{ for all } i, \text{ and } \delta_m = 0 \text{ for all } m > n.\end{aligned}$$

This is an A_∞ -coalgebra structure on P , as may be checked via some lengthy calculations.

4. A_∞ -CODERIVATIONS ON A PROJECTIVE RESOLUTION

In this section, we work with an arbitrary shifted bimodule resolution P (i.e. $P = P'[-1]$ for some projective A -bimodule resolution P' of A). We will show that Hochschild cocycles on P give rise to A_∞ -coderivations and Hochschild coboundaries correspond to inner A_∞ -coderivations. We will give some examples and prove our main theorem.

4.1. Main Theorem. We will prove the following isomorphism of Gerstenhaber algebras. The notation $\text{Coder}_\infty^\infty(P)$ and $\text{Inn}_\infty^\infty(P)$ were introduced in Section 2.1, and the associative and Lie products on the quotient space $\text{Coder}_\infty^\infty(P)/\text{Inn}_\infty^\infty(P)$ were introduced in Sections 2.1 and 2.4.

Theorem 4.1.1. *Let P be a shifted projective bimodule resolution of A with A_∞ -coalgebra structure δ . There is an isomorphism of Gerstenhaber algebras*

$$\text{HH}^*(A) \cong (\text{Coder}_\infty^\infty(P)/\text{Inn}_\infty^\infty(P))[1].$$

The proof is deferred to Section 4.5 and consists of several steps, relying on some lemmas and preliminary results. Our method of proof is very direct. Indeed, one can view Theorem 4.1.1 as a statement about the existence of solutions to an infinite sequence of equations. We show directly that, in fact, one can produce solutions to all of the given equations.

In Lemma 4.3.1 we explicitly identify the inner A_∞ -coderivations among all A_∞ -coderivations. We then give in Theorem 4.4.6 a construction of an A_∞ -coderivation corresponding to a Hochschild cocycle, and show the connection to the homotopy lifting method of [26]. For this purpose, in Sections 4.3–4.5, we fix a shifted projective bimodule resolution P of A and a weakly counital A_∞ -coalgebra structure on P . By Theorem 3.1.1, such a structure always exists.

4.2. A remark on methods. Let us consider an A -bimodule resolution $\mu_{P'} : P' \rightarrow A$. For any positive n , the tensor power $\mu_{P'}^{\otimes n} : (P')^{\otimes n} \rightarrow A$ is also a resolution of A , and there are quasi-isomorphisms

$$1^{\otimes m} \otimes_A \mu_{P'}^{\otimes (n-m)} : P^{\otimes n} \rightarrow P^{\otimes m}$$

for each $m \leq n$. In the shifted case, where $P = P'[-1]$ and $\mu_P = \mu_{P'}$ after forgetting the grading, we still have quasi-isomorphisms $1^{\otimes_A m} \otimes_A \mu_P^{\otimes_A(n-m)} : P^{\otimes_A n} \rightarrow P^{\otimes_A m}$.

Since each positive power $P^{\otimes_A w}$ is bounded above and projective, the induced maps

$$(1^{\otimes_A m} \otimes \mu^{\otimes_A(n-m)})_* : \text{Hom}_{\mathcal{Q}}(P^{\otimes_A w}, P^{\otimes_A n}) \rightarrow \text{Hom}_{\mathcal{Q}}(P^{\otimes_A w}, P^{\otimes_A m})$$

are all quasi-isomorphisms. In particular, each map $(\mu^{\otimes_A n})_*$ to $\text{Hom}_{\mathcal{Q}}(P^{\otimes_A w}, A)$ is a quasi-isomorphism. So, if we would like to know whether a given cocycle f in a hom complex $\text{Hom}_{\mathcal{Q}}(P^{\otimes_A w}, P^{\otimes_A n})$ is a coboundary, it suffices to check whether f is a coboundary after composing with some power of μ_P .

We will make copious use of this fact throughout the section in order to prove that a given map f admits a bounding morphism ϕ , i.e. ϕ with $\partial(\phi) = f$. Most often, we will check after composing with the highest possible power $\mu^{\otimes_A n}$. For example, we have the following useful lemma.

Lemma 4.2.1. *Let P be a shifted bimodule resolution of A which is endowed with a weakly counital A_∞ -coalgebra structure, as in Theorem 3.1.1. Then the two maps $(1 \otimes \mu_P)\delta_2$ and $-(\mu_P \otimes 1)\delta_2 : P \rightarrow P$ are homotopic to the identity.*

Proof. Recall that δ_2 satisfies the equation $(\mu_P \otimes \mu_P)\delta_2 = \mu_P$. Hence the endomorphisms $(1 \otimes \mu_P)\delta_2$, $-(\mu_P \otimes 1)\delta_2$, and the identity map 1_P all agree after applying $\mu_P : P \rightarrow A$. It follows that all of the given maps are homotopic in $\text{End}_{\mathcal{Q}}(P)$. \square

4.3. Inner coderivations. Recall that $P = (P, \mu_P)$ is a shifted bimodule resolution with a corresponding weakly unital A_∞ -coalgebra structure. We first give a characterization of inner A_∞ -coderivations.

Lemma 4.3.1. *Suppose that $\alpha \in \text{Coder}_{\mathcal{Q}}^\infty(P)$. Then $\alpha \in \text{Inn}_{\mathcal{Q}}^\infty(P)$ if, and only if, $\alpha_0 \sim 0$.*

Proof. If $\alpha \in \text{Inn}_{\mathcal{Q}}^\infty(P)$, then by definition $\alpha = [\delta, \beta]$ for some $\beta \in \text{Der}_{\mathcal{Q}}(\Omega C)$, and hence $\alpha_0 = \partial(\beta_0) \sim 0$.

Suppose now that $\alpha_0 \sim 0$. Then $\alpha_0 = \partial(\beta_0)$ for some β_0 . Define $\beta^{(0)} \in \text{Der}_{\mathcal{Q}}(\Omega C)$ by $\beta_0^{(0)} = \beta_0$ and $\beta_{>0}^{(0)} = 0$. Then $(\alpha - [\delta, \beta^{(0)}])_0 = 0$.

Let $\alpha^{(1)} = \alpha - [\delta, \beta^{(0)}]$ so that $\alpha_0^{(1)} = 0$. We may assume that α is homogeneous of some degree n . We will prove by induction on m that if $\alpha^{(m)} \in \text{Coder}_{\mathcal{Q}}^\infty(P)_n$ is such that $\alpha_k^{(m)} = 0$ for $0 \leq k \leq m-1$, then there is some $\beta^{(m)} \in \text{Der}_{\mathcal{Q}}(\Omega C)$ such that $\beta_i^{(m)} = 0$ for $i \notin \{m-1, m\}$ and $(\alpha^{(m)} - [\delta, \beta^{(m)}])_k = 0$ for $0 \leq k \leq m$. The assertion of the lemma will follow as then $\alpha = [\delta, \sum_{i=0}^\infty \beta^{(i)}]$. Let us prove this fact.

For simplicity, we will drop superscripts, writing α in place of $\alpha^{(m)}$: Assume that $\alpha \in \text{Coder}_{\mathcal{Q}}^\infty(P)_n$ is such that $\alpha_k = 0$ for $0 \leq k \leq m-1$. Applying Definition 2.3.4, we thus have $\partial(\alpha_m) = 0$ and

$$\partial(\alpha_{m+1}) + \sum_{i=0}^{m-1} (1^{\otimes_A i} \otimes_A \delta_2 \otimes_A 1^{\otimes_A(m-i-1)})\alpha_m - (-1)^n (\alpha_m \otimes_A 1 + 1 \otimes_A \alpha_m)\delta_2 = 0.$$

Suppose that $2 \nmid m$. Compose this equality with $\mu_P^{\otimes A(m+1)}$. Since $\mu_P^{\otimes A(m+1)}\partial(\alpha_{m+1})$ is a coboundary, we get

$$\begin{aligned} 0 &\sim \mu_P^{\otimes A(m+1)} \sum_{i=0}^{m-1} (1^{\otimes A^i} \otimes_A \delta_2 \otimes_A 1^{\otimes A(m-i-1)}) \alpha_m - (-1)^n \mu_P^{\otimes A(m+1)} (\alpha_m \otimes_A 1 + 1 \otimes_A \alpha_m) \delta_2 \\ &= \sum_{i=0}^{m-1} (-1)^{m-i-1} (\mu_P^{\otimes A^i} \otimes_A \mu_P \otimes_A \mu_P^{\otimes A(m-i-1)}) \alpha_m - \mu_P^{\otimes Am} \alpha_m (1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2 \\ &= \mu_P^{\otimes Am} \alpha_m - \mu_P^{\otimes Am} \alpha_m (1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2. \end{aligned}$$

Now $(1 \otimes_A \mu_P) \delta_2$ and $-(\mu_P \otimes_A 1) \delta_2$ are both homotopic to the identity map. It follows that $(1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2 \sim 2$. Thus $\mu_P^{\otimes Am} \alpha_m \sim 0$, and consequently $\alpha_m \sim 0$. Now we can define $\beta_i = 0$ for $i \neq m$ and choose β_m such that $\alpha_m = \partial(\beta_m)$.

