FINITE GROUPS ACTING LINEARLY: HOCHSCHILD COHOMOLOGY AND THE CUP PRODUCT

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Abstract. When a finite group acts linearly on a complex vector space, the natural semi-direct product of the group and the polynomial ring over the space forms a skew group algebra. This algebra plays the role of the coordinate ring of the resulting orbifold and serves as a substitute for the ring of invariant polynomials from the viewpoint of geometry and physics. Its Hochschild cohomology predicts various Hecke algebras and deformations of the orbifold. In this article, we investigate the ring structure of the Hochschild cohomology of the skew group algebra. We show that the cup product coincides with a natural smash product, transferring the cohomology of a group action into a group action on cohomology. We express the algebraic structure of Hochschild cohomology in terms of a partial order on the group (modulo the kernel of the action). This partial order arises after assigning to each group element the codimension of its fixed point space. We describe the algebraic structure for Coxeter groups, where this partial order is given by the reflection length function; a similar combinatorial description holds for an infinite family of complex reflection groups.

1. Introduction

Physicists often regard space as a Calabi-Yau manifold M endowed with symmetries forming a group G. The orbifold M/G is not smooth in general, and they regularly shift focus from the orbifold to a desingularisation and its coordinate ring. In the affine case, we take M to be a finite dimensional, complex vector space V upon which a finite group G acts linearly. The orbifold V/G may then be realized as an algebraic variety whose coordinate ring is the ring of invariant polynomials $S(V^*)^G$ on the dual space V^* (see Harris [18]). The variety V/G is nonsingular exactly when the action of G on V is generated by reflections. When the orbifold V/G is singular, we seek to replace the space of invariant polynomials with a natural algebra attached to V/G playing the role of a coordinate ring.

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In situations arising naturally in physics, a noncommutative substitute for the commutative invariant ring $S(V^*)^G$ is provided by the skew group algebra

$$S(V^*)\#G = S(V^*) \rtimes G,$$

the natural semi-direct product of G with the symmetric algebra $S(V^*)$. One resolves the singularities of V/G with a smooth Calabi-Yau variety X whose coordinate ring behaves as the skew group algebra. Indeed, the McKay equivalence implies (in certain settings) that Hochschild cohomology sees no difference:

$$\mathrm{HH}^{\bullet}(S(V^*)\#G) \cong \mathrm{HH}^{\bullet}(\mathcal{O}_X)$$

as algebras under both the cup product and Lie bracket. Note that the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V^*)\#G)$ can also be used to recover the orbifold (or stringy) cohomology $\mathrm{H}^{\bullet}_{orb}(V/G)$ (which is isomorphic to the singular cohomology of X). (See [3], [6], [7], [15], [19].) The cohomology and deformations of $S(V)\#G$ appear in various other areas of mathematics as well—for example, combinatorics, representation theory, Lie theory, noncommutative algebra, and invariant theory (see, for example, Etingof and Ginzburg [11]).

In this paper, we consider any finite group G acting linearly on V and explore the rich algebraic structure of the Hochschild cohomology of $S(V)\#G$ under the cup product. This structure is interesting not only in its own right, but also because of possible applications in algebra and representation theory. For example, the graded Lie bracket, which predicts potential deformations (like sympletic reflection algebras and graded Hecke algebras), is a graded derivation on Hochschild cohomology with respect to the cup product. The representation theory of finite dimensional algebras provides an application of the cup product in a different setting: Often, one may associate an algebraic variety to each module over the algebra using the ring structure of its Hochschild cohomology; the collection of such varieties provide a coarse invariant of the representation theory of the algebra (see, for example, Snashall and Solberg [26]).

For any algebra A over a field k, Hochschild cohomology $\mathrm{HH}^{\bullet}(A)$ is the space $\mathrm{Ext}^{\bullet}_{A\otimes A^{op}}(A, A)$, which is a Gerstenhaber algebra under the two compatible operations, cup product and bracket. Both operations are defined initially on the bar resolution, a natural $A \otimes A^{op}$-free resolution of A. For $A = S(V)\#G$, an explicit description of Hochschild cohomology $\mathrm{HH}^{\bullet}(A)$ arises not from the bar resolution of A, but instead from a Koszul resolution: $\mathrm{HH}^{\bullet}(S(V)\#G)$ is isomorphic to the G-invariant subalgebra of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$, which is computed with a Koszul resolution of $S(V)$. We analyze the cup product on $\mathrm{HH}^{\bullet}(S(V)\#G)$ by taking advantage of these two different manifestations of cohomology arising from two different resolutions. We show that the cup product on $\mathrm{HH}^{\bullet}(S(V)\#G)$ may be written as the cup product on $S(V)$ twisted by the action of the group. This perspective yields convenient descriptions of the ring structure of cohomology. (We study the Gerstenhaber bracket in a future work.)

Over the real numbers, the cup product on $\mathrm{HH}^{\bullet}(S(V)\#G)$ has been studied in related settings (e.g., see [22]). But note that this analysis of Hochschild cohomology does not readily extend to our setting. In this paper, we do not assume that G acts symplectically or even faithfully, as there are interesting applications in which it does not (see [9, 10, 24]). Over the real numbers, V and V^* are naturally G-isomorphic, which may simplify several aspects of the theory. We develop the theory in the richer setting of complex affine space. Note that our results actually hold over any field containing the eigenvalues of the action of G on V in which $|G|$ is invertible. Anno [1] also gave a cup product formula in the geometric setting over fairly general fields; we give a natural interpretation of the resulting ring structure from a purely algebraic and combinatorial point of view.

In Section 2, we define the "codimension poset" which arises from assigning to each group element the codimension of its fixed point space. We also posit a few observations needed later on the geometry of finite group actions. In Sections 3 and 4, we establish definitions and notation and recall necessary facts about the bar and Koszul resolutions. We review the structure of $\mathrm{HH}^{\bullet}(S(V)\#G)$ as a graded vector space as well, originally found independently by Farinati [13] and Ginzburg and Kaledin [15] for faithful group actions. In Section 5, we define a combinatorial map in terms of Demazure operators and quantum differentiation which allows for conversion between complexes. This combinatorial conversion map (introduced in [25]) induces isomorphisms on cohomology.

In Sections 6 and 7, we transform the "cohomology of a group action" into a "group action on cohomology". One may first take Hochschild cohomology and then form the skew group algebra, or one may reverse this order of operation. In Section 6, we compare resulting algebras for $S(V)$:

$$\mathrm{HH}^{\bullet}(S(V)\#G) \quad \text{versus} \quad \mathrm{HH}^{\bullet}(S(V))\#G \quad \text{versus} \quad \mathrm{HH}^{\bullet}(S(V)) \otimes \mathbb{C}[G] \ .$$

Since $\mathrm{HH}^{\bullet}(S(V)\#G)$ is the G-invariant subalgebra of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$, we focus on this latter ring. We show that the smash product on $\mathrm{HH}^{\bullet}(S(V))\#G$ induces a smash product on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$, which we then view as an algebra under three operations:

- the cup product $\smile$ induced from the bar resolution of $S(V)$,
- the smash product $\diamond$ induced from $\mathrm{HH}^{\bullet}(S(V))\#G$, and
- the usual multiplication in the tensor algebra product $\mathrm{HH}^{\bullet}(S(V)) \otimes \mathbb{C}[G]$.

In Section 7 (see Theorem 7.1), we show that these three algebraic operations *coincide*. This yields a simple formula for the cup product on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ (see Theorem 7.3; cf. Anno [1]) and implies that $\mathrm{HH}^{\bullet}(S(V)\#G)$ is isomorphic to an *algebra subquotient* of $\mathrm{HH}^{\bullet}(S(V))\#G$ (see Corollary 7.6). These results express the cup product on $\mathrm{HH}^{\bullet}(S(V)\#G)$ (at the cochain level) as the cup product on $\mathrm{HH}^{\bullet}(S(V))$ twisted by the group G.

In Section 8, we identify an interesting graded subalgebra (the "volume subalgebra") of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ whose dimension is the order of G. When G is a subgroup of the symplectic group $\mathrm{Sp}(V)$, its G-invariant subalgebra is isomorphic to the cohomology of the G-invariant subalgebra of the Weyl algebra. In this case, it is also isomorphic to the orbifold cohomology of V/G. (See Remark 8.2.) We thus display the orbifold cohomology $\mathrm{H}^{\bullet}_{\mathrm{orb}}(V/G)$ as a natural *subalgebra* of $\mathrm{HH}^{\bullet}(S(V)\#G)$.

In Section 9, we describe generators of cohomology (as an algebra) via the codimension poset. The partial order is defined on G modulo the kernel of its action on V. We view division in the volume algebra as a purely geometric construction by interpreting results in terms of this poset. Generators for the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ arise from minimal elements in the poset (with the identity removed); see Corollaries 9.3 and 9.4.

In Section 10, we explore implications for reflection groups. The reflection length of a group element is the length of a shortest word expressing that element as a product of reflections. For Coxeter groups and many complex reflection groups, reflection length coincides with the codimension of the fixed point space. In this case, the codimension poset appears as a well-studied poset (arising from reflection length) in the theory of reflection groups. We show (in Corollary 10.6) that for Coxeter groups and many other complex reflection groups, the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is generated as an algebra in homological degrees 0 and 1, in analogy with the Hochschild-Kostant-Rosenberg Theorem for smooth commutative algebras (such as $S(V)$).

Finally, in Section 11, we return to $\mathrm{HH}^{\bullet}(S(V)\#G)$ as the G-invariant subalgebra of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$. We point out the simple cup product structure in cohomological degrees 0 and 1. In Theorem 11.4, we use standard group-theoretic techniques (Green/Mackey functors and transfer maps) to describe the product on $\mathrm{HH}^{\bullet}(S(V)\#G)$. We use the natural symplectic action of the symmetric group to give a nontrivial example.

