# Irreducible Representations of Crossed Products

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**Abstract**

We prove that if the dimension of any irreducible module for a finite dimensional algebra over an algebraically closed field divides the dimension of the algebra, then the same is true of any crossed product of that algebra with a group algebra or its dual, provided the characteristic of the field does not divide the order of the group. Kaplansky's Conjecture regarding dimensions of irreducible modules for Hopf algebras then follows for those finite dimensional semisimple Hopf algebras constructed by a sequence of crossed products involving group algebras and their duals. We show that any semisimple Hopf algebra of prime power dimension in characteristic 0 is of this type, so that Kaplansky's Conjecture holds for these Hopf algebras.

## Introduction

In 1975, Kaplansky conjectured that the dimension of any irreducible $H$-module divides the dimension of $H$, where $H$ is a finite dimensional semisimple Hopf algebra over an algebraically closed field [9]. This is well known for semisimple group algebras (a classical result of Frobenius), and for the dual of a group algebra, whose irreducible modules are all of dimension one. Recently, Nichols and Richmond have proven that if $H$ has an irreducible module of dimension 2, then 2 divides the dimension of $H$ [16]. Zhu has proven that the dimension of any irreducible $D(H)$-submodule of $H$ divides the dimension of $H$, where $D(H)$ is the quantum double of $H$ and the field has characteristic 0 [22]. Aside from these cases and some examples however, the general case has remained unknown.

We prove this conjecture for those Hopf algebras which can be constructed by a sequence of crossed products involving group algebras and their duals. All finite dimensional semisimple Hopf algebras over algebraically closed fields known at this time are of this form, so Kaplansky's Conjecture holds for all known examples. Further, we show that all semisimple Hopf algebras of prime power dimension over algebraically closed fields of characteristic 0 are of this form, establishing Kaplansky's Conjecture for these Hopf algebras.

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More specifically, let $G$ be a finite group, $k$ an algebraically closed field whose characteristic does not divide the order of $G$, and $A$ a finite dimensional $k$-algebra. We first show that if the divisibility property for dimensions of irreducible modules holds for $A$, then it holds for any crossed product $A\#_\sigma kG$. This follows from an application to $A\#_\sigma kG$ of Clifford Theory for group-graded rings [7]. We give an explicit construction of $A\#_\sigma kG$-modules from $A$-modules and modules for twisted group algebras of subgroups of $G$.

Next we consider a crossed product $B = A\#_\sigma (kG)^*$ where $(kG)^*$ is the Hopf algebra dual of $kG$. By duality [2], the smash product $B\#kG$ is isomorphic to a matrix algebra $M_n(A)$. If the divisibility property for dimensions of irreducible modules holds for $A$, then it holds for $M_n(A)$, and our construction of $B\#kG$-modules from $B$-modules then shows that it holds for $B$.

We define lower and upper semisolvable Hopf algebras to be those having certain normal series with factors either commutative or cocommutative. Our results involving $kG$ and $(kG)^*$ show that Kaplansky's Conjecture holds for a finite dimensional semisimple lower or upper semisolvable Hopf algebra $H$ when the characteristic of $k$ does not divide the dimension of $H$. We show that any semisimple Hopf algebra of prime power dimension over an algebraically closed field of characteristic 0 is upper and lower semisolvable, and thus Kaplansky's Conjecture holds for these Hopf algebras.

All algebras and modules will be finite dimensional over the algebraically closed field $k$, and tensor products will be over $k$ unless otherwise indicated.

# 1 Group crossed products

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$, $G$ a finite group, and $A\#_\sigma kG$ a crossed product. That is, there is an invertible map $\sigma : G \times G \to A$ and a $k$-linear map $kG \otimes A \to A$ denoted $g \otimes a \mapsto g \cdot a$ satisfying

$$\begin{aligned} g\cdot(h\cdot a) &= \sigma(g,h)(gh\cdot a)\sigma(g,h)^{-1}, \\ (g\cdot\sigma(h,\ell))\sigma(g,h\ell) &= \sigma(g,h)\sigma(gh,\ell), \\ \sigma(g,1) &= \sigma(1,g) \;=\; 1, \\ g\cdot(ab) = (g\cdot a)(g\cdot b) \quad &, \quad g\cdot 1 \;=\; 1 \;,\; 1\cdot a \;=\; a, \end{aligned}$$

for all $g, h, \ell \in G$ and $a, b \in A$. In particular, $kG$ measures $A$ and $\sigma$ is a 2-cocycle. We denote the element $a \otimes g$ of $A \otimes kG$ by $a\#g$, and define a product on this vector space by