Suppose now that $2 \mid m$. Here we still have $\partial(\alpha_m) = 0$. Define $\beta \in \text{Der}_{\mathcal{Q}}(\Omega C)$ by taking $\beta_i = 0$ for $i \neq m-1$ and choose β_{m-1} in such a way that $\partial(\beta_{m-1}) = 0$ and $\mu_P^{\otimes A(m-1)} \beta_{m-1} + \mu_P^{\otimes Am} \alpha_m = 0$. We can choose $\beta_{m-1} = -(1^{\otimes A(m-1)} \otimes_A \mu_P) \alpha_m$, for example. In this case $(\alpha - [\delta, \beta])_k = 0$ for $0 \leq k \leq m-1$. Let us consider

$$H = \alpha_m - \sum_{i=0}^{m-2} (1^{\otimes A^i} \otimes_A \delta_2 \otimes_A 1^{\otimes A(m-i-2)}) \beta_{m-1} - (-1)^n (\beta_{m-1} \otimes_A 1 + 1 \otimes_A \beta_{m-1}) \delta_2.$$

Applying ∂ to the above equation, since $\partial(\alpha_m) = 0$, $\partial(\beta_{m-1}) = 0$, and $\partial(\delta_2) = 0$, we find that $\partial(H) = 0$. Applying $\mu_P^{\otimes Am}$ to the above definition of H , since $(\mu_P \otimes_A \mu_P) \delta_2 = \mu_P$ by hypothesis, we have

$$\mu_P^{\otimes Am} H \sim \mu_P^{\otimes Am} \alpha_m - \mu_P^{\otimes A(m-1)} \beta_{m-1} + \mu_P^{\otimes A(m-1)} \beta_{m-1} (1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2.$$

Since $\mu_P^{\otimes Am} \alpha_m = -\mu_P^{\otimes A(m-1)} \beta_{m-1}$ and $(1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2 \sim 2$, it follows that $\mu_P^{\otimes Am} H \sim 0$. Thus we can choose β_m in such a way that $(\alpha - [\delta, \beta'])_m = 0$, for β' with $\beta'_i = 0$ whenever $i \notin \{m-1, m\}$, $\beta'_{m-1} = \beta_{m-1}$ and $\beta'_m = \beta_m$. \square

4.4. Lifting cocycles to A_∞ -coderivations. In order to prove our main Theorem 4.1.1, we will need to prove that any Hochschild cocycle on P can be lifted to an A_∞ -coderivation of (P, δ) . For this purpose we need some additional technical lemmas and notation.

First, let us introduce for each $t \geq 1$ and integer r a map $\psi_t^r \in \text{End}_{\mathcal{Q}}(P)$ of degree $1-t$. We take

$$\psi_t^r = \sum_{u=0}^t (-1)^{ru} (\mu_P^{\otimes Au} \otimes_A 1 \otimes_A \mu_P^{\otimes A(t-u)}) \delta_{t+1}$$

for $t > 1$, or $t = 1$ and r even, and otherwise, for all l ,

$$\psi_1^{2l+1} = (1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2 - 1.$$

These maps will be used in Lemma 4.4.4, an essential step in the proof of Theorem 4.4.6 that states a Hochschild cocycle can be lifted to an A_∞ -coderivation. We first prove some properties of the maps ψ_t^r .

Lemma 4.4.1. *The above functions ψ_t^r satisfy the equations*

$$\partial(\psi_t^r) + \sum_{s=1}^{t-1} (-1)^s \psi_s^r \psi_{t-s}^{r-s} = 0.$$

Proof. Let us introduce $\mu_k^l = \mu_P^{\otimes_A k} \otimes_A 1 \otimes_A \mu_P^{\otimes_A (l-k)}$. Then, applying (2.3.2), we have

$$(4.4.2) \quad \partial(\psi_t^r) = \sum_{u=0}^t \sum_{s=1}^{t-1} \sum_{v=0}^{t-s} (-1)^{ru+t+1} \mu_u^t (1^{\otimes_A v} \otimes_A \delta_{s+1} \otimes_A 1^{\otimes_A (t-s-v)}) \delta_{t-s+1}.$$

Note that

$$\begin{aligned} & \mu_u^t (1^{\otimes_A v} \otimes_A \delta_{s+1} \otimes_A 1^{\otimes_A (t-s-v)}) \delta_{t-s+1} \\ &= \begin{cases} (-1)^{vs+t-s} (\mu_{u-v}^s \delta_{s+1}) (\mu_v^{t-s} \delta_{t-s+1}), & \text{if } v \leq u \leq v+s, \\ (-1)^{t-v} \mu_{u-1}^{t-1} \delta_t, & \text{if } s=1 \text{ and } 0 \leq v \leq u-2, \\ (-1)^{t-v-1} \mu_u^{t-1} \delta_t, & \text{if } s=1 \text{ and } u+1 \leq v \leq t-1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Substituting the obtained values into (4.4.2) and considering separately the cases $2 \mid t$ and $2 \nmid t$, we find that $\partial(\psi_t^r) = \sum_{s=1}^{t-1} (-1)^{s+1} \psi_s^r \psi_{t-s}^{r-s}$. \square

We next use the maps ψ_t^r in the definition of additional maps ϕ_t that will be used in the proof of Lemma 4.4.4.

Lemma 4.4.3. *There exist maps $\phi_t : P \rightarrow P$ ($t \geq 1$) such that $|\phi_t| = -t$ and*

$$\partial(\phi_t) = \psi_t^{t-1} - \sum_{s=1}^{t-1} \psi_s^{t-1} \phi_{t-s}$$

for all $t \geq 1$.

Proof. For $t=0$ we simply need to show that ψ_1^0 is a coboundary. Since δ_2 is a cocycle and $\psi_1^0 = (1 \otimes_A \mu_P + \mu_P \otimes_A 1) \delta_2$, we have $\partial(\psi_1^0) = 0$. Since $\mu_P \psi_1^0 = 0$, ψ_1^0 is indeed a coboundary, and there is a map ϕ_1 satisfying the required conditions. Let us now proceed by induction on t . Suppose that $t \geq 2$ and we have constructed ϕ_i for $1 \leq i \leq t-1$ satisfying the required conditions. Let

$$\Phi = \psi_t^{t-1} - \sum_{s=1}^{t-1} \psi_s^{t-1} \phi_{t-s}.$$

Since Φ has degree $1-t < 0$, it is enough to verify that it is a chain map. Using Lemma 4.4.1, we get

$$\begin{aligned} \partial(\Phi) &= \sum_{u=1}^{t-1} (-1)^{u+1} \psi_u^{t-1} \psi_{t-u}^{t-u-1} - \sum_{s=1}^{t-1} \sum_{u=1}^{s-1} (-1)^{u+1} \psi_u^{t-1} \psi_{s-u}^{t-u-1} \phi_{t-s} + \sum_{s=1}^{t-1} (-1)^s \psi_s^{t-1} \partial(\phi_{t-s}) \\ &= \sum_{s=1}^{t-1} (-1)^{s+1} \psi_s^{t-1} (\psi_{t-s}^{t-s-1} - \sum_{r=1}^{t-s-1} \psi_r^{t-s-1} \phi_{t-s-r} - \partial(\phi_{t-s})) = 0. \end{aligned}$$