2. Poset and volume forms

We begin by collecting several geometric observations needed later. Let G be a finite group and V a $\mathbb{C}G$-module of finite dimension n. Denote the image of $v \in V$ under the action of $g \in G$ by ${}^{g}v$. We work with the induced group action on maps: For any function θ and any element $h \in \mathrm{GL}(V)$ acting on its domain and range, we define the map ${}^{h}\theta$ by $({}^{h}\theta)(v) := {}^{h}(\theta({}^{h^{-1}}v))$. Let V^* denote the contragredient (or dual) representation. For any basis $v_1, \ldots, v_n$ of V, let $v_1^*, \ldots, v_n^*$ be the dual basis of V^*. Let V^G be the set of G-invariants in V: $V^G = \{v \in V : {}^{g}v = v \text{ for all } g \in G\}$. For any $g \in G$, set $V^g = \{v \in V : {}^{g}v = v\}$, the fixed point set of g in V. Since G is finite, we assume G acts by isometries on V (i.e., G preserves a given inner product). All tensor and exterior products will be taken over $\mathbb{C}$ unless otherwise indicated.

We regard G modulo the kernel of its action on V as a poset using the following lemma, which is no surprise (see [4] or [15], for example). Note that $(V^g)^\perp = \mathrm{im}(1-g)$ for all g in G, where $(V^g)^\perp$ denotes the orthogonal complement to V^g.

Lemma 2.1. *Let $g, h \in G$. The following are equivalent:*
(i) $(V^g)^\perp \cap (V^h)^\perp = 0$,
(ii) $V^g + V^h = V$,
(iii) $\operatorname{codim} V^g + \operatorname{codim} V^h = \operatorname{codim} V^{gh}$,
(iv) $(V^g)^\perp \oplus (V^h)^\perp = (V^{gh})^\perp$.
Any of these properties implies that $V^g \cap V^h = V^{gh}$.

Proof. Taking orthogonal complements yields the equivalence of (i) and (ii).

Assume (ii) holds and write $u \in V^{gh}$ as $v + w$ where $v \in V^g$ and $w \in V^h$. Then ${}^h v + {}^h w = {}^{g^{-1}} v + {}^{g^{-1}} w$ and ${}^h v + w = v + {}^{g^{-1}} w$, i.e., ${}^{(h-1)} v = {}^{(g^{-1}-1)} w$. But $(V^h)^\perp = \mathrm{im}(h-1)$ while $(V^g)^\perp = (V^{g^{-1}})^\perp = \mathrm{im}(g^{-1}-1)$, and the intersection of these spaces is 0 by (i), hence ${}^{(h-1)} v = 0 = {}^{(g^{-1}-1)} w$. Therefore, $v \in V^h$, $w \in V^g$, and $u \in V^g \cap V^h$, and thus $V^{gh} \subset V^g \cap V^h$. The reverse inclusion is immediate, hence $V^{gh} = V^g \cap V^h$. We take orthogonal complements and observe that $(V^{gh})^\perp = (V^g)^\perp \oplus (V^h)^\perp$. A dimension count then gives (iii).

To show (iii) implies (iv), note $(V^{gh})^\perp \subset (V^g)^\perp + (V^h)^\perp$ since $V^g \cap V^h \subset V^{gh}$. By (iii), this containment is forced to be an equality and the sum is direct: $(V^g)^\perp \oplus (V^h)^\perp = (V^{gh})^\perp$. As (iv) trivially implies (i), we are finished. $\square$

Let K be the kernel of the representation of G acting on V:

$$K := \{k \in G : V^k = V\}.$$

Definition 2.2. *Define a binary relation $\leq$ on G by $g \leq h$ whenever*

$$\operatorname{codim}(V^g) + \operatorname{codim}(V^{g^{-1}h}) = \operatorname{codim}(V^h).$$

By Lemma 2.1, this codimension condition holds exactly when

$$(V^g)^\perp \oplus (V^{g^{-1}h})^\perp = (V^h)^\perp .$$

This induces a binary relation $\leq$ on the quotient group G/K as well: For g, h in G, define $gK \leq hK$ when $g \leq h$. Note the relation does not depend on choice of representatives of cosets, as $V^g = V^h$ whenever $gK = hK$ for g, h in G.

The relation $\leq$ appears in work of Brady and Watt [4] on orthogonal transformations. Their arguments apply equally well to our setting of isometries with respect to some inner product and the quotient group G/K. (Note that if G does not act faithfully, the binary relation $\leq$ on G may not be anti-symmetric and thus may not define a partial order on G.)

Lemma 2.3 (Brady and Watt [4]). *The relation $\leq$ is a partial order on G/K. If G acts faithfully, the relation $\leq$ is a partial order on G.*

We shall use the following elements in the sequel.

Definition 2.4. *For each $g \in G$, let $\mathrm{vol}_g^{\perp}$ be a choice of nonzero element in the one-dimensional space $\bigwedge^{\operatorname{codim} V^g}((V^g)^{\perp})^*$.*

We show in the next lemma how these choices determine a multiplicative cocycle. A function $\vartheta : G \times G \to \mathbb{C}$ is a **multiplicative 2-cocycle** on G if

$$\vartheta(gh, k)\vartheta(g, h) = \vartheta(g, hk)\vartheta(h, k)$$

for all $g, h, k \in G$. We may use any such cocycle to define a new algebra structure on the group algebra $\mathbb{C}[G]$, a generalization of a twisted group algebra (in which the values of ϑ may include 0): Let $\mathbb{C}^{\vartheta}[G]$ be the $\mathbb{C}$-algebra with basis G and multiplication $g \cdot_{\vartheta} h = \vartheta(g, h)\, gh$ for all $g, h \in G$. Associativity is equivalent to the 2-cocycle identity. If $\vartheta(g, 1_G) = 1 = \vartheta(1_G, g)$ for all $g \in G$, where 1_G denotes the identity element of G, then $\mathbb{C}^{\vartheta}[G]$ has multiplicative identity 1_G.

We canonically embed each space $\bigwedge((V^g)^{\perp})^*$ into $\bigwedge V^*$.

Proposition 2.5. *For all g and h in G,*

$$\mathrm{vol}_g^{\perp} \wedge \mathrm{vol}_h^{\perp} = \vartheta(g, h)\, \mathrm{vol}_{gh}^{\perp}$$

in $\bigwedge V^*$ where $\vartheta : G \times G \to \mathbb{C}$ is a (multiplicative) 2-cocycle on G with

$$\vartheta(g, h) \neq 0 \quad \text{if and only if} \quad g \leq gh\ .$$

Under wedge product, the algebra $\bigwedge\{\mathrm{vol}_g^{\perp} : g \in G\}$ is isomorphic to the (generalized) twisted group algebra $\mathbb{C}^{\vartheta}[G]$.

Proof. Let g, h be any pair of elements in G. Then $V^g \cap V^h \subset V^{gh}$, and hence $(V^{gh})^{\perp} \subset (V^g)^{\perp} + (V^h)^{\perp}$. If the sum is direct, then by Lemma 2.1, we have equality of vector subspaces: $(V^{gh})^{\perp} = (V^g)^{\perp} \oplus (V^h)^{\perp}$. If the sum is not direct, then $\mathrm{vol}_g^{\perp} \wedge \mathrm{vol}_h^{\perp} = 0$. In either case, the product $\mathrm{vol}_g^{\perp} \wedge \mathrm{vol}_h^{\perp}$ is a (possibly zero) scalar multiple of $\mathrm{vol}_{gh}^{\perp}$. Hence, there is a scalar $\vartheta(g, h) \in \mathbb{C}$ such that

$$\mathrm{vol}_g^{\perp} \wedge \mathrm{vol}_h^{\perp} = \vartheta(g, h)\, \mathrm{vol}_{gh}^{\perp}\, .$$

Note that $\vartheta(g,h) \neq 0$ if and only if $\operatorname{codim} V^g + \operatorname{codim} V^h = \operatorname{codim} V^{gh}$ (by Lemma 2.1) if and only if $g \leq gh$. By associativity of the exterior algebra, the function $\vartheta : G \times G \to \mathbb{C}$ is a (multiplicative) 2-cocycle on G. □

We shall also need the following easy lemma, which is a consequence of the fact that $(V^g)^\perp = \operatorname{im}(1-g)$ for all g. Let $I((V^g)^\perp)$ denote the ideal of $S(V)$ generated by $(V^g)^\perp$.

Lemma 2.6. *For all g in G:*

- *$f - {}^g f$ lies in the ideal $I((V^g)^\perp)$ for all $f \in S(V)$, and*
- *$dv \wedge \mathrm{vol}_g^\perp = {}^g dv \wedge \mathrm{vol}_g^\perp$ for all $dv \in \bigwedge^\bullet V$.*

3. Skew group algebra and Hochschild cohomology

In this section, we recall the basic definitions of the skew group algebra and Hochschild cohomology, as well as a fundamental theorem describing the cohomology of the skew group algebra as a space of invariants. We work over the complex numbers $\mathbb{C}$.

Let A denote any $\mathbb{C}$-algebra on which G acts by automorphisms. The **skew group algebra** (or **smash product**) $A\#G$ is the vector space $A \otimes \mathbb{C}G$ with multiplication given by

$$(a \otimes g)(b \otimes h) = a({}^g b) \otimes gh$$

for all $a, b \in A$ and $g, h \in G$.