$$(a\#g)(b\#h) = a(g\cdot b)\sigma(g,h)\#gh.$$

This results in an associative algebra, denoted $A\#_\sigma kG$ [15, 18]. This algebra is a fully $G$-graded algebra, with $g$-component $A\#_\sigma kg$ for each $g \in G$ [3, 7]. The algebra $A$ embeds in $A\#_\sigma kG$ as the identity component. Clifford Theory for group-graded rings implies a correspondence between $A\#_\sigma kG$-modules, and those of $A$ and of twisted group algebras for subgroups of $G$ [6, 7]. Alternatively, the correspondence may be generalized directly from Clifford Theory for group algebras [5] to crossed products $A\#_\sigma kG$. We will describe this correspondence explicitly for a crossed product, in terms of the cocycle $\sigma$. These constructions do not require $k$ to be algebraically closed.

We first introduce an action of $G$ on the set of isomorphism classes of finite dimensional $A$-modules, the conjugation action of $G$. If $T$ is an $A$-module with $\rho : A \to \mathrm{End}_k(T)$ defining the action of $A$ on $T$, and $g \in G$, the *conjugate* module $g \cdot T$ is defined to be the $A$-module with underlying vector space $T$ and action $g \cdot \rho : A \to \mathrm{End}_k(T)$ defined by

$$g \cdot \rho(a) = \rho(g^{-1} \cdot a)$$

for all $a \in A$. We may also express the module $g \cdot T$ as $(A\#_\sigma kg) \otimes_A T \cong (1\#g^{-1})^{-1} \otimes_A T$, where $A$ acts by left multiplication on the first factor, as

$$(1\#g^{-1})(a\#1)(1\#g^{-1})^{-1} = (g^{-1} \cdot a)\#1.$$

Here $(1\#g^{-1})^{-1} = \sigma(g, g^{-1})^{-1}\#g$. We note that $\dim(g \cdot T) = \dim(T)$, whereas in the general situation of a group-graded algebra, conjugate modules do not always have the same dimension [3]. If $h \in G$, then $h \cdot g \cdot \rho$ is equivalent to $hg \cdot \rho$ as a representation, since $g^{-1} \cdot h^{-1} \cdot a = \sigma(g^{-1}, h^{-1})((hg)^{-1} \cdot a)\sigma(g^{-1}, h^{-1})^{-1}$.

Let $U$ be an $A\#_\sigma kG$-module. We denote by $U\downarrow_A$ the module $U$ restricted to $A$. If $U$ is irreducible, then $U\downarrow_A$ is semisimple and isomorphic to a direct sum of copies of an irreducible $A$-module $T$ and its distinct $G$-conjugates, each occurring the same number of times ([6, Exercises 18.9 and 18.10] or [7, Theorem 12.4]). That is, $U\downarrow_A \cong (T_1 \oplus \cdots \oplus T_k)^{\oplus e}$ for some positive integer $e$, where $T = T_1, \ldots, T_k$ are the distinct (mutually nonisomorphic) $G$-conjugates of $T$.

Let $G_T$ be the *stabilizer* in $G$ of the irreducible $A$-module $T$:

$$G_T = \{g \in G \mid g \cdot T \cong T\}.$$

Then the $A$-submodule $T^{\oplus e}$ of $U\downarrow_A$ may be considered to be an irreducible $A\#_\sigma kG_T$-submodule of the restriction of $U$ to $A\#_\sigma kG_T$, as Proposition 1.1 below implies. The proposition follows from [6, Exercises 18.11 and 18.12] or [7, Corollary 11.16]. If $V$ is an $A\#_\sigma kG_T$-module, we will denote by $V\uparrow^G$ the induced $A\#_\sigma kG$-module $(A\#_\sigma kG) \otimes_{A\#_\sigma kG_T} V$, where the action is given by left multiplication by the first factor. In this case, note that $V\uparrow^G \cong \sum_{g \in G/G_T} g \cdot V$ as an $A\#_\sigma kG_T$-module, since $A\#_\sigma kG$ is free as a right $A\#_\sigma kG_T$-module. Therefore $\dim(V\uparrow^G) = |G : G_T| \dim(V)$.

**Proposition 1.1** *Let $T$ be an irreducible $A$-module, and $G_T$ its stabilizer in $G$. Induction of modules provides an equivalence between the category of $A\#_\sigma kG_T$-modules whose restriction to $A$ is isomorphic to a direct sum of copies of $T$, and the category of $A\#_\sigma kG$-modules whose restriction to $A$ is isomorphic to a direct sum of copies of conjugates of $T$.*

We next describe a construction of $A\#_\sigma kG_T$-modules from modules for a twisted group algebra $k^\beta G_T$ of $G_T$. Given a 2-cocycle $\beta : G_T \times G_T \to k^\times$, where elements of $G_T$ act trivially on $k^\times$, the *twisted group algebra* $k^\beta G_T$ is the crossed product $k\#_\beta kG_T$.