\square

Lemma 4.4.4. *Let i be any nonnegative integer. Suppose that $\alpha \in \text{Der}_{\mathcal{Q}}(\Omega P)_n$ is such that $[\delta, \alpha]_j = 0$ for $j \leq 2i$. Then $\mu_P^{\otimes_A (2i+1)} [\delta, \alpha]_{2i+1} \sim 0$.*

Proof. We apply the weakly counital property to find

$$\begin{aligned}
\mu_P^{\otimes A(2i+1)}[\delta, \alpha]_{2i+1} &= \mu_P^{\otimes A(2i+1)}\partial(\alpha_{2i+1}) - (-1)^n \sum_{j=0}^{2i} (-1)^{(2i+1-j)n} \mu_P^{\otimes Aj} \alpha_j \psi_{2i+1-j}^{2i-j} \\
(4.4.5) \quad &\sim - \sum_{j=0}^{2i} (-1)^{(2i-j)n} \mu_P^{\otimes Aj} \alpha_j \psi_{2i+1-j}^{2i-j}.
\end{aligned}$$

Let us introduce $A_t = \sum_{j=0}^{2i-t} (-1)^{(2i-j)n} \mu_P^{\otimes Aj} \alpha_j \left(\psi_{2i+1-j}^{2i-j} - \sum_{s=2i-t-j+1}^{2i-j} \psi_s^{2i-j} \phi_{2i+1-j-s} \right)$, where the functions ϕ_u are as in Lemma 4.4.3. Note that $-A_0$ is the final term in the above sequence (4.4.5), and $A_k = 0$ for $k > 2i$. By considering the ranges of the sums involved in the definition of the A_t , one calculates the difference

$$\begin{aligned}
A_t - A_{t+1} &= (-1)^{tn} \mu_P^{\otimes A(2i-t)} \alpha_{2i-t} \left(\psi_{t+1}^t - \sum_{s=1}^t \psi_s^t \phi_{t-s+1} \right) + \sum_{j=0}^{2i-t-1} (-1)^{(2i-j)n} \mu_P^{\otimes Aj} \alpha_j \psi_{2i-t-j}^{2i-j} \phi_{t+1} \\
&= (-1)^{tn} \mu_P^{\otimes A(2i-t)} \alpha_{2i-t} \partial(\phi_{t+1}) + \sum_{j=0}^{2i-t-1} (-1)^{(2i-j)n} \mu_P^{\otimes Aj} \alpha_j \psi_{2i-t-j}^{2i-j} \phi_{t+1} \\
&= (-1)^{(t+1)n} \mu_P^{\otimes A(2i-t)} \partial(\alpha_{2i-t} \phi_{t+1}) \sim 0
\end{aligned}$$

for $0 \leq t \leq 2i$. Thus, $\mu_P^{\otimes A(2i+1)}[\delta, \alpha]_{2i+1} \sim -A_0 \sim -A_{2i+1} = 0$. \square

Now we are ready to prove existence of A_∞ -coderivations corresponding to Hochschild cocycles.

Theorem 4.4.6. *Let P be any shifted projective bimodule resolution of A , with A_∞ -coalgebra structure as in Theorem 3.1.1. For any chain map $f : P \rightarrow A$ of degree n , there exists an A_∞ -coderivation α_f of the same degree with $(\alpha_f)_0 = f$.*

Proof. Let us set $(\alpha_f)_0 = f$, and note that $(\alpha_f \circ \delta)_0 = f \circ \delta_1 = fd_{P[-1]} = 0$ since f is a cocycle. On the other hand, $(\delta \circ \alpha_f)_0 = \delta_0 \circ (\alpha_f)_1 = 0$ for any choice of $(\alpha_f)_1$ since $\delta_0 = 0$.

Now assume that $i > 0$ and $(\alpha_f)_k$ has been defined for $0 \leq k \leq i-1$ in such a way that $[\delta, \alpha_f]_k = 0$ for $0 \leq k \leq i-1$. We are going to prove that it is possible to define $(\tilde{\alpha}_f)_k$ for $0 \leq k \leq i$ in such a way that $(\tilde{\alpha}_f)_k = (\alpha_f)_k$ for $k < i-1$ and $[\delta, \tilde{\alpha}_f]_k = 0$ for $0 \leq k \leq i$ with any choice of $(\tilde{\alpha}_f)_k$ for $k > i$. Since $[\delta, [\delta, \alpha]] = 0$ for any $\alpha \in \text{Der}_{\mathcal{Q}}(\Omega P)$, we have

$$0 = [\delta, [\delta, \alpha_f]]_i = (\delta \circ [\delta, \alpha_f])_i - (-1)^n ([\delta, \alpha_f] \circ \delta)_i = \partial([\delta, \alpha_f]_i).$$

Note now that $[\delta, \alpha_f]_i = \partial((\alpha_f)_i) + \Phi_{\alpha_0, \dots, \alpha_{i-1}}$, where $\Phi_{\alpha_0, \dots, \alpha_{i-1}} : P \rightarrow P^{\otimes Ai}$ is a map depending on $(\alpha_f)_k$ with $0 \leq k \leq i-1$. Hence, we have $\partial(\Phi_{\alpha_0, \dots, \alpha_{i-1}}) = 0$ and $(\alpha_f)_i$ such that $[\delta, \alpha_f]_i = 0$ exists if and only if $\mu_P^{\otimes Ai} \Phi_{\alpha_0, \dots, \alpha_{i-1}} \sim 0$.

In the case $2 \nmid i$ we have $\mu_P^{\otimes Ai} [\delta, \alpha_f]_i \sim 0$ for any choice of $(\alpha_f)_k$ ($k \geq i$) by Lemma 4.4.4, that is $\mu_P^{\otimes Ai} \Phi_{\alpha_0, \dots, \alpha_{i-1}} \sim \mu_P^{\otimes Ai} [\delta, \alpha_f]_i \sim 0$.

Suppose now that $2 \mid i$. Let us introduce $u = (1^{\otimes A(i-1)} \otimes_A \mu_P) \Phi_{\alpha_0, \dots, \alpha_{i-1}} : P \rightarrow P^{\otimes A(i-1)}$. By construction, we have $\mu_P^{\otimes A(i-1)} u = \mu_P^{\otimes Ai} \Phi_{\alpha_0, \dots, \alpha_{i-1}}$ and $\partial(u) = 0$. Let us set $(\tilde{\alpha}_f)_k = (\alpha_f)_k$

for $0 \leq k \leq i-2$ and $(\tilde{\alpha}_f)_{i-1} = (\alpha_f)_{i-1} + u$. We still have $[\delta, \tilde{\alpha}_f]_k = 0$ for $0 \leq k \leq i-1$, since $\partial(\Phi_{\tilde{\alpha}_0, \dots, \tilde{\alpha}_{i-1}}) = 0$. On the other hand, direct calculations show that

$$\begin{aligned}
& \mu_P^{\otimes A^i} \Phi_{\tilde{\alpha}_0, \dots, \tilde{\alpha}_{i-1}} \\
&= \mu_P^{\otimes A^i} \Phi_{\alpha_0, \dots, \alpha_{i-1}} + \mu_P^{\otimes A^i} \sum_{j=0}^{i-2} (1^{\otimes Aj} \otimes_A \delta_2 \otimes_A 1^{\otimes A(i-2-j)})u - (-1)^n \mu_P^{\otimes A^i} (u \otimes_A 1 + 1 \otimes_A u) \delta_2 \\
&= \mu_P^{\otimes A^{i-1}} u + \sum_{j=0}^{i-2} (-1)^{i-j} \mu_P^{\otimes A(i-1)} u - \mu_P^{\otimes A(i-1)} u (1 \otimes_A \mu_P + (-1)^{i-1} \mu_P \otimes_A 1) \delta_2 \\
&= 2\mu_P^{\otimes A(i-1)} u - \mu_P^{\otimes A(i-1)} u (1 \otimes_A \mu_P - \mu_P \otimes_A 1) \delta_2 \\
&\sim 2\mu_P^{\otimes A(i-1)} u - 2\mu_P^{\otimes A(i-1)} u = 0.
\end{aligned}$$

The argument above shows that there is $(\tilde{\alpha}_f)_i$ such that $[\delta, \tilde{\alpha}_f]_k = 0$ for $0 \leq k \leq i$. \square