The **Hochschild cohomology** of a $\mathbb{C}$-algebra A (such as $S(V)$ or $S(V)\#G$), with coefficients in an A-bimodule M, is the graded vector space $\mathrm{HH}^\bullet(A, M) = \mathrm{Ext}^\bullet_{A^e}(A, M)$, where $A^e = A \otimes A^{op}$ acts on A by left and right multiplication. We abbreviate $\mathrm{HH}^\bullet(A) = \mathrm{HH}^\bullet(A, A)$.

To construct the cohomology $\mathrm{HH}^\bullet(A, M)$, one applies the functor $\mathrm{Hom}_{A^e}(-, M)$ to a projective resolution of A as an A^e-module, for example, to the **bar resolution**

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{m} A \to 0, \tag{3.1}$$

where $\delta_p(a_0 \otimes \cdots \otimes a_{p+1}) = \sum_{j=0}^p (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{p+1}$ for all $a_0, \ldots, a_{p+1} \in A$, and m is multiplication. For all $p \geq 0$, $\mathrm{Hom}_{A^e}(A^{(p+2)}, M) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, M)$, and we identify these two vector spaces in what follows.

When M is itself an algebra, the Hochschild cohomology $\mathrm{HH}^\bullet(A, M)$ is a graded associative algebra under the **cup product**, defined at the cochain level on the bar complex (see [14, §7]): Let $f \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, M)$ and $f' \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes q}, M)$; then the cup product $f \smile f'$ in $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes (p+q)}, M)$ is given by

$$(f \smile f')(a_1 \otimes \cdots \otimes a_{p+q}) = f(a_1 \otimes \cdots \otimes a_p) f'(a_{p+1} \otimes \cdots \otimes a_{p+q})$$

for all $a_1, \ldots, a_{p+q} \in A$. We seek to describe the algebra structure under cup product explicitly in the case $A = S(V)$, $M = S(V)\#G$, and in the case $A = M = S(V)\#G$. These two cases are related by a well-known theorem that we state next.

Since $|G|$ is invertible, a result of Ştefan [27, Cor. 3.4] implies in our setting that there is a G-action giving an isomorphism of graded algebras (under the cup product):

Theorem 3.2.

$$\mathrm{HH}^{\bullet}(S(V)\#G) \quad \cong \quad \mathrm{HH}^{\bullet}(S(V), S(V)\#G)^G \ .$$

(Specifically, the action of G on V extends naturally to the bar complex of $S(V)$ and thus induces an action on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$, from which we define G-invariant cohomology in the theorem. In fact, any projective resolution of $S(V)$ compatible with the action of G may be used to define the G-invariant cohomology.) We thus concentrate on describing the cup product on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$.

4. Koszul and bar resolutions

One may use either the Koszul or the bar resolution of $S(V)$ to describe the cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ and thus its G-invariant subalgebra, $\mathrm{HH}^{\bullet}(S(V)\#G)$. The **Koszul resolution** is the following free $S(V)^e$-resolution of $S(V)$:

$$\cdots \xrightarrow{d_3} S(V)^e \otimes \textstyle\bigwedge^2 V \xrightarrow{d_2} S(V)^e \otimes \textstyle\bigwedge^1 V \xrightarrow{d_1} S(V)^e \xrightarrow{m} S(V) \to 0, \tag{4.1}$$

where the differential d is given by

$$d_p(1 \otimes 1 \otimes v_{j_1} \wedge \cdots \wedge v_{j_p}) = \sum_{i=1}^{p} (-1)^{i+1} (v_{j_i} \otimes 1 - 1 \otimes v_{j_i}) \otimes (v_{j_1} \wedge \cdots \wedge \hat{v}_{j_i} \wedge \cdots \wedge v_{j_p}), \tag{4.2}$$

for all $v_{j_1}, \ldots, v_{j_p} \in V$. Let Φ be the canonical inclusion (a chain map) of the Koszul resolution (4.1) into the bar resolution (3.1):

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & S(V)^{\otimes 4} & \xrightarrow{\delta_2} & S(V)^{\otimes 3} & \xrightarrow{\delta_1} & S(V)^e & \xrightarrow{m} & S(V) & \longrightarrow 0 \\
& \uparrow{\scriptstyle \Phi_2} & & \uparrow{\scriptstyle \Phi_1} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & \\
\cdots \longrightarrow & S(V)^e \otimes \bigwedge^2 V & \xrightarrow{d_2} & S(V)^e \otimes \bigwedge^1 V & \xrightarrow{d_1} & S(V)^e & \xrightarrow{m} & S(V) & \longrightarrow 0,
\end{array}$$

that is, for all $p \geq 1$,

$$\Phi_p \ : \ S(V)^e \otimes \textstyle\bigwedge^p V \to S(V)^{\otimes (p+2)},$$

$$\Phi_p(1\otimes 1\otimes v_{j_1}\wedge\cdots\wedge v_{j_p}) = \sum_{\pi\in \mathrm{Sym}_p} \mathrm{sgn}(\pi)\otimes v_{j_{\pi(1)}}\otimes\cdots\otimes v_{j_{\pi(p)}}\otimes 1\ , \tag{4.3}$$

for all $v_{j_1},\ldots,v_{j_p}\in V$, where Sym_p denotes the symmetric group on the set $\{1,\ldots,p\}$. Note that Φ is invariant under the action of $\mathrm{GL}(V)$, i.e., ${}^h\Phi=\Phi$ for all h in $\mathrm{GL}(V)$.

One finds cohomology $\mathrm{HH}^{\bullet}(S(V),M)$ by applying the functor $\mathrm{Hom}_{S(V)^e}(-,M)$ to either of the above two resolutions and dropping the term $\mathrm{Hom}_{S(V)^e}(S(V),M)$. We make the customary identifications: For each p, set $\mathrm{Hom}_{S(V)^e}(S(V)^{\otimes(p+2)},M)=\mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p},M)$ as before, and

$$\mathrm{Hom}_{S(V)^e}(S(V)^e\otimes \textstyle\bigwedge^p V, M) = \mathrm{Hom}_{\mathbb{C}}(\textstyle\bigwedge^p V, M) = \textstyle\bigwedge^p V^*\otimes M\ . \tag{4.4}$$

In the case $M=S(V)\#G$, we write $S(V)\otimes\bigwedge^p V^*\otimes\mathbb{C}[G]$ for the vector space $\bigwedge^p V^*\otimes M$.

We obtain a commutative diagram giving two different cochain complexes describing the cohomology $\mathrm{HH}^{\bullet}(S(V),S(V)\#G)$. In [25], we introduced a "combinatorial converter" map Υ (whose definition is recalled in the next section), which serves as an inverse to the induced map Φ^* and converts between complexes:

$$\begin{CD} \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, S(V)\#G) @>{\delta^*}>> \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes (p+1)}, S(V)\#G) \\ @A{\Upsilon_p}AA{\Phi_p^*} @A{\Upsilon_{p+1}}AA{\Phi_{p+1}^*} \\ S(V)\otimes\bigwedge^p V^*\otimes\mathbb{C}[G] @>{d^*}>> S(V)\otimes \bigwedge^{p+1} V^*\otimes\mathbb{C}[G] \end{CD}\ . \tag{4.5}$$

We use the maps Υ and Φ^* in our analysis of the cup product in later sections.

We describe cohomology explicitly in terms of cocycles and coboundaries. Under the identification (4.4), Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V),S(V)\#G)$ arises from the complex of cochains

$$C^{\bullet} := \bigoplus_{g\in G} S(V)\otimes \textstyle\bigwedge^{\bullet} V^*\otimes g\ . \tag{4.6}$$

One may determine the set of cocycles and coboundaries explicitly as the kernel and image of the induced map d^* (under the above identifications (4.4)). We set

$$Z^{\bullet} := \bigoplus_{g\in G} S(V)\otimes \textstyle\bigwedge^{\bullet-\operatorname{codim} V^g}(V^g)^*\otimes \bigwedge^{\operatorname{codim} V^g}((V^g)^{\perp})^*\otimes g\ , \tag{4.7}$$

a subspace of the space of cocycles, and

$$B^{\bullet} := \bigoplus_{g\in G} I((V^g)^{\perp})\otimes \textstyle\bigwedge^{\bullet-\operatorname{codim} V^g}(V^g)^*\otimes \bigwedge^{\operatorname{codim} V^g}((V^g)^{\perp})^*\otimes g\ ,$$

a subspace of the space of coboundaries, where $I((V^g)^{\perp})$ is the ideal of $S(V)$ generated by $(V^g)^{\perp}$. (We agree that a negative exterior power of a space is defined to be 0.) We regard these subspaces $Z^{\bullet}$ and $B^{\bullet}$ as subsets of the cochains $C^{\bullet}$ after making canonical identifications (we identify $W_1 \otimes W_2$ with $W_1 \wedge W_2$ for any subspaces W_1, W_2 of V intersecting trivially). We refer to cochains, cocycles, and coboundaries as vector forms "tagged" by the group elements indexing the direct summands above.

The next remark explains that we may view $Z^{\bullet}$ as a substitute for the set of cocycles and $B^{\bullet}$ as a substitute for the set of coboundaries. We use this alternate description of cohomology (as $Z^{\bullet}/B^{\bullet}$) in Section 6 to expose a smash product structure.