Let $\rho : A \to \mathrm{End}_k(T)$ express the action of $A$ on the irreducible $A$-module $T$, and for each $g \in G_T$, let $\tau(g) \in \mathrm{End}_k(T)$ be an isomorphism between $g \cdot T$ and $T$. That is,

$$\rho(a)\tau(g) = \tau(g)\rho(g^{-1} \cdot a)$$

for all $a \in A$. Choose $\tau(1)$ to be the identity map. For each $g, h \in G_T$, it may be checked that the invertible map $\tau(g)\tau(h)\rho(\sigma(h^{-1}, g^{-1}))\tau(gh)^{-1}$ commutes with $\rho(a)$ for all $a \in A$; that is, it is an $A$-homomorphism. As $T$ is irreducible, Schur's Lemma implies that there is a scalar $\alpha(g, h) \in k^\times$ such that $\tau(g)\tau(h)\rho(\sigma(h^{-1}, g^{-1}))\tau(gh)^{-1} = \alpha(g, h) \circ \mathrm{id}_T$. So

$$\tau(g)\tau(h) = \alpha(g, h)\tau(gh)\rho(\sigma^{-1}(h^{-1}, g^{-1}))$$

for all $g, h \in G_T$. Using the 2-cocycle relation for $\sigma$ and the definition of $\tau$, we see that $\alpha$ is itself a 2-cocycle (with corresponding trivial action on $k^\times$). We will be interested in the 2-cocycle $\alpha^{-1}$.

Let $M$ be a $k^{\alpha^{-1}}G_T$-module. It may be checked that the vector space $T \otimes M$ has the structure of an $A\#_\sigma kG_T$-module given by

$$(a\#g) \cdot (t \otimes m) = \rho(a)\tau(g)\rho(\sigma(g^{-1}, g))t \otimes g \cdot m$$

for all $a \in A$, $g \in G_T$, $t \in T$, and $m \in M$. Here $g \cdot m$ denotes the action of the image of $g$ in $k^{\alpha^{-1}}G_T$ on $m$. The restriction $(T \otimes M)\downarrow_A$ is a direct sum of copies of $T$. If $f : M \to N$ is a $k^{\alpha^{-1}}G_T$-homomorphism, then $\mathrm{id}_T \otimes f$ is an $A\#_\sigma kG_T$-homomorphism from $T \otimes M$ to $T \otimes N$.

On the other hand, if $V$ is an $A\#_\sigma kG_T$-module whose restriction to $A$ is a direct sum of copies of $T$, the vector space $\mathrm{Hom}_A(T, V)$ becomes a $k^{\alpha^{-1}}G_T$-module by defining

$$(g \cdot \phi)(t) = \alpha^{-1}(g, g^{-1})\#g \cdot \phi(\tau(g^{-1})t)$$

for all $g \in G_T$, $\phi \in \mathrm{Hom}_A(T, V)$, $t \in T$. If $f : V \to W$ is an $A\#_\sigma kG_T$-homomorphism, then $f_*$ is a $k^{\alpha^{-1}}G_T$-homomorphism from $\mathrm{Hom}_A(T, V)$ to $\mathrm{Hom}_A(T, W)$, where $f_*(\phi) = f \circ \phi$.

We have described functors between the categories of $k^{\alpha^{-1}}G_T$-modules and of those $A\#_\sigma kG_T$-modules whose restriction to $A$ is a direct sum of copies of $T$. It may be checked that these functors provide a category equivalence, as stated in the following proposition. Alternatively, the proposition follows from a result of Dade about group-graded rings ([6, pp. 239–240] or [7, Theorem 10.6]).

**Proposition 1.2** *Let $T$ be an irreducible $A$-module, $G_T$ its stabilizer in $G$, and $\alpha$ the 2-cocycle of $G_T$ determined by $T$ as above. There is an equivalence between the category of $k^{\alpha^{-1}}G_T$-modules and the category of those $A\#_\sigma kG_T$-modules whose restriction to $A$ is isomorphic to a direct sum of copies of $T$.*

Propositions 1.1 and 1.2, together with the above description of the category equivalences, immediately imply the following theorem.