For some resolutions, the picture is simpler than this general case. For example, for the bar resolution (Example 3.2.1), A_∞ -coderivations α are in fact essentially just coderivations, that is one may always take α_1 to be a coderivation and $\alpha_i = 0$ for $i \geq 2$. For a resolution P having a coassociative diagonal map (such as the Koszul resolutions of Section 5.3 below), we may take $\delta_i = 0$ for $i \geq 3$, leading to simplified conditions defining the α_i : For all N ,

$$(4.4.7) \quad \partial(\alpha_N) = - \sum_{i+j=N-2} (1^{\otimes Ai} \otimes_A \delta_2 \otimes_A 1^{\otimes Aj}) \alpha_{N-1} + (-1)^{|\alpha|} (1 \otimes_A \alpha_{N-1} + \alpha_{N-1} \otimes_A 1) \delta_2.$$

Written another way,

$$\begin{aligned}
& \sum_{i+j=N-1} (1^{\otimes Ai} \otimes_A \delta_1 \otimes_A 1^{\otimes Aj}) \alpha_N + \sum_{i+j=N-2} (1^{\otimes Ai} \otimes_A \delta_2 \otimes_A 1^{\otimes Aj}) \alpha_{N-1} \\
&= (-1)^{|\alpha|} \alpha_N \delta_1 + (-1)^{|\alpha|} (1 \otimes_A \alpha_{N-1} + \alpha_{N-1} \otimes_A 1) \delta_2.
\end{aligned}$$

4.5. Homotopy liftings and proof of Theorem 4.1.1. Let us now recall the definition of a homotopy lifting from [26], adjusted here to fit our grading and sign choices. To prove our main Theorem 4.1.1, we will need a connection between A_∞ -coderivations and homotopy liftings, given by Lemma 4.5.4 below. First we define homotopy liftings.

Definition 4.5.1. Let P be a shifted projective bimodule resolution of A with A_∞ -coalgebra structure δ , and let $f : P \rightarrow A$ be a chain map. A map $\phi_f : P \rightarrow P$ is called a *homotopy lifting* of the pair (f, δ_2) if $\partial(\phi_f) = (f \otimes_A 1 + 1 \otimes_A f) \delta_2$ and $\mu_P \phi_f + f \phi \sim 0$ for some $\phi : P \rightarrow P$ of degree -1 such that

$$(4.5.2) \quad \partial(\phi) = (\mu_P \otimes_A 1 + 1 \otimes_A \mu_P) \delta_2.$$

Remark 4.5.3. Note that a map $\phi : P \rightarrow P$ of degree -1 satisfying (4.5.2) exists [26]. Moreover, any two such maps are homotopic. Thus, the condition $\mu_P \phi_f + f \phi \sim 0$ holds for some ϕ satisfying (4.5.2) if and only if it holds for any ϕ satisfying (4.5.2).

Lemma 4.5.4. *Let P be a shifted projective bimodule resolution of A with weakly counital A_∞ -coalgebra structure δ . Let α be an A_∞ -coderivation on P of degree n . Then $(-1)^n \alpha_1$ is a homotopy lifting for (α_0, δ_2) .*

Proof. The equality $\partial(\alpha_1) = (-1)^n(\alpha_0 \otimes_A 1 + 1 \otimes_A \alpha_0)\delta_2$ is equivalent to $[\delta, \alpha]_1 = 0$. So we need only find ϕ solving (4.5.2) and $\mu_P\alpha_1 + \alpha_1\phi \sim 0$.

Note now that

$$(4.5.5) \quad \begin{aligned} 0 &= \mu_P^{\otimes A^2}[\delta, \alpha]_2 \sim (\mu_P \otimes_A \mu_P)\delta_2\alpha_1 - (-1)^n\alpha_0\psi_2^2 - \mu_P\alpha_1(1 \otimes_A \mu_P - \mu_P \otimes_A 1)\delta_2 \\ &= \mu_P\alpha_1 - (-1)^n\alpha_0\psi_2^2 - \mu_P\alpha_1(1 \otimes_A \mu_P - \mu_P \otimes_A 1)\delta_2. \end{aligned}$$

There exists $u : P \rightarrow P$ of degree -1 such that $\partial(u) = (1 \otimes_A \mu_P - \mu_P \otimes_A 1)\delta_2 - 2$, by Lemma 4.2.1. We will show that $\phi = \psi_2^2 - (1 \otimes_A \mu_P + \mu_P \otimes_A 1)\delta_2u$ provides the desired function.

We have

$$\begin{aligned} \mu_P\alpha_1(1 \otimes_A \mu_P - \mu_P \otimes_A 1)\delta_2 &= 2\mu_P\alpha_1 + \mu_P\alpha_1\partial(u) \\ &\sim 2\mu_P\alpha_1 - (-1)^n\alpha_0(1 \otimes_A \mu_P + \mu_P \otimes_A 1)\delta_2u. \end{aligned}$$

Thus, one multiplies (4.5.5) by $(-1)^{n+1}$ and applies the above equivalence to find

$$0 \sim (-1)^n\mu_P\alpha_1 + \alpha_0(\psi_2^2 - (1 \otimes_A \mu_P + \mu_P \otimes_A 1)\delta_2u) = (-1)^n\mu_P\alpha_1 + \alpha_0\phi.$$

Finally, the equality

$$\partial(\phi) = \partial(\psi_2^2 - (1 \otimes_A \mu_P + \mu_P \otimes_A 1)\delta_2u) = (\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2$$

follows from the expression

$$\partial(\psi_2^2) = \psi_1^2\psi_1^1 = -(\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2 + (\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2(1 \otimes_A \mu_P - \mu_P \otimes_A 1)\delta_2$$

of Lemma 4.4.1 and the calculation

$$\begin{aligned} -\partial((\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2u) &= -(\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2\partial(u) \\ &= 2(\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2 - (\mu_P \otimes_A 1 + 1 \otimes_A \mu_P)\delta_2(1 \otimes_A \mu_P - \mu_P \otimes_A 1)\delta_2. \end{aligned}$$

□

Now we will prove our main Theorem 4.1.1.

Proof of Theorem 4.1.1. By Theorem 4.4.6, for any Hochschild cocycle f on P , there exists an A_∞ -coderivation α_f such that $(\alpha_f)_0 = f$. By Lemma 4.3.1, this construction gives a well-defined bijection F from $\text{HH}^*(A)$ to $(\text{Coder}_\infty^\infty(P)/\text{Inn}_\infty^\infty(P))[1]$. Since

$$(\alpha_f \smile \alpha_g)_0 = (f \otimes_A g)\delta_2 \quad \text{and} \quad [\alpha_f, \alpha_g]_0 = f(\alpha_g)_1 - (-1)^{(|f|-1)(|g|-1)}g(\alpha_f)_1,$$

F is an isomorphism of Gerstenhaber algebras: The former expression induces precisely the cup product on $\text{HH}^*(A)$. The latter expression induces the Gerstenhaber bracket on $\text{HH}^*(A)$ by [26, Theorem 4] and Lemma 4.5.4. □

4.6. Coderivations on the tensor coalgebra. Next we make a direct comparison of A_∞ -coderivations on the bar resolution $B(A)$ of A with coderivations on the tensor coalgebra of A . This comparison will clarify how we view A_∞ -coderivations on an arbitrary bimodule resolution P as a generalization of such coderivations when they are cocycles, and thus our results as generalizing those of Stasheff [21].

We take $P = B(A)$ here, with A_∞ -coalgebra structure as in Example 3.2.1. Let α be an A_∞ -coderivation on $B(A)$ for which $\alpha_n = 0$ whenever $n \geq 2$ so that $\alpha = \alpha_0 + \alpha_1$. By Lemma 4.5.4, up to a sign, α_1 is a homotopy lifting for (α_0, δ_2) . View $T(A)$ as embedded in $B(A)$ via $A^{\otimes n} \cong k \otimes A^{\otimes n} \otimes k \hookrightarrow A^{\otimes(n+2)}$ for each n . Then $\alpha_1|_{T(A)}$ is a coderivation that is in the kernel of the differential on $T(A)$. Conversely, given a coderivation on $T(A)$ that is a cocycle, composing with projection onto A yields an element of $\text{Hom}_k(T(A), A)$ which

can be extended to an A -bimodule homomorphism α_0 on $B(A)$. We may extend the original coderivation and denote the corresponding function to A by α_1 , and the sum $\alpha_0 \pm \alpha_1$ is thus an A_∞ -coderivation on $B(A)$. Under this correspondence, brackets are preserved by their definitions as graded commutators, and so our Theorem 4.1.1 generalizes Stasheff's description of the Gerstenhaber bracket as a graded commutator of coderivations on the tensor coalgebra [21].