Remark 4.8. Note that one may use the isomorphism $V/(V^g)^{\perp} \cong V^g$ to select a set of a representatives of cohomology classes: Let

$$H^{\bullet} := \bigoplus_{g\in G} S(V^g) \otimes \textstyle\bigwedge^{\bullet - \operatorname{codim} V^g}(V^g)^* \otimes \textstyle\bigwedge^{\operatorname{codim} V^g}((V^g)^{\perp})^* \otimes g\ .$$

Then $H^{\bullet} \cong Z^{\bullet}/B^{\bullet}$ via the canonical map $\mathrm{Proj}_H : Z^{\bullet}/B^{\bullet} \to H^{\bullet}$ induced from the compositions, for each g in G,

$$S(V) \longrightarrow S(V)/I((V^g)^{\perp}) \xrightarrow{\sim} S(V^g)\ .$$

Farinati [13] and Ginzburg and Kaledin [15] showed that

$$\mathrm{HH}^{\bullet}(S(V), S(V)\#G) \cong H^{\bullet} \ \cong Z^{\bullet}/B^{\bullet}.$$

5. Combinatorial converter map

We recall the definition of the combinatorial converter map Υ (in Diagram 4.5) introduced in [25]. A nonidentity element of $\mathrm{GL}(V)$ is a **reflection** if it fixes a hyperplane in V pointwise. Given any basis $v_1, \ldots, v_n$ of V, let $\partial/\partial v_i$ denote the usual partial differential operator with respect to v_i. In addition, given a complex number $\epsilon \neq 1$, we define the **ϵ-quantum partial differential operator** with respect to $v := v_i$ as the scaled Demazure (BGG) operator $\partial_{v,\epsilon} : S(V) \to S(V)$ given by

$$\partial_{v,\epsilon}(f) \ = \ (1-\epsilon)^{-1}\, \frac{f - {}^s f}{v} \ = \ \frac{f - {}^s f}{v - {}^s v}\ , \tag{5.1}$$

where $s \in \mathrm{GL}(V)$ is the reflection whose matrix with respect to the basis $v_1, \ldots, v_n$ is $\mathrm{diag}(1, \ldots, 1, \epsilon, 1, \ldots, 1)$ with ϵ in the i-th slot. Set $\partial_{v,\epsilon} = \partial/\partial v$ when $\epsilon = 1$. The operator $\partial_{v,\epsilon}$ coincides with the usual definition of quantum partial differentiation: One takes the ordinary partial derivative with respect to v but instead of multiplying each monomial by its degree k in v, one multiplies by the quantum integer

$$[k]_{\epsilon} := 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{k-1}.$$

We define the map Υ in terms of these Demazure operators. For each g in G, fix a basis $B_g = \{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of g with corresponding eigenvalues $\epsilon_1, \ldots, \epsilon_n$. Decompose g into (commuting) reflections diagonal in this basis: Let $g = s_1 \cdots s_n$ where each s_i is either the identity or a reflection defined by $s_i(v_j) = v_j$ for $j \neq i$ and $s_i(v_i) = \epsilon_i v_i$. Let $\partial_i := \partial_{v_i, \epsilon_i}$, the quantum partial derivative with respect to B_g. Recall that $C^\bullet$ denotes cochains (see (4.6)).

Definition 5.2. *We define a map Υ from the dual Koszul complex to the dual bar complex with coefficients in $S(V)\#G$:*

$$\Upsilon_p : \quad C^p \to \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, S(V)\#G) \ .$$

For g in G with basis $B_g = \{v_1, \ldots, v_n\}$ of V as above, and $\alpha = f_g \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_p}^* \otimes g$ with $f_g \in S(V)$ and $1 \leq j_1 < \ldots < j_p \leq n$, define $\Upsilon(\alpha) : S(V)^{\otimes p} \to S(V)\#G$ by

$$\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_p) \ = \left(\prod_{k=1,\ldots,p} {}^{s_1 s_2 \cdots s_{j_k - 1}}(\partial_{j_k} f_k) \right) f_g \otimes g \ .$$

By Theorem 5.5 below, Υ is a cochain map. Thus Υ induces a map on the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$, which we denote by Υ as well.

We make the following remark, which will be needed in our analysis of the cup product in Section 7.

Remark 5.3. For the fixed basis $B_g = \{v_1, \ldots, v_n\}$ and $\alpha = f_g \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_p}^* \otimes g$ in C^p (with $j_1 < \ldots < j_p$), note that

$$\Upsilon(\alpha)(v_{i_1} \otimes \cdots \otimes v_{i_p}) = 0 \qquad \text{unless } i_1 = j_1, \ldots, i_p = j_p \ .$$

In general, $\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_p) = 0$ whenever $\dfrac{\partial}{\partial v_{j_k}}(f_k) = 0$ for some k.

We shall use the following consequence of the definitions.

Proposition 5.4. *For any choices of bases defining Υ,*

$$\Phi^* \Upsilon = 1$$

as a map on cochains $C^\bullet$.

Proof. Consider a nonzero cochain α in C^p. Without loss of generality, suppose that $\alpha = f_g \otimes v_1^* \wedge \cdots \wedge v_p^* \otimes g$ for some g in G, where f_g is in $S(V)$ and $B_g =$

$\{v_1, \ldots, v_n\}$ is the fixed basis of eigenvectors of g. Then for all $1 \le i_1, \ldots, i_p \le n$,

$$\begin{aligned}
(\Phi^*\Upsilon)(\alpha)(v_{i_1} \wedge \cdots \wedge v_{i_p}) &= \Upsilon(\alpha)(\Phi(v_{i_1} \wedge \cdots \wedge v_{i_p})) \\
&= \Upsilon(\alpha) \left(\sum_{\pi \in \mathrm{Sym}_p} \mathrm{sgn}(\pi)\ v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(p)}} \right) \\
&= \sum_{\pi \in \mathrm{Sym}_p} \mathrm{sgn}(\pi)\ \Upsilon(\alpha)(v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(p)}})\ ,
\end{aligned}$$

while
$$\alpha(v_{i_1} \wedge \cdots \wedge v_{i_p}) = f_g\ (v_1^* \wedge \cdots \wedge v_p^*)(v_{i_1} \wedge \cdots \wedge v_{i_p}) \otimes g\ .$$

By Remark 5.3, both expressions are zero unless $\{v_{i_1}, \ldots v_{i_p}\} = \{v_1, \ldots, v_p\}$, in which case both yield $\mathrm{sgn}(\pi)\, f_g \otimes g$, where π is the permutation sending $(i_1, \ldots, i_p)$ to $(1, \ldots, p)$. □

We summarize results needed from [25]:

Theorem 5.5. *The combinatorial converter map*

$$\begin{array}{rrcl}
\Upsilon : & \textit{Dual Koszul Complex} & \to & \textit{Dual Bar Complex} \\
& C^p & \to & \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, S(V)\#G)
\end{array}$$

induces isomorphisms of cohomology independent of choices of bases:

- *For any basis B_g of eigenvectors for any g in G, Υ is a cochain map.*
- *Although the cochain map Υ depends on choices B_g for g in G, the induced map Υ on cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is independent of choices.*
- *The map Υ induces an automorphism of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ with inverse automorphism Φ^*. Specifically, Υ and Φ^* convert between expressions of cohomology in terms of the Koszul resolution and the bar resolution.*
- *The map Υ on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is G-invariant and hence induces an automorphism on $\mathrm{HH}^{\bullet}(S(V)\#G) \cong \mathrm{HH}^{\bullet}(S(V), S(V)\#G)^G$.*

Remark 5.6. We do not symmetrize Υ, in comparison with similar maps in the literature (see Anno [1] and Halbout and Tang [17]). Since they are chain maps, our maps are the same on cohomology as their symmetrized versions. Symmetrization may be more elegant, however unsymmetrized maps can be more convenient for computation.

6. Smash product structure

In this section, we transform the "cohomology of a group action" into a "group action on cohomology" by viewing both as algebras. We relate Hochschild cohomology of the skew group algebra, $\mathrm{HH}^\bullet(S(V)\#G)$, to the skew group algebra of a Hochschild cohomology algebra, $\mathrm{HH}^\bullet(S(V))\#G$: We manifest the first algebra as a subquotient of the second. We thus twist the cup product on $\mathrm{HH}^\bullet(S(V))$ by the group action and obtain a natural smash product on $\mathrm{HH}^\bullet(S(V)\#G)$. In the next section, we show that the cup product on $\mathrm{HH}^\bullet(S(V)\#G)$ is precisely this natural smash product.

We first embed $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ as a graded vector space into

$$\mathrm{HH}^\bullet(S(V))\,\#\,G\ ,$$

the skew group algebra determined by the action of G on $\mathrm{HH}^\bullet(S(V))$ induced from its action on V. Note that for a general algebra S with action of G by automorphisms, $\mathrm{HH}^\bullet(S, S\#G)$ is a *G-graded* algebra; we show that in the special case $S = S(V)$, it is not only G-graded, but is very close to being a smash product itself. To see this, we first identify the Hochschild cohomology $\mathrm{HH}^\bullet(S(V))$ with the set of vector forms on V (cf. [31]):

$$\mathrm{HH}^\bullet(S(V)) = S(V)\otimes \textstyle\bigwedge^\bullet V^*\ .$$

The group G acts on this tensor product diagonally, and the skew group algebra $\mathrm{HH}^\bullet(S(V))\#G$ is the $\mathbb{C}$-vector space of cochains,

$$\mathrm{HH}^\bullet(S(V))\otimes\mathbb{C}[G] = S(V)\otimes \textstyle\bigwedge^\bullet V^*\otimes\mathbb{C}[G] = C^\bullet\ ,$$

together with smash product

$$(f_g\otimes dv_g\otimes g)\diamond(f'_h\otimes dv'_h\otimes h) = f_g\,{}^g f'_h\otimes(dv_g\wedge {}^g dv'_h)\otimes gh\ , \tag{6.1}$$

where $f_g, f'_h\in S(V)$, $g,h\in G$, and $dv_g, dv'_h\in\bigwedge^\bullet V^*$.