**Theorem 1.3** *Let $T$ be an irreducible $A$-module, $G_T$ its stabilizer in $G$, and $\alpha$ the 2-cocycle of $G_T$ determined by $T$ as above. There is an equivalence between the category of $k^{\alpha^{-1}}G_T$-modules and the category of those $A\#_\sigma kG$-modules whose restriction to $A$ is isomorphic to a direct sum of copies of conjugates of $T$. This equivalence is given by sending a $k^{\alpha^{-1}}G_T$-module $M$ to the module $(T \otimes M)\uparrow^G$ induced from the $A\#_\sigma kG_T$-module $T \otimes M$ defined by $(a\#g) \cdot (t \otimes m) = \rho(a)\tau(g)\rho(\sigma(g^{-1}, g))t \otimes g \cdot m$ for all $a \in A$, $g \in G_T$, $t \in T$, and $m \in M$.*

As a consequence of the theorem, irreducible $k^{\alpha^{-1}}G_T$-modules are in one-to-one correspondence with those irreducible $A\#_\sigma kG$-modules having $T$ as a direct summand. We also need the fact that the dimension of an irreducible module for a twisted group algebra over $k$ divides the order of the group, provided the characteristic of $k$ is relatively prime to the order of the group. This follows from the theory of Schur representation groups [8]. We note that Passman now has a simpler proof of Corollary 1.4 below that does not use Theorem 1.3. However, we need Theorem 1.3 in the next section.

**Corollary 1.4** *Let $A$ be a finite dimensional $k$-algebra such that the dimension of any irreducible $A$-module divides the dimension of $A$. Let $G$ be a finite group such that the characteristic of $k$ does not divide $|G|$, and let $B = A\#_\sigma kG$ be a crossed product. Then the dimension of any irreducible $B$-module divides the dimension of $B$.*

Proof: Let $U$ be an irreducible $B$-module. By the discussion at the beginning of the section, $U\downarrow_A$ is a direct sum of conjugates of an irreducible $A$-module $T$, each occurring the same number of times. Let $G_T$ be the stabilizer of $T$, and $\alpha : G_T \times G_T \to k^\times$ the 2-cocycle determined by $T$ as above. By Theorem 1.3, there is an irreducible $k^{\alpha^{-1}}G_T$-module $M$ with $U \cong (T \otimes M)\uparrow^G$, so that $\dim(U) = [G : G_T] \dim(T) \dim(M)$. Now $\dim(M)$ divides $|G_T|$ as noted above, so $\dim(U)$ divides $|G| \dim(T)$. But $\dim(T)$ divides $\dim(A)$ by hypothesis, so $\dim(U)$ divides $|G| \dim(A) = \dim(B)$. □

# 2 The dual of a group algebra

In this section we will consider $A\#_\sigma(kG)^*$-modules, where $A$ is a finite dimensional $k$-algebra, $G$ a finite group, and $(kG)^*$ the Hopf algebra dual of $kG$. We refer the reader to Section 3 for the definition of the general crossed product $A\#_\sigma H$ for a Hopf algebra $H$. Here we use duality to achieve an analogous result to Corollary 1.4 regarding dimensions of irreducible $A\#_\sigma(kG)^*$-modules. That is, $G$ acts on $B = A\#_\sigma(kG)^*$ by

$$g \cdot (a\#f) = a\#(g \cdot f),$$

where $g \cdot f(x) = f(xg^{-1})$. Thus we may form the smash product $B\#kG$, that is a crossed product with trivial cocycle. By duality $B\#kG \cong M_n(A)$, where $n$ is the order of $G$ [2].

If the dimension of any irreducible $A$-module divides the dimension of $A$, then the same is true of $M_n(A)$, as an irreducible $M_n(A)$-module is a direct sum of $n$ copies of an irreducible $A$-module. In particular, the dimension of an irreducible $M_n(A)$-module is $n$ times the dimension of the corresponding $A$-module.

At this point it is natural to ask if a converse to Corollary 1.4 is true. That is, suppose $B$ is a finite dimensional $k$-algebra and the dimension of any irreducible $B\#_\sigma kG$-module divides $\dim(B\#_\sigma kG)$. Does it follow that the dimension of any irreducible $B$-module divides the dimension of $B$? The following example, which was pointed out to us by Guralnick, shows that the answer is no. Let $B$ be an algebra for which there exists an irreducible module whose dimension does not divide $\dim(B)$, for example $B = k \oplus M_2(k)$. Then it is possible to choose an abelian group $G$ of large enough order that irreducible $B \otimes kG$-modules will have dimension dividing $\dim(B \otimes kG)$. However duality provides a positive answer for the smash product of $B = A\#_\sigma(kG)^*$ with $kG$.