4.7. Examples. We now give some examples. In Example 4.7.1 we consider a Koszul algebra and its coassociative Koszul resolution, while in Examples 4.7.2 and 4.7.3 we consider resolutions with nonzero higher A_∞ -coalgebra maps.

Example 4.7.1. Let \mathbf{k} be an arbitrary field and let $A = \mathbf{k}[x]/(x^2)$. Let P be the following resolution of A (cf. Example 3.2.2):

$$P : \quad \cdots \xrightarrow{\cdot v} A \otimes A \xrightarrow{\cdot u} A \otimes A \xrightarrow{\cdot v} A \otimes A \xrightarrow{\cdot u} A \otimes A,$$

where $u = x \otimes 1 - 1 \otimes x$ and $v = x \otimes 1 + 1 \otimes x$. For each i , let e_i denote the element $1 \otimes 1$ in $A \otimes A$. An embedding of P into the bar resolution on A (defined in Example 3.2.1) is given by

$$\iota(e_i) = 1 \otimes x \otimes x \otimes \cdots \otimes x \otimes 1 \quad \text{in } A^{\otimes(i+2)}$$

for each i . An induced map $\delta_2 : P \rightarrow P \otimes_A P$ is thus given by

$$\delta_2(e_i) = \sum_{j+l=i} (-1)^j e_j \otimes e_l$$

for all i . We may take $\delta_n = 0$ for $n \geq 3$. One may check directly that δ gives P the structure of an A_∞ -coalgebra. See also Section 5.3 for a general construction for Koszul algebras.

Now consider the Hochschild 1-cocycle on A given by $\alpha_0(e_1) = x$ (and $\alpha_0(e_i) = 0$ for $i \neq 1$). Let

$$\alpha_1(e_i) = -ie_i$$

and $\alpha_m(e_i) = 0$ for all i and all $m \geq 2$. Straightforward calculations show that (α_i) is an A_∞ -coderivation. Consider the Hochschild 2-cocycle given by $\beta_0(e_2) = x$ (and $\beta_0(e_i) = 0$ for $i \neq 2$). Let

$$\beta_1(e_{2i}) = -e_{2i-1} \quad \text{and} \quad \beta_1(e_{2i+1}) = 0$$

for all i . Let $\beta_m(e_i) = 0$ for all $m \geq 2$ and all i . Then (β_i) is an A_∞ -coderivation.

Example 4.7.2. Let \mathbf{k} be an arbitrary field and let $A = \mathbf{k}[x]/(x^3)$. We take resolution P with A_∞ -coalgebra structure δ as given in Example 3.2.2. Consider the Hochschild 1-cocycle given by $\alpha_0(e_1) = x$ (and $\alpha_0(e_i) = 0$ for $i \neq 1$). Let

$$\begin{aligned} \alpha_1(e_{2i}) &= -3ie_{2i}, \\ \alpha_1(e_{2i+1}) &= (-3i - 1)e_{2i+1} \end{aligned}$$

for all i and $\alpha_m(e_i) = 0$ for all i and all $m \geq 2$. Straightforward calculations show that (α_i) is an A_∞ -coderivation. Consider the Hochschild 2-cocycle given by $\beta_0(e_2) = x$ (and $\beta_0(e_i) = 0$ for $i \neq 2$). Let

$$\beta_1(e_{2i}) = -e_{2i-1} \quad \text{and} \quad \beta_1(e_{2i+1}) = 0$$

for all i . Let $\beta_m(e_i) = 0$ for all $m \geq 2$ and all i . Then (β_i) is an A_∞ -coderivation.

Next is our first example of an A_∞ -coderivation α with α_2 nonzero. See also Example 5.2.5.

Example 4.7.3. Let \mathbf{k} be a field of characteristic 3, and $A = \mathbf{k}[x]/(x^3)$. We again take resolution P and A_∞ -coalgebra structure as in Example 3.2.2. Consider the Hochschild 1-cocycle given by $\alpha_0(e_1) = x^2$ (and $\alpha_0(e_i) = 0$ for $i \neq 1$). Let

$$\begin{aligned}\alpha_1(e_{2i}) &= 0, \\ \alpha_1(e_{2i+1}) &= -e_{2i+1}x - xe_{2i+1}, \\ \alpha_2(e_{2i}) &= 0, \\ \alpha_2(e_{2i+1}) &= \sum_{j+k=i} e_{2j+1} \otimes_A e_{2k+1}\end{aligned}$$

for all i , and $\alpha_m(e_i) = 0$ for all $m > 2$ and all i . Straightforward calculations and induction on m show that (α_i) is an A_∞ -coderivation.

We end this section with some examples in which Gerstenhaber brackets are found using these techniques.

Example 4.7.4. Let \mathbf{k} be an arbitrary field. First let $A = \mathbf{k}[x]/(x^2)$, as in Example 4.7.1. By Lemma 4.5.4 and the Gerstenhaber bracket formula of [26] that is recalled in the proof of Theorem 4.1.1 above, the cocycles α_0, β_0 of Example 4.7.1 have Gerstenhaber bracket

$$[\alpha, \beta]_0 = -\alpha_0\beta_1 + \beta_0\alpha_1 = \beta_0,$$

as may be determined by evaluating on e_2 .

Next let $A = \mathbf{k}[x]/(x^3)$, as in Example 4.7.2. The cocycles α_0, β_0 of Example 4.7.2 similarly have Gerstenhaber bracket $[\alpha, \beta]_0 = \beta_0$.

5. CONNECTED GRADED ALGEBRAS

In this section, we will show that many connected graded algebras have the property that the A_∞ -coderivations on P corresponding to Hochschild cocycles are locally finite, that is, they factor through the direct sum $\bigoplus_{n \geq 0} P^{\otimes n}$. These are the connected graded algebras satisfying a certain finiteness condition, (HF) below, which is always satisfied by Koszul algebras and Noetherian connected graded algebras. We will also give some explicit formulas and further results on A_∞ -coderivations for Koszul algebras.

5.1. Cohomology and connected graded algebras. Let A be a connected graded \mathbf{k} -algebra, so that $A = \bigoplus_{n \geq 0} A_n$ with $A_0 \cong \mathbf{k}$. We suppose additionally that A satisfies the following homological finiteness condition:

(HF) $\dim \text{Ext}_A^i(\mathbf{k}, \mathbf{k}) < \infty$ for each i .

We note that any connected graded algebra which satisfies (HF) must be finitely generated, as $\text{Ext}_A^1(\mathbf{k}, \mathbf{k})$ is identified with the dual of a minimal space of generators (see e.g. [15, Section 5]). We also note that if A is Noetherian, or Koszul, then A satisfies the condition (HF).

Let B be a connected graded algebra. We are thinking here of $B = A$ or $B = A^e$. Then any finitely generated graded B -module M admits a minimal (graded) projective resolution $P(M) \rightarrow M$, with $P(M)_{-i} = B \otimes V_{-i}$ for some graded vector space V_{-i} . The minimality assumption gives $\text{Ext}_B^i(M, \mathbf{k}) = (V_{-i})^*$ (cf. [2, Section 2]). We refer to the gradings on $P(M)$ induced by the grading on B as the *internal grading*. When $P(M)$ is finitely generated in each degree, we also refer to the induced grading on each $\text{Ext}_B^i(M, \mathbf{k})$ as the internal grading.

Recall that for any augmented algebra A we have $\text{Ext}_{A^e}^*(A, \mathbf{k}) \cong \text{Ext}_A^*(\mathbf{k}, \mathbf{k})$ [27, Lemma 9.1.9]. In particular, for our connected graded algebra A the condition (HF) implies $\dim \text{Ext}_{A^e}^i(A, \mathbf{k}) < \infty$ for each i , and hence the minimal bimodule resolution $P(A) \rightarrow A$ of A is such that each $P(A)_{-i}$ is finitely generated.