We regard $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ as a *vector space subquotient* of $\mathrm{HH}^\bullet(S(V))\#G$ by identifying with $Z^\bullet/B^\bullet$ (see Remark 4.8 and the comments before it):

$$\mathrm{HH}^\bullet(S(V), S(V)\#G) = Z^\bullet/B^\bullet \quad\subset\quad C^\bullet/B^\bullet = \big(\mathrm{HH}^\bullet(S(V))\otimes\mathbb{C}[G]\big)/B^\bullet\ . \tag{6.2}$$

We next recognize this subquotient of vector spaces as a *subquotient of algebras* under the smash product.

Proposition 6.3. *Under the smash product of* $\mathrm{HH}^\bullet(S(V))\#G$*, the space $Z^\bullet$ forms a subalgebra of $C^\bullet$ and the space $B^\bullet$ forms an ideal of $Z^\bullet$: For all cochains α and β in* $\mathrm{HH}^\bullet(S(V))\#G$*,*

- *If α and β lie in $Z^\bullet$, then $\alpha\diamond\beta$ also lies in $Z^\bullet$,*
- *If α lies in $Z^\bullet$ and β lies in $B^\bullet$, then $\alpha\diamond\beta$ and $\beta\diamond\alpha$ also lie in $B^\bullet$.*

Proof. Let α and β be cocycles in $Z^\bullet$. Without loss of generality, suppose that

$$\alpha = f_g \otimes dv_g \otimes \mathrm{vol}_g^\perp \otimes g \quad \text{and} \quad \beta = f'_h \otimes dv'_h \otimes \mathrm{vol}_h^\perp \otimes h$$

for g, h in G, f_g, f'_h in $S(V)$, and dv_g, dv'_h in $\bigwedge V^*$. By Lemma 2.6 (see (6.1)),

$$\begin{aligned}\alpha \diamond \beta &= f_g\, {}^g f'_h \otimes (dv_g \wedge \mathrm{vol}_g^\perp \wedge {}^g(dv'_h \wedge \mathrm{vol}_h^\perp)) \otimes gh \\ &= f_g\, {}^g f'_h \otimes (dv_g \wedge \mathrm{vol}_g^\perp \wedge dv'_h \wedge \mathrm{vol}_h^\perp) \otimes gh \\ &= \pm\ f_g\, {}^g f'_h \otimes (dv_g \wedge dv'_h \wedge \mathrm{vol}_g^\perp \wedge \mathrm{vol}_h^\perp) \otimes gh\ .\end{aligned}$$

Assume this product is nonzero. Then by Proposition 2.5 and Lemma 2.1,

$$(V^g)^\perp \oplus (V^h)^\perp = (V^{gh})^\perp = (V^{hg})^\perp$$

and $\mathrm{vol}_g^\perp \wedge \mathrm{vol}_h^\perp$ is a scalar multiple of $\mathrm{vol}_{gh}^\perp$. Hence $\alpha \diamond \beta$ lies in $Z^\bullet$.

Now assume further that β is a coboundary in $B^\bullet$, i.e., that f'_h lies in the ideal $I((V^h)^\perp)$ of $S(V)$. Note that ${}^g((V^h)^\perp) \subset {}^g((V^{hg})^\perp) = (V^{ghgg^{-1}})^\perp = (V^{gh})^\perp$. Hence ${}^g f'_h$, and thus the product $f_g\, {}^g f'_h$, lies in $I((V^{gh})^\perp)$. Therefore, $\alpha \diamond \beta$ is an element of $B^\bullet$. The argument for $\beta \diamond \alpha$ is similar (and easier). $\square$

The proposition above immediately implies that the smash product on the skew group algebra $\mathrm{HH}^\bullet(S(V))\#G$ induces a smash algebra product on the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$, as we see in the next two results.

Corollary 6.4. *The vector space subquotient $Z^\bullet/B^\bullet$ of $\mathrm{HH}^\bullet(S(V))\#G$ is an algebra subquotient (subalgebra of a quotient of algebras) under the induced smash product.*

Note that cohomology classes (and cocycles) in general are represented by *sums* of elements of the form of α and β given in the next proposition.

Proposition 6.5. *The cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ identifies naturally as a graded vector space with an algebra subquotient of the smash product $\mathrm{HH}^\bullet(S(V))\#G$. Under this identification, $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ inherits the smash product: For cohomology classes in $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ represented by cocycles in $Z^\bullet$,*

$$\alpha = \sum_{g\in G} f_g \otimes dv_g \otimes g \quad \text{and} \quad \beta = \sum_{h \in G} f'_h \otimes dv'_h \otimes h\ ,$$

where each $f_g, f'_h \in S(V)$ and each $dv_g, dv'_h \in \bigwedge^\bullet V^*$, the smash product

$$\alpha \diamond \beta = \sum_{\substack{g,h\in G \\ g \le gh}} f_g\, {}^g f'_h \otimes (dv_g \wedge {}^g dv'_h) \otimes\ gh \tag{6.6}$$

is also a cocycle representing a class of $\mathrm{HH}^\bullet(S(V), S(V)\#G)$.

Proof. We saw in (6.2) that $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is isomorphic to the vector space subquotient

$$Z^{\bullet}/B^{\bullet} \quad \text{of} \quad \big(\mathrm{HH}^{\bullet}(S(V))\#G\big)/B^{\bullet}\ .$$

(In fact, we may identify $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ with a *subset* of $\mathrm{HH}^{\bullet}(S(V))\#G$; see Remarks 4.8 and 6.7.) By Proposition 6.3, the smash product on $\mathrm{HH}^{\bullet}(S(V))\#G$ may be restricted to the subset $Z^{\bullet}$ and induces a multiplication on the subquotient $Z^{\bullet}/B^{\bullet}$ (Corollary 6.4). Hence $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is isomorphic to an algebra carrying a natural smash product, and thus it inherits (under this isomorphism) a natural smash product of its own. Formula (6.1) gives a cohomology class representative of the smash product of two cocycles. □

The proposition above explains that the vector space inclusion of (6.2) yields an injection of algebras: The algebra $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$, under smash product $\diamond$, is isomorphic to an algebra subquotient of $\mathrm{HH}^{\bullet}(S(V))\#G$. But for any $\mathbb{C}$-algebra A carrying an action of G, the smash product on the skew group algebra $A\#G$ maps G-invariants to G-invariants. Hence, the algebra $\mathrm{HH}^{\bullet}(S(V)\#G)$ also inherits a smash product and is isomorphic (under smash product) to an algebra subquotient of $\mathrm{HH}^{\bullet}(S(V))\#G$. We shall see in Corollary 7.6 below that the same is true under cup product.

Remark 6.7. One may identify $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ with the subset $H^{\bullet}$ of the smash product $\mathrm{HH}^{\bullet}(S(V))\#G$ by fixing a set of cohomology class repesentatives as in Remark 4.8. But note that $H^{\bullet}$ is not closed under the smash product as a subset of $\mathrm{HH}^{\bullet}(S(V))\#G$—we must take a quotient by coboundaries and again take chosen representatives for this quotient. The induced smash product explicitly becomes

$$(\alpha, \beta) \mapsto \mathrm{Proj}_H(\alpha \diamond \beta)\ .$$

7. Equivalence of cup and smash products

In the previous section, we twisted the cup product on $\mathrm{HH}^{\bullet}(S(V))$ by the action of G in a natural way to define a multiplication on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ and also on $\mathrm{HH}^{\bullet}(S(V)\#G)$. In fact, we showed that as a graded vector space, the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V)\#G)$ maps isomorphically to an algebra subquotient of $\mathrm{HH}^{\bullet}(S(V))\#G$ and thus inherits a natural smash product structure. We now regard $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ as an algebra under three operations:

- the cup product $\smile$ induced from the bar resolution of $S(V)$,
- a "smash" product $\diamond$ induced from $\mathrm{HH}^{\bullet}(S(V))\#G$, and
- the usual multiplication in the algebra tensor product $\mathrm{HH}^{\bullet}(S(V)) \otimes \mathbb{C}[G]$.

We show in this section that these three basic algebraic operations coincide. This allows us to describe generators for Hochschild cohomology in the next section. The equality of cup and smash products (given in the theorem below) also explains how the cup product, defined on the bar resolution, may be expressed in terms of the Koszul resolution.

Theorem 7.1. *For all cocycles α and β in $Z^{\bullet}$,*

$$\alpha \smile \beta = \alpha \diamond \beta \ .$$

On the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$, this product is induced from the usual multiplication on the tensor product $\mathrm{HH}^{\bullet}(S(V)) \otimes \mathbb{C}[G]$ of algebras.

Proof. We need only verify the statement for cocycles in $Z^{\bullet}$ of the form $\alpha = f_g \otimes dv_g \otimes g$ and $\beta = f'_h \otimes dv'_h \otimes h$, for some g, h in G, where $f_g, f'_h \in S(V)$ and $dv_g \in \bigwedge^p V^*$, $dv'_h \in \bigwedge^q V^*$ (as any cocycle in $Z^{\bullet}$ will be the sum of such elements).