**Theorem 2.1** *Let $A$ be a finite dimensional $k$-algebra such that the dimension of any irreducible $A$-module divides the dimension of $A$. Let $G$ be a finite group such that the characteristic of $k$ does not divide $|G|$, and let $B = A\#_\sigma(kG)^*$ be a crossed product. Then the dimension of any irreducible $B$-module divides the dimension of $B$.*

Proof: Let $T$ be an irreducible $B$-module. The action of $G$ on $B$ described above yields an action of $G$ on the isomorphism classes of $B$-modules, as described in Section 1. Let $G_T$ be the subgroup of all $g \in G$ such that $g \cdot T \cong T$, and let $\alpha : G_T \times G_T \to k^\times$ be the 2-cocycle determined by $T$ as in Section 1. Let $M$ be any irreducible $k^{\alpha^{-1}}G_T$-module. By Theorem 1.3, there is a corresponding irreducible $B\#kG$-module $U = (T \otimes M)\uparrow^G$, so that

$$\dim(U) = [G : G_T] \dim(T) \dim(M).$$

By duality, we have $B\#kG \cong M_n(A)$, where $n = |G|$ [2]. Let $X$ be an irreducible $A$-module corresponding to the $M_n(A)$-module $U$, so that $\dim(U) = |G| \dim(X)$. Comparing to the above expression for $\dim(U)$, we have

$$\dim(T) = |G_T| \dim(X) / \dim(M).$$

As $M$ is an irreducible $k^{\alpha^{-1}}G_T$-module, $\dim(M)$ divides $|G_T|$, and so $|G_T|/\dim(M)$ divides $|G_T|$. By hypothesis, $\dim(X)$ divides $\dim(A)$. Therefore $\dim(T)$ divides $|G_T| \dim(A)$, which divides $|G| \dim(A) = \dim(B)$. □

An inductive argument using Corollary 1.4 and Theorem 2.1 yields the following consequence.

**Corollary 2.2** *Let $A$ be a finite dimensional algebra with a sequence of subalgebras $k = A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_2 \subseteq A_1 = A$ such that for each $i$, $A_i$ is isomorphic to $A_{i+1}\#_\sigma kG$ or $A_{i+1}\#_\sigma (kG)^*$ for some group $G$ and cocycle $\sigma$. Suppose further that the characteristic of $k$ does not divide the order of any of the groups $G$ involved. Then the dimension of any irreducible $A$-module divides the dimension of $A$.*

# 3 Semisolvable Hopf algebras

Let $H$ be a finite dimensional Hopf algebra over the algebraically closed field $k$. (The assumption that $k$ be algebraically closed is not needed until Theorem 3.4.) A *lower normal series* for $H$ is a series of proper subHopf algebras

$$k = H_{n+1} \subseteq H_n \subseteq \cdots \subseteq H_2 \subseteq H_1 = H,$$

where $H_{i+1}$ is normal in $H_i$ for each $i$. That is, $(ad_\ell H_i)(H_{i+1}) \subseteq H_{i+1}$ and $(ad_r H_i)(H_{i+1}) \subseteq H_{i+1}$, where $ad_\ell$ and $ad_r$ are the left and right adjoint actions [15]. As $H_{i+1}$ is normal in $H_i$, $H_i H_{i+1}^+ = H_{i+1}^+ H_i$ is a Hopf ideal of $H_i$, where $H_{i+1}^+$ is the augmentation ideal of $H_{i+1}$. The *factors* of this normal series are the quotients $\overline{H_i} = H_i/H_i H_{i+1}^+$. An *upper normal series* for $H$ is a series defined inductively as follows. Let $H_{(0)} = H$. Let $H_i$ be a normal subHopf algebra of $H_{(i-1)}$, and define $H_{(i)} = H_{(i-1)}/H_{(i-1)}H_i^+$. We assume $H_n = H_{(n-1)}$ for some positive integer $n$ so that $H_{(n)} = k$. The *factors* are the subHopf algebras $H_i$ of the quotients $H_{(i-1)}$. We use the following lemma to relate normal series of $H$ and $H^*$.

**Lemma 3.1** *Let $H$ be a finite dimensional Hopf algebra and $K$ a normal subHopf algebra of $H$. Then $J = (H/HK^+)^*$ is isomorphic to a normal subHopf algebra of $H^*$, and $H^*/H^*J^+$ is isomorphic to $K^*$.*

Proof: By the Nichols-Zoeller Theorem [17], $H$ is free over $K$. As $K$ is normal in $H$, it follows from [20, Lemma 1.3 (2)] that $HK^+$ is a normal Hopf ideal of $H$. Equivalently, $J = (H/HK^+)^*$ is isomorphic to a normal subHopf algebra of $H^*$ as $H$ is finite dimensional [15, p. 36].