The important point here is that any A satisfying (HF) admits a graded bimodule resolution P' with the following property:

(HF') Each P'_{-i} is finitely generated, in internal degrees $i \leq d \leq l(i)$, for some positive integer $l(i)$ which depends on i .

(The lower bound i is obtained by induction and the fact that A is generated in positive degree.) Indeed, the existence of a graded bimodule resolution P' satisfying (HF') is equivalent to condition (HF). We note that such a P' is free in each degree, by a Nakayama type argument. We say a shifted bimodule resolution P satisfies (HF') if the associated non-shifted bimodule resolution P' satisfies (HF').

5.2. Local finiteness of A_∞ -coderivations. Let P be a graded, shifted, bimodule resolution of A . Then $T(P)$ has an induced internal grading, and we may speak of homogenous endomorphisms of $T(P)$ with respect to the internal grading (of some particular degree). For the completed algebra $\widehat{T}(P)$, we say a continuous endomorphism of $\widehat{T}(P)$ is homogenous with respect to the internal grading if each of the truncations $\widehat{T}(P)/I^{n'} \rightarrow \widehat{T}(P)/I^n$ is homogenous, where I denotes the ideal generated by P .

We fix now A a connected graded algebra satisfying the finiteness condition (HF), and P a shifted bimodule resolution satisfying (HF'). Recall that elements in $\text{Der}(\Omega)$, for a topological algebra Ω , are by definition continuous.

Lemma 5.2.1. *Any degree n derivation $f \in \text{Der}_{\mathcal{Q}}(\widehat{T}(P))$ which is homogenous with respect to the internal grading is locally finite. Rather, any such f is in the image of the completion map $\text{Der}_{\mathcal{Q}}(T(P)) \rightarrow \text{Der}_{\mathcal{Q}}(\widehat{T}(P))$.*

Proof. Let d denote the degree of f with respect to the internal grading. It suffices to show that for each i there is a corresponding N such that $f_k : P \rightarrow P^{\otimes A^k}$ vanishes on P_{-i+1} whenever $k > N$. Let l be the maximal internal degree of a generator for P_{-i+1} , i.e. the maximal nonvanishing degree of the reduction $k \otimes_{A^e} P_{-i+1}$. The restriction of f_k to P_{-i+1} corresponds to a degree $n - k + 1$ map from the unshifted resolution $P'_{-i} \rightarrow (P')^{\otimes A^k}$, which we also denote by f_k by abuse of notation. Since each P'_{-j} is generated in internal degrees at least j , the space $((P')^{\otimes A^k})_{-i+n-k+1}$ vanishes in degrees less than $k + i - n - 1$. We can therefore take $N = d + l - i + n + 1$ to find that $f_k|_{P'_{-i}}$ vanishes for all $k > N$. Thus f is locally finite.

The local finiteness implies that $f|_P : P \rightarrow \widehat{T}(P)$ has image in $T(P)$, and thus defines a unique derivation $F : T(P) \rightarrow T(P)$ such that the completion \widehat{F} recovers f . \square

In our construction of the A_∞ -coalgebra structure on P , in the proof of Proposition 3.1.1, we are free to assume that all of the δ_i preserve the internal grading. By the previous lemma, the A_∞ -coalgebra structure on P will therefore be locally finite, and we may consider the discrete cobar construction $\Omega^{\text{disc}} P$ of Definition 2.1.4. We have the completion map

$$(5.2.2) \quad \text{cpl} : \text{Der}_{\mathcal{Q}}(\Omega^{\text{disc}} P) \rightarrow \text{Der}_{\mathcal{Q}}(\Omega P), \quad f \mapsto \widehat{f},$$

which is a map of dg Lie algebras. The completion map is also a map of dg algebras, under the convolution product of Section 2.4.

Let us take $\text{Coder}_{lf}^\infty(P)$ and $\text{Inn}_{lf}^\infty(P)$ to be the cocycles and coboundaries in $\text{Der}_\mathcal{Q}(\Omega^{\text{disc}}P)$, respectively.

Theorem 5.2.3. *For A a connected graded algebra satisfying (HF), and P a shifted resolution satisfying (HF'), the completion map (5.2.2) is a quasi-isomorphism. Consequently, there is an identification of Gerstenhaber algebras*

$$(5.2.4) \quad \text{HH}^*(A) \cong \text{Coder}_{lf}^\infty(P)/\text{Inn}_{lf}^\infty(P)[1].$$

Proof. We have the diagram

$$\begin{array}{ccc} \text{Der}_\mathcal{Q}(\Omega^{\text{disc}}P) & \xrightarrow{\text{cpl}} & \text{Der}_\mathcal{Q}(\Omega P) \\ & \searrow z_{lf} & \swarrow z \\ & \text{Hom}_\mathcal{Q}(P, A) & \end{array},$$

where the maps to $\text{Hom}_\mathcal{Q}(P, A)$ take f to f_0 . The map z is a quasi-isomorphism by Theorem 4.1.1. (Or, more precisely, by Lemma 4.3.1 and Theorem 4.4.6.) To see that cpl is a quasi-isomorphism, it suffices to show that z_{lf} is a quasi-isomorphism. In this case, the identification (5.2.4) will follow by Theorem 4.1.1 and the fact that cpl is a dg (Lie) algebra map.

To see that z_{lf} is a quasi-isomorphism, we note that $\text{Hom}_\mathcal{Q}^n(P, A)$ is spanned by homogenous elements with respect to the internal grading, and for homogenous $\alpha_0 \in \text{Hom}_\mathcal{Q}^n(P, A)$ we can produce a lift $\alpha \in \text{Der}_\mathcal{Q}(\Omega P)$ which is also homogenous with respect to the internal degree. Indeed, our construction of such a lift in the proof of Theorem 4.4.6 can be done in a homogeneous manner. In this case α is locally finite, by Lemma 5.2.1, and we therefore have a lift of α_0 in $\text{Coder}_{lf}^\infty(P)$. This shows that any cocycle in $\text{Hom}_\mathcal{Q}(P, A)$ lifts to a cocycle in $\text{Der}_\mathcal{Q}(\Omega^{\text{disc}}P)$.

To see that z_{lf} is a quasi-isomorphism we must show that if α is a locally finite A_∞ -coderivation such that α_0 is a coboundary, then α admits a locally finite bounding element β .

Since each P_{-i} is generated in finitely many internal degrees, any $f \in \text{Der}_\mathcal{Q}(\Omega P)$ can be written uniquely as a convergent sum $\sum_{k=-\infty}^\infty f(k)$ such that $f(k)$ is homogenous of internal degree k and the composites $\pi_n f(k) : P_{-i} \rightarrow \bigoplus_{l \leq n} P^{\otimes l}$ vanish for all but finitely many k . We have $[\delta, \sum_k f(k)] = \sum_k [\delta, f(k)]$, and $[\delta, f(k)] = [\delta, f](k)$ since δ is homogenous of degree 0. So, an element α is bounded by some β if and only if $\alpha(k) = [\delta, \beta(k)]$ for each k .

Now, let $P(i)$ denote the homogeneous degree i portion of P with respect to the internal degree. Since P_{-i+1} is generated in a finite number of degrees which are at least i , we have $P(i) = \bigoplus_{l \leq i+1} P_l(i)$, and one can conclude from Lemma 5.2.1 that a derivation f is locally finite if and only if for each i there is an integer $N(i)$ such that $f|_{P(i)} = \sum_{k=-N(i)}^{k=N(i)} f(k)|_{P(i)}$. For a locally finite cocycle α , we have $\alpha|_{P(i)} = \sum_{k=-M(i)}^{M(i)} \alpha(k)|_{P(i)}$ for some integers $M(i)$, and we claim that if α_0 is a coboundary then there exists a bounding element β for α such that

$$\beta|_{P(i)} = \sum_{k=-N(i)}^{N(i)} \beta(k)|_{P(i)}, \quad \text{for } N(i) = \max\{M(j) : 0 \leq j \leq i\}.$$

Indeed, one can use homogeneity of δ and freeness of P to prove that such a $\beta = \sum_{n=1}^{\infty} \beta_n$ exists via induction on n (cf. the proof of Lemma 4.3.1). Such a β provides the desired locally finite bounding element. \square

We next give a small example illustrating local finiteness.