As Φ^* and Υ are inverse maps on the cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ converting between Koszul and bar cochain complexes (see Diagram 4.5),

$$\alpha \smile \beta = \Phi^*(\Upsilon(\alpha) \smile \Upsilon(\beta))$$

where $\smile$ (on the right hand side) denotes cup product on the bar complex. Suppose $B_g = \{v_1, \ldots, v_n\}$, a basis of eigenvectors of g (see Definition 5.2). Without loss of generality, assume that $dv_g = v_1^* \wedge \cdots \wedge v_p^*$ where the span of $v_1, \ldots, v_p$ includes $(V^g)^{\perp}$ (see (4.7)). (In general, dv_g will be a sum of such elements with indices relabeled.) By the second part of Lemma 2.6,

$$\begin{aligned} \alpha \diamond \beta &= (f_g \otimes dv_g \otimes g) \diamond (f'_h \otimes dv'_h \otimes h) \\ &= f_g \, {}^g f'_h \otimes (dv_g \wedge {}^g dv'_h) \otimes gh \\ &= f_g \, {}^g f'_h \otimes (dv_g \wedge dv'_h) \otimes gh \ . \end{aligned}$$

We compare the values of $\alpha \smile \beta$ and $\alpha \diamond \beta$ on any $v_{i_1} \wedge \cdots \wedge v_{i_{p+q}}$. Now

$$(\alpha \diamond \beta)(v_{i_1} \wedge \cdots \wedge v_{i_{p+q}}) = f_g \, {}^g f'_h \ (dv_g \wedge dv'_h)(v_{i_1} \wedge \cdots \wedge v_{i_{p+q}}) \otimes gh$$

while

$$\begin{aligned} &(\alpha \smile \beta)(v_{i_1} \wedge \cdots \wedge v_{i_{p+q}}) \\ &\quad = \Phi^*(\Upsilon(\alpha) \smile \Upsilon(\beta))(v_{i_1} \wedge \cdots \wedge v_{i_{p+q}}) \\ &\quad = (\Upsilon(\alpha) \smile \Upsilon(\beta))\big(\sum_{\pi \in \mathrm{Sym}_{p+q}} \mathrm{sgn}(\pi) v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(p+q)}} \big) \\ &\quad = \sum_{\pi \in \mathrm{Sym}_{p+q}} \mathrm{sgn}(\pi) \ \Upsilon(\alpha)(v_{i_{\pi(1)}} \otimes \cdots \otimes v_{i_{\pi(p)}}) \ \Upsilon(\beta)(v_{i_{\pi(p+1)}} \otimes \cdots \otimes v_{i_{\pi(p+q)}}) \ . \end{aligned}$$

By Remark 5.3, both $\alpha \smile \beta$ and $\alpha \diamond \beta$ are readily seen to be zero on $v_{i_1} \wedge \cdots \wedge v_{i_{p+q}}$ unless $\{v_{i_1}, \ldots, v_{i_{p+q}}\}$ contains $\{v_1, \ldots, v_p\}$. Thus, we may assume (after relabeling indices and also possibly changing signs throughout) that $v_1 = v_{i_1}, \dots, v_{p+q} = v_{i_{p+q}}$.

Again by Remark 5.3, $(\alpha \smile \beta)(v_1 \wedge \cdots \wedge v_{p+q})$ is equal to

$$\sum_{\pi \in \mathrm{Sym}_p} \mathrm{sgn}(\pi)\ \Upsilon(\alpha)(v_{\pi(1)} \otimes \cdots \otimes v_{\pi(p)}) \sum_{\pi \in \mathrm{Sym}_q} \mathrm{sgn}(\pi)\ \Upsilon(\beta)(v_{p+\pi(1)} \otimes \cdots \otimes v_{p+\pi(q)})\ .$$

Proposition 5.4 then implies that this value of the cup product is just

$$\begin{aligned}\Phi^*(\Upsilon(\alpha))(v_1 \wedge \cdots \wedge v_p) \cdot \Phi^*(\Upsilon(\beta))(v_{p+1} \wedge \cdots \wedge v_{p+q})&\\ = \alpha(v_1 \wedge \cdots \wedge v_p) \cdot \beta(v_{p+1} \wedge \cdots \wedge v_{p+q})&\\ = f_g\, {}^g f'_h\ dv_g(v_1 \wedge \cdots \wedge v_p)\ dv'_h(v_{p+1} \wedge \cdots \wedge v_{p+q}) \otimes gh&\\ = f_g\, {}^g f'_h\ (dv_g \wedge\ dv'_h)(v_1 \wedge \cdots \wedge v_{p+q}) \otimes gh&\\ = (\alpha \diamond \beta)(v_1 \wedge \cdots \wedge v_{p+q})&\end{aligned}$$

as elements of $S(V)\#G$. Thus $\alpha \smile \beta$ and $\alpha \diamond \beta$ agree as cochains.

If this product is nonzero, then $f'_h - {}^g f'_h$ lies in the ideal $I((V^g)^\perp) \subset I((V^{gh})^\perp)$ by Lemma 2.1, Proposition 2.5, and the first part of Lemma 2.6. Hence,

$$f_g\, {}^g f'_h\ \otimes (dv_g \wedge\ dv'_h) \otimes gh \quad \text{and} \quad f_g\, f'_h\ \otimes (dv_g \wedge\ dv'_h) \otimes gh$$

represent the same class in the cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G) = Z^{\bullet}/B^{\bullet}$. Thus the usual multiplication in the tensor product of algebras $\mathrm{HH}^{\bullet}(S(V)) \otimes \mathbb{C}[G]$ gives the cup product on cohomology. $\square$

We next give an example to show that the cup and smash products do not agree on arbitrary cochains.

Example 7.2. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by elements a_1, a_2, a_3. Let $V = \mathbb{C}^3$ with basis v_1, v_2, v_3 on which G acts as follows:

$${}^{a_i}v_j = (-1)^{\delta_{ij}} v_j.$$

Let $\alpha = 1 \otimes v_3^* \otimes a_1a_3$ and $\beta = 1 \otimes v_1^* \wedge v_2^* \otimes a_1a_2$ in $C^{\bullet}$. Then

$$\alpha \diamond \beta = -1 \otimes v_1^* \wedge v_2^* \wedge v_3^* \otimes a_2a_3 = -\alpha \smile \beta.$$

Our results imply the following explicit formula for the cup product, first given by Anno [1], expressed here in terms of the poset in Definition 2.2 and the multiplicative cocycle $\vartheta : G \times G \to \mathbb{C}$ of Proposition 2.5. Note that the condition $g < gh$ in the sum below is included merely for computational convenience, as $\vartheta(g, h)$ is nonzero if and only if $g < gh$. Also note that cohomology classes (and cocycles) in general are represented by *sums* of such α and β given in the theorem below.

Theorem 7.3. *Consider cohomology classes in* $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ *represented by*

$$\alpha = \sum_{g\in G} f_g \otimes dv_g \otimes \mathrm{vol}_g^{\perp} \otimes g \quad \textit{and} \quad \beta = \sum_{h\in G} f'_h \otimes dv'_h \otimes \mathrm{vol}_h^{\perp} \otimes h$$

in $Z^{\bullet}$ *of degrees* p *and* q*, resp. (see (4.7)). Set* $m = (\operatorname{codim} V^g)(q - \operatorname{codim} V^h)$*. The cup product* $\alpha \smile \beta$ *is represented by the cocycle*

$$\sum_{\substack{g,h\in G\\ \text{with } g\leq gh}} (-1)^m\, \vartheta(g,h)\ f_g\, f'_h \otimes (dv_g \wedge dv'_h) \otimes \mathrm{vol}_{gh}^{\perp} \otimes\, gh\ . \tag{7.4}$$

Proof. The formula follows directly from Proposition 2.5, Proposition 6.5, and Theorem 7.1 (see its proof). Note that the factor $(-1)^m$ arises when we replace $\mathrm{vol}_g^{\perp} \wedge dv'_h$ with $dv'_h \wedge \mathrm{vol}_g^{\perp}$. □

Example 7.5. Let G, V be as in Example 7.2. Let $\alpha = 1 \otimes v_1^* \otimes a_1$, $\beta = 1 \otimes v_3^* \wedge v_2^* \otimes a_2$ in $H^{\bullet}$. Let $\mathrm{vol}_{a_1}^{\perp} = v_1^*$, $\mathrm{vol}_{a_2}^{\perp} = v_2^*$, and $\mathrm{vol}_{a_1a_2}^{\perp} = v_1^* \wedge v_2^*$, so that $\vartheta(a_1, a_2) = 1$. Then

$$\begin{aligned} \alpha \smile \beta \;\; = \;\; \alpha \diamond \beta \;\; &= \;\; (-1)^{1(2-1)} \otimes {}^{a_1}(v_3^*) \wedge v_1^* \wedge v_2^* \otimes a_1a_2 \\ &= \;\; -1 \otimes v_3^* \wedge \mathrm{vol}_{a_1a_2}^{\perp} \otimes\, a_1a_2. \end{aligned}$$

Corollary 7.6. *The algebra* $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ *is isomorphic to an algebra subquotient of* $\mathrm{HH}^{\bullet}(S(V))\#G$*. The algebra* $\mathrm{HH}^{\bullet}(S(V)\#G)$ *is isomorphic to its* G*-invariant subalgebra. Hence, it is also an algebra subquotient of* $\mathrm{HH}^{\bullet}(S(V))\#G$*.*

Proof. The space $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ may be written as a (graded) subspace of a vector space quotient of $\mathrm{HH}^{\bullet}(S(V))\#G$ (see (6.2)). Proposition 6.5 states that this subquotient is actually a subquotient of algebras under the smash product (see Proposition 6.3 and Corollary 6.4). But the cup product on $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is the same as this induced smash product by Theorem 7.1. By Theorem 3.2, the cohomology algebra $\mathrm{HH}^{\bullet}(S(V)\#G)$ is isomorphic to the G-invariant subalgebra of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ (both spaces regarded as algebras under their respective cup products). Hence the algebra $\mathrm{HH}^{\bullet}(S(V)\#G)$ is also a subquotient of $\mathrm{HH}^{\bullet}(S(V))\#G$. □

8. Volume algebra

Results in the last section reveal an interesting subalgebra of the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ isomorphic to algebras that have appeared in the literature before (see Remark 8.2 below); we call it the **volume algebra**. We use this subalgebra to give algebra generators of cohomology in the next section.