Let $\overline{H} = H/HK^+$ and $\pi : H \to \overline{H}$ the natural projection. By [19, Lemma 1.3] or [15, Proposition 3.4.3] we have $K = H^{co\overline{H}}$, the coinvariants in $H$ of the $\overline{H}$-comodule given by $(\mathrm{id} \otimes \pi) \circ \Delta$, where $\Delta$ is the coproduct on $H$. That is, $K$ is the set of all $h \in H$ such that $\sum h_1 \otimes \pi(h_2) = h \otimes 1$. Thus the following equalizer diagram is exact:

$$K \xrightarrow{i} H \underset{g}{\overset{f}{\rightrightarrows}} H \otimes \overline{H},$$

where $i$ is inclusion, $f(h) = h \otimes \overline{1}$ and $g(h) = (\mathrm{id} \otimes \pi) \circ \Delta(h) = \sum h_1 \otimes \overline{h}_2$, as in [15, pp. 34–35]. Dualizing this diagram as in [15, p. 143] or [19, Theorem 2.4 (2)] gives the coequalizer diagram

$$H^* \otimes J \underset{g^*}{\overset{f^*}{\rightrightarrows}} H^* \xrightarrow{i^*} K^*,$$

as $J = \overline{H}^*$. Thus $K^*$ can be defined by this exact sequence. Now $\mathrm{Ker}(i^*) = \mathrm{Im}(f^* - g^*) = H^*J^+$, and so $K^* \cong H^*/H^*J^+$. □

**Theorem 3.2** *Let $H$ be a finite dimensional Hopf algebra. If $H$ has a lower normal series with factors $\overline{H_i}$, then $H^*$ has an upper normal series with factors $(H^*)_i \cong (\overline{H_i})^*$. If $H$ has an upper normal series with factors $H_i$, then $H^*$ has a lower normal series with factors $\overline{(H^*)_i} \cong (H_i)^*$.*

Proof: Let $k = H_{n+1} \subseteq H_n \subseteq \ldots \subseteq H_2 \subseteq H_1 = H$ be a lower normal series for $H$ with factors $\overline{H_i} = H_i/H_iH_{i+1}^+$. If $n \leq 1$, there is nothing to prove. If $n > 1$, let $J = (H/HH_2^+)^*$, a normal subHopf algebra of $H^*$ with $H^*/H^*J^+ \cong (H_2)^*$ by Lemma 3.1. By induction $(H_2)^*$ has an upper normal series with factors isomorphic to $(\overline{H_i})^*$ for $i = 2, \ldots, n$. Letting $(H^*)_1 = J$, $H^*$ has an upper normal series with factors $(H^*)_i \cong (\overline{H_i})^*$ for $i = 1, \ldots, n$.

Let $H_{(0)} = H$ , $H_{(1)} = H/HH_1^+$ , $\ldots$ , $H_{(n)} = H_{(n-1)}/H_{(n-1)}H_n^+ = k$ be an upper normal series for $H$. If $n \leq 1$, there is nothing to prove. If $n > 1$, let $J = (H_{(1)})^* = (H/HH_1^+)^*$. By Lemma 3.1, $J$ is a normal subHopf algebra of $H^*$, and $H^*/H^*J^+ \cong (H_1)^*$. By induction $J = (H_{(1)})^*$ has a lower normal series with factors isomorphic to $(H_i)^*$ for $i = 2, \ldots, n$. Letting $(H^*)_2 = J$, $H^*$ has a lower normal series with factors $\overline{(H^*)_i} \cong (H_i)^*$ for $i = 1, \ldots, n$. □

We say that $H$ is *lower solvable* (respectively *lower cosolvable*) if there is a lower normal series for $H$ all factors of which are commutative (respectively cocommutative), and that $H$ is *lower semisolvable* if there is a lower normal series for $H$ each factor of which is either commutative or cocommutative. We say that $H$ is *upper solvable* (respectively *upper cosolvable, upper semisolvable*) if $H$ has an upper normal series each factor of which is commutative (respectively cocommutative, either commutative or cocommutative). As the dual of a commutative Hopf algebra is cocommutative and vice versa, the next result follows immediately from Theorem 3.2.

**Corollary 3.3** *Let $H$ be a finite dimensional Hopf algebra. Then $H$ is lower (respectively upper) solvable if and only if $H^*$ is upper (respectively lower) cosolvable, and $H$ is lower semisolvable if and only if $H^*$ is upper semisolvable.*