Example 5.2.5. Let \mathbf{k} be a field of arbitrary characteristic, and $A = \mathbf{k}[x]$. Let K be the resolution of A as an A^e -module given by

$$K : \quad 0 \rightarrow A^e \xrightarrow{\cdot(x \otimes 1 - 1 \otimes x)} A^e.$$

Then K embeds into the bar resolution: Identify the degree 0 component $K_0 = A^e$ with that in the bar resolution (3.2.1), and in degree 1, send $e_1 := 1 \otimes 1$ in $K_1 = A^e$ to $1 \otimes x \otimes 1$ in $A^{\otimes 3}$ in the bar resolution. The coalgebra structure δ_2 on the bar resolution restricts to a coalgebra structure for K . We find that

$$\begin{aligned} \delta_2(e_0) &= e_0 \otimes e_0, \\ \delta_2(e_1) &= e_0 \otimes e_1 - e_1 \otimes e_0. \end{aligned}$$

Let n be any positive integer and consider the Hochschild 1-cocycle given by $\alpha_0(e_1) = x^n$ (and $\alpha_0(e_0) = 0$). Let

$$\begin{aligned} \alpha_1(e_1) &= - \sum_{a+b=n-1} x^a e_1 x^b, \\ \alpha_2(e_1) &= \sum_{a+b+c=n-2} x^a e_1 \otimes_A x^b e_1 x^c, \\ &\vdots \\ \alpha_i(e_1) &= (-1)^i \sum_{a_1+\dots+a_{i+1}=n-i} x^{a_1} e_1 \otimes_A x^{a_2} e_1 \otimes_A \dots \otimes_A x^{a_i} e_1 x^{a_{i+1}}, \\ &\vdots \\ \alpha_n(e_1) &= (-1)^n e_1 \otimes_A \dots \otimes_A e_1, \end{aligned}$$

$\alpha_m(e_0) = 0$ for all m , and $\alpha_m(e_1) = 0$ for all $m > n$. Lengthy calculations show that (α_i) is an A_∞ -coderivation; see also the general formula (5.3.1) below for Koszul algebras.

5.3. Koszul algebras. Now we focus on Koszul algebras and show that the grading on such algebras appears in the structure of A_∞ -coderivations corresponding to Hochschild cocycles. These A_∞ -coderivations will all be locally finite, that is, they take values in $\bigoplus_{k \geq 0} P^{\otimes_A k}$, by Lemma 5.2.1. We recall definitions of Koszul algebras and their resolutions next. Our Koszul algebras A will always be graded and connected. As discussed above, the grading on A imposes an additional grading on $\text{Ext}_A^*(\mathbf{k}, \mathbf{k})$ so that for each i , $\text{Ext}_A^i(\mathbf{k}, \mathbf{k}) = \bigoplus_j \text{Ext}_A^{i,j}(\mathbf{k}, \mathbf{k})$.

By definition, A is a *Koszul algebra* if $\text{Ext}_A^{i,j}(\mathbf{k}, \mathbf{k}) = 0$ whenever $i \neq j$. This implies that A is quadratic, that is all its relations are in degree 2. It is equivalent to exactness of the following sequence, so that it is a projective A^e -module resolution of A [13]: Let $V = A_1$ and let $R \subset V \otimes V$ be the subspace of quadratic relations, where $\otimes = \otimes_{\mathbf{k}}$, so that $A \cong T_{\mathbf{k}}(V)/(R)$. The Koszul resolution is defined from this information:

$$K : \quad \dots \longrightarrow A \otimes R \otimes A \longrightarrow A \otimes V \otimes A \longrightarrow A \otimes A$$

with $K_0 = A \otimes A$, $K_1 = A \otimes V \otimes A$, and for each $i \geq 2$, $K_i = A \otimes K'_i \otimes A$ where

$$K'_i = \bigcap_{j+l=i-2} (V^{\otimes j} \otimes R \otimes V^{\otimes l}).$$

Each K_i embeds canonically into $A^{\otimes(i+2)}$, and K is in this way a subcomplex of the bar resolution of Example 3.2.1. It is shown in [3] that through this embedding, there is a coassociative diagonal map Δ_2 on K . This map may be modified by including some signs as in Example 3.2.1 to obtain a map δ_2 on K :

$$\delta_2(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i (a_0 \otimes \cdots \otimes a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1})$$

for all $a_0, \dots, a_{n+1} \in A$ on the bar resolution (it may be checked that δ_2 in fact restricts to a map from K_n to K_{n-1}). Take $\delta_i = 0$ for all $i \geq 3$.

We set $P_i := K_{1-i}$, so that P is a shifted projective resolution of A as an A^e -module. We will use the A_∞ -coalgebra structure on P as given by the maps δ_n above. In particular, we have $P_i = 0$ and $\delta_i = 0$ for $i > 2$.

Note that the grading on A induces a grading on P . Thus, P has a $\mathbb{Z} \times \mathbb{Z}$ -grading such that elements from $A_p \otimes K'_i \otimes A_q$ have degree $(i, p + i + q)$. A homogeneous element with respect to this grading can be represented by a map from K'_n to A_p for some integers n and p . The corresponding element has degree $(n, p - n)$ and belongs to $\mathrm{HH}^n(A)$.

Let us take an element of $\mathrm{HH}^*(A)$ that is represented by a map from K'_n to A_p and construct $\alpha \in \mathrm{Coder}_\infty(P)$ representing it following the prescription in the proof of Theorem 4.4.6. Start with α_0 , a map from P_{1-n} to A . For each k , we construct $\alpha_k|_{P_{-i}}$ by induction on i . For each $s \geq 0$, denote by $\pi_s : V^{\otimes s} \rightarrow A_s$ the canonical projection and let $\tau : V \rightarrow K_1$ be the canonical embedding.

Let us set $\alpha_k|_{P_{-i}} = 0$ for $i < n - 1$. Let us now choose some vector space basis x_1, \dots, x_l of K'_n . Represent $\alpha_0(x_t)$ by an element $y_t \in V^{\otimes p}$, via a choice of section $A_p \rightarrow V^{\otimes p}$ of π_p . We will use the canonical identification of $(K_1 \otimes A) \otimes_A (A \otimes K_1)$ with $K_1 \otimes A \otimes K_1$. Let us define $\alpha_k|_{P_{1-n}}$ by the equalities

$$(5.3.1) \quad \alpha_k(1 \otimes x_t \otimes 1) = \sum_{i_0 + \cdots + i_k = p-k} (-1)^k (\pi_{i_0} \otimes_A \tau \otimes_A \pi_{i_1} \otimes_A \tau \otimes_A \cdots \otimes_A \tau \otimes_A \pi_{i_k}) y_t$$

for $1 \leq t \leq l$.

Proposition 5.3.2. *Let A be a Koszul algebra, with shifted Koszul resolution P , and let $\alpha_0 : P_{1-n} \rightarrow A$ be a Hochschild n -cocycle. Then α_k ($k \geq 1$), as given above on P_{-i} for $i \leq n - 1$, can be extended to components P_{-i} with $i > n - 1$ in such a way that $\alpha = \sum_{k \geq 0} \alpha_k$ is an A_∞ -coderivation.*

Proof. As above, we first define $\alpha_k|_{P_{-i}} = 0$ for $i < n - 1$ and $\alpha_k|_{P_{1-n}}$ by (5.3.1). We will check that such α_k satisfies the equality

$$\partial(\alpha_k)|_{P_{-i}} = \left((-1)^{n+1} (\alpha_{k-1} \otimes_A 1 + 1 \otimes_A \alpha_{k-1}) \delta_2 - \sum_{i=0}^{k-2} (1^{\otimes_A i} \otimes_A \delta_2 \otimes_A 1^{\otimes_A (k-2-i)}) \alpha_{k-1} \right) \Big|_{P_{-i}}$$

for $i \leq n-1$. Indeed, the left hand side evaluated on x_t in K'_n (that is, on $1 \otimes x_t \otimes 1$ in P_{1-n}) is

$$\begin{aligned} & \sum_{i+j=k-1} (1^{\otimes A^i} \otimes_A \delta_1 \otimes_A 1^{\otimes A^j}) \alpha_k(1 \otimes x_t \otimes 1) + (-1)^n \alpha_k \delta_1(1 \otimes x_t \otimes 1) \\ &= \sum_{i+j=k-1} (1^{\otimes A^i} \otimes_A \delta_1 \otimes_A 1^{\otimes A^j}) \sum_{i_0+\dots+i_k=p-k} (-1)^k (\pi_{i_0} \otimes_A \tau \otimes_A \dots \otimes_A \tau \otimes_A \pi_{i_k}) y_t \\ & \quad + (-1)^n \alpha_k(x_t \otimes 1 + (-1)^n \otimes x_t). \end{aligned}$$