We define the volume algebra with the next proposition. Recall that for each g in G, the form $\mathrm{vol}_g^{\perp}$ is a choice of nonzero element in the one-dimensional space $\bigwedge^{\operatorname{codim} V^g}((V^g)^{\perp})^*$. We show that the elements $\mathrm{vol}_g^{\perp} \otimes g := 1 \otimes \mathrm{vol}_g^{\perp} \otimes g$ in $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ generate a subalgebra which captures the binary relation $\leq$ on G and reflects the poset structure of G/K (see Definition 2.2). (Note that if k acts trivially on V, then $\mathrm{vol}_k^{\perp} = 1$ up to a nonzero constant.)

Proposition 8.1. *The $\mathbb{C}$-vector space*

$$A_{\mathrm{vol}} := \mathrm{Span}_{\mathbb{C}}\{\mathrm{vol}_g^{\perp} \otimes g \mid g \in G\}$$

is a subalgebra of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$. The induced multiplication on this subalgebra is given by

$$(\mathrm{vol}_g^{\perp} \otimes g)(\mathrm{vol}_h^{\perp} \otimes h) = \vartheta(g,h)\,\mathrm{vol}_{gh}^{\perp} \otimes gh$$

where $\vartheta : G \times G \to \mathbb{C}$ is a (multiplicative) cocycle on G. For any g, h in G, the above product is nonzero ($\vartheta(g,h) \neq 0$) if and only if $g \leq gh$. The algebra A_{vol} is isomorphic to the (generalized) twisted group algebra $\mathbb{C}^{\vartheta}[G]$.

Proof. Lemma 2.6 and Proposition 2.5 imply that for any pair g, h in G,

$$\begin{aligned}(\mathrm{vol}_g^{\perp} \otimes g) \ \diamond\ (\mathrm{vol}_h^{\perp} \otimes h) &= (\mathrm{vol}_g^{\perp} \wedge {}^{g}(\mathrm{vol}_h^{\perp})) \otimes\ gh \\ &= (\mathrm{vol}_g^{\perp} \wedge \mathrm{vol}_h^{\perp}) \otimes\ gh \\ &= \vartheta(g,h)\,\mathrm{vol}_{gh}^{\perp} \otimes gh\ ,\end{aligned}$$

for a cocycle ϑ which is nonzero on the pair (g,h) exactly when $g \leq gh$. Hence by Theorem 7.1, A_{vol} is closed under cup product and isomorphic to $\mathbb{C}^{\vartheta}[G]$. ☐

The above proposition explains how A_{vol} algebraically captures the geometric relation on G given by $\leq$. Indeed, the space A_{vol} forms a *graded algebra* where

$$\deg(\mathrm{vol}_g^{\perp} \otimes g) = \operatorname{codim} V^g \quad \text{for each } g \in G\ .$$

Moreover, for all g, h in G,

$$g \leq h \qquad \text{if and only if} \qquad \mathrm{vol}_g^{\perp} \otimes g \ \text{ divides }\ \mathrm{vol}_h^{\perp} \otimes h \quad \text{in } A_{\mathrm{vol}}\ .$$

Let S be the subset $\{\mathrm{vol}_g^{\perp} \otimes g : g \in G\}$ of A_{vol}. Then S is a poset under division:

$$\mathrm{vol}_g^{\perp} \otimes g \ \leq_S\ \mathrm{vol}_h^{\perp} \otimes h \qquad \text{if and only if} \qquad \mathrm{vol}_g^{\perp} \otimes g \ \text{ divides }\ \mathrm{vol}_h^{\perp} \otimes h \quad \text{in}\ \ A_{\mathrm{vol}}\ .$$

If G acts faithfully, we have isomorphic posets:

$$(G, \leq) \cong (S, \leq_S) \ .$$

Remark 8.2. The subalgebra A_{vol} of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ appears in other arenas. If V is a symplectic vector space and $G \subset \mathrm{Sp}(V)$, the algebra A_{vol} is isomorphic to the graded algebra $\mathrm{gr}_F \mathbb{C}[G]$ associated to the filtration F on G assigning to each group element g the codimension of V^g. Its G-invariant subalgebra A_{vol}^G is isomorphic to the orbifold cohomology of V/G in this case (see Fantechi and Göttsche [12, §2] or Ginzburg-Kaledin [15]) and also to the Hochschild cohomology of the G-invariant subalgebra of the Weyl algebra $A(V)$ (see Suarez-Alvarez [28]):

$$\mathrm{HH}^{\bullet}(A(V)^G) \cong \mathrm{HH}^{\bullet}\big(A(V)\#G\big) \ \cong \ A_{\mathrm{vol}}^G \ \cong \ \mathrm{H}^{\bullet}_{\mathrm{orb}}\big(V/G\big) \ .$$

Work of several authors has shown that for $G = \mathrm{Sym}_n$ acting symplectically on $\mathbb{C}^{2n}$, this orbifold cohomology is isomorphic to the cohomology of a Hilbert scheme which is a crepant resolution of the orbifold [12, 20, 30]. Lehn and Sorger gave a description of $A_{\mathrm{vol}}^{\mathrm{Sym}_n}$ in terms of generators and relations [20, Remark 6.3].

9. Cohomology generators given by the poset

In this section, we give generators for the algebra $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ in terms of the partial order $\leq$ on the quotient group G/K, where K is the kernel of the representation of the group G acting on V. In Section 7, we showed that the cup product and an induced smash product on the algebra $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ agree. Hence, we simply discuss generation of cohomology as an algebra, without explicit reference to the product. We explain in the next three results how generators for $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ are tagged by K together with minimal elements in the poset G/K. Actually, $1_{G/K} := K$ is the unique minimal element in the poset G/K, and we remove it before seeking minimal elements.

Theorem 9.1. *The subalgebra A_{vol} of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is generated as an algebra over $\mathbb{C}$ by*

$$\{\mathrm{vol}_g^{\perp} \otimes g \mid g \in K \textit{ or } g \in [G/K] \textit{ with } gK \textit{ minimal in the poset } G/K - \{1_{G/K}\}\} \ ,$$

where $[G/K]$ is any set of coset representatives for G/K.

Proof. Assume $h \notin K$. We first observe that we may write $\mathrm{vol}_h^{\perp} \otimes h$ as the product of two volume forms, one tagged by any other coset representive of hK and the other tagged by an element of K: Suppose $hK = h'K$ for some h' in G. Then

$V^h = V^{h'}$ and $\mathrm{vol}_h^\perp = \mathrm{vol}_{h'}^\perp$ up to a nonzero scalar. Let $k = h^{-1}h'$, so $\mathrm{vol}_k^\perp$ is a nonzero scalar itself. Then (see Equation 6.1),

$$(\mathrm{vol}_h^\perp \otimes h) \diamond (\mathrm{vol}_k^\perp \otimes k) = (\mathrm{vol}_h^\perp \wedge {}^h(\mathrm{vol}_k^\perp)) \otimes\ hk,$$

which is a nonzero scalar multiple of $\mathrm{vol}_h^\perp \otimes h'$, and thus of $\mathrm{vol}_{h'}^\perp \otimes h'$. We may therefore assume that h itself lies in $[G/K]$.

Suppose that hK is not minimal in the partial order on $G/K - \{1_{G/K}\}$. Then there exists $g \in [G/K]$ with $K \neq gK \neq hK$ and $gK < hK$, i.e., $(V^g)^\perp \oplus (V^{g^{-1}h})^\perp = (V^h)^\perp$. By Theorem 7.3,

$$(\mathrm{vol}_g^\perp \otimes g)\ \diamond\ (\mathrm{vol}_{g^{-1}h}^\perp \otimes g^{-1}h) = \vartheta(g, g^{-1}h)\, \mathrm{vol}_h^\perp \otimes h\ ,$$

which is nonzero since $gK < hK$. By symmetry in the definition of the partial order, $g^{-1}hK < hK$ in the poset $G/K - \{1_{G/K}\}$ as well. Hence, we have written $\mathrm{vol}_h^\perp \otimes h$ as a product of volume forms each tagged by a group element less than h in the partial order. As the set $G/K - \{1_{G/K}\}$ is finite, the partial order is well-founded, i.e., every descending chain contains a minimal element. Hence, by induction, $\mathrm{vol}_h^\perp \otimes h$ may be written (up to a scalar) as the product of elements in the set given in the statement of the theorem. □

We now turn to the task of describing generators for the full Hochschild cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ as an algebra. We regard $\mathrm{HH}^\bullet(S(V))$ as a subalgebra of $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ by identifying

$$\mathrm{HH}^\bullet(S(V)) = S(V) \otimes \textstyle\bigwedge^\bullet V^* \quad \text{with} \quad S(V) \otimes \textstyle\bigwedge^\bullet V^* \otimes 1_G\ .$$

Then $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ becomes a module over $\mathrm{HH}^\bullet(S(V))$ under cup product.

Theorem 9.2. *The Hochschild cohomology algebra* $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ *is generated by its subalgebras* $\mathrm{HH}^\bullet(S(V))$ *and* A_{vol}.