We next define the crossed product of an algebra $A$ with a Hopf algebra $H$. We use summation notation for the coproduct of $H$ [15, 1.4.2]. Suppose there is a convolution-invertible $k$-linear map $\sigma : H \otimes H \to A$ and a $k$-linear map $H \otimes A \to A$ denoted $h \otimes a \mapsto h \cdot a$

satisfying

$$\begin{aligned} h\cdot(\ell\cdot a) &= \sum\sigma(h_1,\ell_1)(h_2\ell_2\cdot a)\sigma^{-1}(h_3,\ell_3), \\ \sum[h_1\cdot\sigma(\ell_1,m_1)]\sigma(h_2,\ell_2m_2) &= \sum\sigma(h_1,\ell_1)\sigma(h_2\ell_2,m), \\ \sigma(h,1) &= \sigma(1,h) \;=\; \epsilon(h)1, \\ h\cdot(ab)=\sum(h_1\cdot a)(h_2\cdot b) \quad &, \quad h\cdot 1 \;=\; \epsilon(h)1 \;,\; 1\cdot a \;=\; a, \end{aligned}$$

for all $h, \ell, m \in H$ and $a, b \in A$. In particular, $H$ measures $A$ and $\sigma$ is a 2-cocycle. We denote the element $a \otimes h$ of $A \otimes H$ by $a\#h$, and define a product on this vector space by

$$(a\#h)(b\#\ell) = \sum a(h_1\cdot b)\sigma(h_2,\ell_1)\#h_3\ell_2.$$

This results in an associative algebra, denoted $A\#_\sigma H$ [15].

We will need some more results from [19]. First, if $K$ is a normal subHopf algebra of the finite dimensional Hopf algebra $H$, then $H \cong K\#_\sigma(H/HK^+)$ for some 2-cocycle $\sigma$. It follows that if $H$ has a lower normal series $k = H_{n+1} \subseteq H_n \subseteq \cdots \subseteq H_1 = H$, then $H \cong \left(\left(\left(H_n\#_{\sigma_{n-1}}\overline{H_{n-1}}\right)\#_{\sigma_{n-2}}\overline{H_{n-2}}\right)\#_{\sigma_{n-3}}\cdots\right)\#_{\sigma_1}\overline{H_1}$. The second result we will need is transitivity of crossed products: Let $A$ be a finite dimensional algebra, $H$ a finite dimensional Hopf algebra, $B = A\#_\sigma H$ a crossed product, and $\overline{H}$ a quotient of $H$. Then $B \cong B^{co\overline{H}}\#_\tau\overline{H}$ for some 2-cocycle $\tau$. In particular, if $\overline{H} = H/HK^+$ for a normal subHopf algebra $K$ of $H$, then $B^{co\overline{H}} = A\#_\sigma K$, and so $A\#_\sigma H \cong (A\#_\sigma K)\#_\tau\overline{H}$.

We next generalize Corollary 1.4 and Theorem 2.1 to crossed products involving upper or lower semisolvable Hopf algebras. The following theorem in particular implies that the dimension of any irreducible $H$-module divides the dimension of $H$ when $H$ is a finite dimensional semisimple upper or lower semisolvable Hopf algebra such that the characteristic of $k$ does not divide the dimension of $H$.

**Theorem 3.4** *Let $A$ be a finite dimensional algebra over an algebraically closed field $k$ such that the dimension of any irreducible $A$-module divides the dimension of $A$. Let $H$ be a finite dimensional semisimple lower or upper semisolvable Hopf algebra over $k$ such that the characteristic of $k$ does not divide the dimension of $H$, and let $B = A\#_\sigma H$ be a crossed product. Then the dimension of any irreducible $B$-module divides the dimension of $B$.*

Proof: First assume $H$ is lower semisolvable, with lower normal series

$$k = H_{n+1} \subseteq H_n \subseteq \cdots \subseteq H_2 \subseteq H_1 = H,$$

each factor of which is either commutative or cocommutative. If $n \leq 1$, then $H$ is either commutative or cocommutative. If $H$ is commutative, then $H \cong (kG)^*$ for some finite group $G$ ([4, 10] or [15, Theorem 2.3.1]). Therefore by Theorem 2.1, the dimension of any irreducible $B$-module divides $\dim(B)$. If $H$ is cocommutative, we consider the cases of prime characteristic and characteristic 0 separately. If $k$ is of characteristic 0, then consider $H^*$, a finite dimensional commutative semisimple Hopf algebra by [11]. By the above argument, $H^* \cong (kG)^*$ and so $H \cong kG$ for some finite group $G$. If $k$ is of prime characteristic $p$, then $H \cong (kL)^* \# kG$ for a finite group $G$ and a finite abelian $p$-group $L$ by [15, Corollary 5.6.4] and [15, Theorem 5.7.1], due independently to Sweedler and Demazure-Gabriel. As $p$ does not divide the dimension of $H$, $L$ is in fact trivial, so $H \cong kG$ in this case as well. By Corollary 1.4, the dimension of any irreducible $B$-module divides $\dim(B)$.