Now $\alpha_k|_{K_{n-1}} = 0$, so $\alpha_k(x_t \otimes 1) = 0 = \alpha_k(1 \otimes x_t)$. The above expression may now be rewritten as

$$\sum_{\substack{i_0+\dots+i_k=p-k \\ j \in \{0, \dots, k-1\}}} (-1)^k (\pi_{i_0} \otimes_A \tau_{i_0} \otimes_A \dots \otimes_A \delta_1 \tau_{i_j} \otimes_A \dots \otimes_A \pi_{i_k}) y_t$$

where $\tau_{i_l} = \tau$ for all l . The right hand side evaluated on x_t in K'_n is

$$\begin{aligned} & (-1)^{n+1} (\alpha_{k-1} \otimes_A 1 + 1 \otimes_A \alpha_{k-1}) ((1 \otimes 1) \otimes_A (1 \otimes x_t \otimes 1) + (-1)^n (1 \otimes x_t \otimes 1) \otimes_A (1 \otimes 1)) \\ & - \sum_{i=0}^{k-2} (1^{\otimes A^i} \otimes_A \delta_2 \otimes_A 1^{\otimes A^{k-2-i}}) \sum_{i_0+\dots+i_{k-1}=p-k+1} (-1)^{k-1} (\pi_{i_0} \otimes_A \tau \otimes_A \dots \otimes_A \tau \otimes_A \pi_{i_{k-1}}) y_t \\ &= \sum_{i_0+\dots+i_{k-1}=p-k+1} (-1)^k (\pi_{i_0} \otimes_A \tau \otimes_A \dots \otimes_A \tau \otimes_A \pi_{i_{k-1}}) y_t \otimes_A (1 \otimes 1) \\ & + \sum_{i_0+\dots+i_{k-1}=p-k+1} (-1)^{k-1} (1 \otimes 1) \otimes_A (\pi_{i_0} \otimes_A \tau \otimes_A \dots \otimes_A \tau \otimes_A \pi_{i_{k-1}}) y_t \\ & + \sum_{\substack{i_0+\dots+i_{k-1}=p-k+1 \\ j \in \{0, \dots, k-2\}}} (-1)^k (\pi_{i_0} \otimes_A \tau_{i_0} \otimes_A \dots \otimes_A \delta_2 \tau_{i_j} \otimes_A \dots \otimes_A \pi_{i_{k-1}}) y_t. \end{aligned}$$

The first two sums above also appear within the third, with opposite signs, and so they cancel. Comparing the left hand side obtained previously with what remains of the right hand side here, since

$$\begin{aligned} \delta_1(1 \otimes v \otimes 1) &= v \otimes 1 - 1 \otimes v \quad \text{and} \\ \delta_2(1 \otimes v \otimes 1) &= (1 \otimes 1) \otimes_A (1 \otimes v \otimes 1) - (1 \otimes v \otimes 1) \otimes_A (1 \otimes 1), \end{aligned}$$

each is a telescoping sum in which there is further cancellation, with remaining terms coinciding. Thus we have shown that α_k satisfies, on such P_{-i} , the condition to be the k -th component of an A_∞ -coderivation. \square

We use these observations in the next result.

Proposition 5.3.3. *Let A be a Koszul algebra, with shifted Koszul resolution P , and let α_0 be a Hochschild cocycle mapping P_{1-n} to A_p . The corresponding A_∞ -coderivation α defined by (5.3.1) satisfies $\alpha_k(P) \subset P_{\leq 0}^{\otimes A^k}$ for all $k \geq 1$.*

Proof. For $k=1$ the required condition is satisfied automatically: Specifically, we may define α_1 via (5.3.1) on P_{-i} for $i \leq n-1$ as above and extend to P_{-i} for $i > n-1$ where degree

conditions force its image to be in $P_{\leq 0}$. On the k -th step we need to construct α_k in such a way that

$$\partial(\alpha_k) = (-1)^{n+1}(\alpha_{k-1} \otimes 1 + 1 \otimes \alpha_{k-1})\delta_2 - \sum_{i=0}^{k-2} (1^{\otimes A^i} \otimes_A \delta_2 \otimes_A 1^{\otimes A^{(k-2-i)}})\alpha_{k-1}.$$

Assume that $\alpha_{k-1}(P) \subset P_{\leq 0}^{\otimes A^{(k-1)}}$. Then it is easy to show using the formulas for δ_2 that the right hand side of the last equality lies in $P_{\leq 0}^{\otimes A^k}$. Since we have already constructed $\alpha_k|_{P_{1-n}}$ in such a way that $\alpha_k(P_{1-n}) \subset \left(P_{\leq 0}^{\otimes A^k}\right)_0$ and $P_{\leq 0}^{\otimes A^k}$ is exact in negative degrees, we can produce α_k in such a way that $\alpha_k(P) \subset P_{\leq 0}^{\otimes A^k}$.

Note that if $2 \nmid k$, then during the construction of α_{k+1} as in Theorem 4.4.6, we replace α_k by

$$\alpha_k + \left(1^{\otimes A^k} \otimes_A \mu_P\right) \left(\sum_{i=0}^{k-1} (1^{\otimes A^i} \otimes_A \delta_2 \otimes_A 1^{\otimes A^{(k-1-i)}})\alpha_k + (-1)^n(\alpha_k \otimes_A 1 + 1 \otimes_A \alpha_k)\delta_2 \right),$$

but one can show that this does not change our conditions. \square

Remark 5.3.4. By Proposition 5.3.3 for Koszul algebras, we obtain an element $\alpha \in \text{Coder}_{\mathcal{D}}^{\infty}(P)$ such that $\alpha_k(P) \subset P_{\leq 0}^{\otimes A^k}$ and all the components of α are homogeneous of degree $p - n$ as A -bimodule homomorphisms. In particular, we have $\alpha_k = 0$ for $k > p$ and $\alpha_k|_{P_{-i}} = 0$ for $i < n - 1$. For an element of $\text{HH}^*(A)$ represented by a map from K'_n to A_p then, we have constructed maps $\alpha_p|_{\oplus_{i \geq n} K'_i} : \oplus_{i \geq n} K'_i \rightarrow (\oplus_{i \geq 1} K'_i)^{\otimes p}$. The necessary inclusion $\alpha_p(\oplus_{i \geq n} K'_i) \subset (\oplus_{i \geq 1} K'_i)^{\otimes p}$ follows from a degree argument. Note that in fact $\oplus_{i \geq 0} K'_i$ has the structure of dg coalgebra C , called the Koszul dual coalgebra of A . The map α induces a map from C to the p -th component of the bar resolution of C . One can show that this map is a cocycle and that this cocycle is a coboundary if and only if α_0 is a coboundary. Moreover, if $\beta \in \text{Coder}_{\mathcal{D}}^{\infty}(P)$ is another coderivation corresponding to an element of $\text{HH}^*(A)$ represented by a map from K'_m to A_p , then the formula for the bracket $[\alpha, \beta]_{p+q}$ is exactly the formula for the Gerstenhaber bracket of the elements α_p and β_q . This gives a new proof of the fact that $\text{HH}^*(A)$ and $\text{HH}^*(C)$ are isomorphic as Gerstenhaber algebras (cf. [10]). We do not give the details of the proof here because this fact is well known.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NC 27599, USA
Email address: cnegron@email.unc.edu

DEPARTMENT OF MATHEMATICS AND MECHANICS, SAINT PETERSBURG STATE UNIVERSITY, SAINT PETERSBURG, RUSSIA
Email address: wolf86_666@list.ru

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA
Email address: sjw@math.tamu.edu