We next assume $n > 1$. Then $H \cong H_2 \#_\tau \overline{H_1}$ for some cocycle $\tau$, and by transitivity of crossed products,

$$B \cong (A \#_\sigma H_2) \#_\mu \overline{H_1}$$

for some cocycle $\mu$ [19]. As $H = H_1$ is semisimple, both $\overline{H_1}$ and $H_2$ are semisimple; this second fact follows from the Nichols-Zoeller Theorem [17] and [15, Corollary 2.2.2]. Clearly $H_2$ is lower semisolvable while $\overline{H_1}$ is either commutative or cocommutative. The characteristic of $k$ cannot divide $\dim(H_2)$ or $\dim(\overline{H_1})$ by the Nichols-Zoeller Theorem [17]. By induction, the dimension of any irreducible $A \#_\sigma H_2$-module divides $\dim(A \#_\sigma H_2)$. As $\overline{H_1}$ is either commutative or cocommutative, the arguments above now show that the dimension of any irreducible $B$-module divides $\dim(B)$.

Next assume $H$ is upper semisolvable, and let

$$H_{(0)} = H \ , \ H_{(1)} = H/HH_1^+ \ , \ \ldots \ , \ H_{(n)} = H_{(n-1)}/H_{(n-1)}H_n^+ = k$$

be an upper normal series for $H$ with each $H_i$ commutative or cocommutative. If $n \leq 1$, then $H$ is commutative or cocommutative and the first paragraph above applies. Assume $n > 1$. Then $H \cong H_1 \#_\tau H_{(1)}$ for some cocycle $\tau$, and by transitivity of crossed products

$$B \cong (A \#_\sigma H_1) \#_\mu H_{(1)}$$

for some cocycle $\mu$ [19]. As $H$ is semisimple, both $H_1$ and $H_{(1)}$ are semisimple. Clearly $H_{(1)}$ is upper semisolvable while $H_1$ is either commutative or cocommutative. By the arguments in the first paragraph above, the dimension of any irreducible $A \#_\sigma H_1$-module divides $\dim(A \#_\sigma H_1)$. By induction, the dimension of any irreducible $B$-module divides $\dim(B)$. □

Now we assume that $k$ has characteristic 0. The next theorem in particular implies that all semisimple Hopf algebras of prime power dimension over $k$ are both upper and lower solvable and cosolvable.

**Theorem 3.5** *Let $H$ be a semisimple Hopf algebra of dimension $p^n$ for a prime $p$ over an algebraically closed field $k$ of characteristic 0. Then $H$ has both an upper and a lower normal series each factor of which is isomorphic to $k[\mathbf{Z}/p\mathbf{Z}]$.*

Proof: First we will prove by induction that $H$ has an upper normal series each factor of which is isomorphic to $k[\mathbf{Z}/p\mathbf{Z}]$. If $\dim(H) = p$, then $H \cong k[\mathbf{Z}/p\mathbf{Z}]$ by [21], so assume $\dim(H) = p^n$ with $n > 1$. By [14], $H$ has a central group-like element $g \neq 1$. We may assume $g$ has order $p$, for if not, $g$ generates a group which contains an element of order $p$. Let $H_1 \cong k[\mathbf{Z}/p\mathbf{Z}]$ be the subHopf algebra of $H$ generated by $g$, and let $H_{(1)} = H/HH_1^+$. Now $H_{(1)}$ has dimension $p^{n-1}$ and is semisimple, so by induction, $H_{(1)}$ has an upper normal series each factor of which is isomorphic to $k[\mathbf{Z}/p\mathbf{Z}]$. Therefore $H$ does as well.

As $H$ is semisimple and $k$ has characteristic 0, $H^*$ is also semisimple [11]. The above procedure applied to $H^*$, also of dimension $p^n$, shows that $H^*$ has an upper normal series each factor of which is isomorphic to $k[\mathbf{Z}/p\mathbf{Z}]$. By Theorem 3.2, $H$ has a lower normal series each factor of which is isomorphic to $(k[\mathbf{Z}/p\mathbf{Z}])^* \cong k[\mathbf{Z}/p\mathbf{Z}]$. □

**Corollary 3.6** *Let $H$ be a semisimple Hopf algebra of dimension $p^n$ for a prime $p$ over an algebraically closed field $k$ of characteristic 0. Then the dimension of any irreducible $H$-module divides the dimension of $H$.*

Proof: By Theorem 3.5, $H$ is both upper and lower semisolvable. By Theorem 3.4 with $A = k$ and $\sigma$ trivial, the dimension of any irreducible $H$-module divides $\dim(H)$. □

Masuoka has classified semisimple Hopf algebras of dimension $p^3$ over $k$ [12, 13]. In particular, for odd $p$ there are $p+1$ such Hopf algebras not isomorphic to group algebras or their duals, and for $p = 2$ there is one such. Masuoka has also shown that a Hopf algebra of dimension $p^2$ is isomorphic to a group algebra [14].

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