Proof. Let α be an arbitrary element of $\mathrm{HH}^\bullet(S(V), S(V)\#G)$. Without loss of generality, assume $\alpha = f_h \otimes dy \wedge \mathrm{vol}_h^\perp \otimes h$ for some h in G and $dy \in \bigwedge^\bullet (V^h)^*$. By Theorem 7.1 (see (7.4)),

$$\alpha = (f_h \otimes dy \otimes 1_G) \diamond (\mathrm{vol}_h^\perp \otimes h)$$

where $f_h \otimes dy \otimes 1_G$ identifies with $f_h \otimes dy$ in $\mathrm{HH}^\bullet(S(V))$. Hence, α lies in the product $\mathrm{HH}^\bullet(S(V)) \cdot A_{\mathrm{vol}}$. □

Recall that for k in K, $\mathrm{vol}_k^\perp = 1$ up to a nonzero constant in $\mathbb{C}$. The last two theorems then imply:

Corollary 9.3. *The Hochschild cohomology algebra* $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ *is generated by its subalgebra* $\mathrm{HH}^{\bullet}(S(V))$ *and*

$$\{\mathrm{vol}_g^{\perp} \otimes g \mid g \in K \text{ or } g \in [G/K] \text{ with } gK \text{ minimal in the poset } G/K - \{1_{G/K}\}\}\ ,$$

where $[G/K]$ *is any set of coset representatives of* G/K*.*

If G acts faithfully on V, then we may simply take minimal elements in the poset $G - \{1_G\}$ (see Lemma 2.3) to obtain a generating set under cup product:

Corollary 9.4. *Assume* G *acts faithfully on* V*. The Hochschild cohomology algebra* $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ *is generated by* $\mathrm{HH}^{\bullet}(S(V))$ *and*

$$\{\mathrm{vol}_g^{\perp} \otimes g \mid g \text{ is minimal in the poset } G - \{1_G\}\}.$$

10. Reflection groups

In the previous sections, we described the algebraic structure of the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$. These results have a special interpretation for reflection groups and Coxeter groups in particular. We are interested in comparing the codimension of the fixed point space V^g of a group element g with its "reflection length" in the group.

Recall that a nonidentity element of $\mathrm{GL}(V)$ is a reflection if it fixes a hyperplane in V pointwise. A **reflection group** is a finite group generated by reflections. A reflection group is called a **Coxeter group** when it is generated by reflections acting on a *real* vector space. In this section, we restrict ourselves to the case when G is a reflection group. We define a length function with respect to the set of *all* reflections inside G. (Note that this definition may differ from the length function defined in terms of a fixed choice of generators for the group G, for example, a choice of simple reflections for a Weyl group.)

Definition 10.1. *For each* g *in* G*, let* $l(g)$ *be the minimal number* k *such that* $g = s_1 \cdots s_k$ *for some reflections* s_i *in* G*. We set* $l(1_G) = 0$*. We call* $l : G \to \mathbb{N}$ *the* **reflection length function** *(or "absolute length function") of* G*.*

The reflection length function induces a partial order $\leq_l$ on G:

$$g \leq_l h \quad \text{when} \quad l(g) + l(g^{-1}h) = l(h)\ .$$

The poset formed by the reflection length function plays an important role in the emerging theory of Artin groups of finite type. This theory relies on a key result for Coxeter groups asserting that the closed interval from the identity group element to a Coxeter element forms a lattice. Brady and Watt [5] gave a case-free proof of this fact by relating the two partial orders $\leq$ and $\leq_l$. The poset $\leq_l$ defined from the reflection length function has received attention not only for Coxeter groups, but for other complex reflection groups G as well. One may define a Coxeter element and again consider the interval (often a lattice) from the identity to the Coxeter element, the so-called *poset of noncrossing partitions for G*. See, for example, Bessis and Reiner [2].

Note that the length of a linear transformation with respect to the ambient group $\mathrm{GL}(V)$ coincides with the codimension of the fixed point space: Any element g in the unitary group $\mathrm{U}(V)$ can be written as the product of m reflections in $\mathrm{GL}(V)$ and no fewer if and only if $\operatorname{codim} V^g = m$ (see Brady and Watt [4]).

Reflection length in the group G is bounded below by the codimension of the fixed point space:

$$l(g) \geq \operatorname{codim} V^g \quad \text{for all } \; g \text{ in } G \ . \tag{10.2}$$

Indeed, if $l(g) = m$ and $g = s_1 \cdots s_m$ is a product of reflections in G, then

$$V^g \supset V^{s_1} \cap \cdots \cap V^{s_m} \ ;$$

but each V^{s_i} is a hyperplane, so

$$\operatorname{codim} V^g \leq \operatorname{codim}(V^{s_1} \cap \cdots \cap V^{s_m}) \leq m \ .$$

For certain reflection groups, reflection length coincides with codimension of fixed point space. The arguments of Carter [8] for Weyl groups hold for Coxeter groups as well:

Lemma 10.3. *Let G be a (finite) Coxeter group. Then reflection length coincides with codimension of fixed point space: For all g in G,*

$$\operatorname{codim} V^g = l(g) \ .$$

The lemma above implies that for Coxeter groups (which act faithfully), the partial order $\leq$ above (see Definition 2.2) describing the ring structure of Hochschild cohomology coincides with the partial order $\leq_l$ induced from the reflection length function. When these two posets agree (i.e., when $\leq \, = \, \leq_l$), we may express the ring structure of Hochschild cohomology in an elegant way. In particular, the Hochschild cohomology algebra $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is generated in degrees 0 and 1, as we see in the next corollary. Note the analogy with the Hochschild-Kostant-Rosenberg Theorem, which implies that the Hochschild cohomology of a smooth commutative algebra is also generated in degrees 0 and 1.

Theorem 10.4. *Suppose G is a (finite) reflection group for which the reflection length function gives codimension of fixed point spaces:*

$$l(g) = \operatorname{codim} V^g \quad \textit{for all } \; g \textit{ in } G \ .$$

Then $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ *is generated (as an algebra) in degrees* 0 *and* 1*.*

Proof. As G is a reflection group, it acts faithfully on V by definition. The elements of $G-\{1_G\}$ minimal in the partial order $\leq_l$ induced by the length function are the reflections. Indeed, suppose the length of h in G is $m>0$ and write $h = s_1 \cdots s_m$ for some reflections s_i in G. As h can not be expressed as the product of fewer than m reflections, $l(s_2 \cdots s_m) = m-1$. Hence,

$$l(s_1) + l(s_1^{-1}h) = l(s_1) + l(s_2 \cdots s_m) = 1 + (m-1) = l(h)$$

and $s_1 \leq_l h$. Note that for any reflection s, the relation $g \leq_l s$ implies that either $g = s$ or $g = 1_G$.

By hypothesis, the length function and codimension function induce the same partial order. Hence, the reflections are precisely the minimal elements of $G-\{1_G\}$ in the partial order $\leq$. By Corollary 9.4, $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is generated by $\mathrm{HH}^{\bullet}(S(V))$ and by all $\mathrm{vol}_s^{\perp} \otimes s$ where s is a reflection in G. The elements $\mathrm{vol}_s^{\perp} \otimes s$ each have cohomological degree 1, and $\mathrm{HH}^{\bullet}(S(V)) = S(V) \otimes \bigwedge^{\bullet} V^*$ is generated as an algebra (under cup product) by $\mathrm{HH}^0(S(V)) \cong S(V)$ and $\mathrm{HH}^1(S(V)) \cong S(V) \otimes V^*$. The statement follows. $\square$

The above corollary applies not only to Coxeter groups, but to other complex reflection groups as well. Let $G(r,1,n)$ be the infinite family of complex reflection groups, each of which is the symmetry group of a regular ("Platonic") polytope in complex space $V = \mathbb{C}^n$. The group $G(r,1,n)$ consists of all those n by n complex matrices which have in each row and column a single nonzero entry, necessarily a primitive r-th root of unity. This group is a natural wreath product of the symmetric group and the cyclic group of order r: $G \cong \mathbb{Z}/r\mathbb{Z} \wr \mathrm{Sym}_n$. In fact, $G(1,1,n)$ is the symmetric group Sym_n and $G(2,1,n)$ is the Weyl group of type B_n.

Lemma 10.5. *For the infinite family* $G(r,1,n)$*, the reflection length function coincides with codimension of fixed point spaces:*

$$l(g) = \operatorname{codim} V^g \quad \text{for all} \quad g \text{ in } G\ .$$

Proof. Let ξ be a primitive r-th root of unity in $\mathbb{C}$. Every element in $G(r,1,n)$ is conjugate to a product $h = c_1 \cdots c_k$ of disjoint cycles c_k of the form

$$c_k = \xi_j^a \, (i, i+1, \ldots, j)$$

(i.e., h is block diagonal, with k-th block c_k) where $\xi_j := \mathrm{diag}(1, \ldots, 1, \xi, 1, \ldots, 1)$ is the diagonal reflection with ξ in the j-th entry, $a \geq 0$, and where $(i, i+1, \ldots, j)$ is the matrix (in the natural reflection representation) of the corresponding cycle in Sym_n. (See, for example, Section 2B of Ram and the first author [23].) Consider a fixed cycle $c = c_k$ as above. We may write the cycle $(i, i+1, \ldots, j)$ as a product of $j-i$ transpositions (reflections) in Sym_n, a subgroup of $G(r,1,n)$. Hence if $\xi^a = 1$, then c may be expressed as the product of $j-i$ reflections in $G(r,1,n)$ while $\operatorname{codim} V^c = j-i$. If $\xi^a \neq 1$, then c may be expressed as the product of $j-i+1$ reflections in $G(r,1,n)$ while $\operatorname{codim} V^c = j-i+1$. In either case, $l(c) \leq \operatorname{codim} V^c$. But reflection length is bounded below by codimension, $\operatorname{codim} V^c \leq l(c)$, and hence $l(c) = \operatorname{codim} V^c$. Then

$$l(h) \leq l(c_1) + \ldots + l(c_k) = \operatorname{codim} V^{c_1} + \ldots + \operatorname{codim} V^{c_k} = \operatorname{codim} V^h \leq l(h)$$

and thus $l(h) = \operatorname{codim} V^h$. $\square$

As Theorem 10.4 applies to Coxeter groups by Lemma 10.3 and to the infinite family $G(r,1,n)$ by Lemma 10.5, we have the following analog of the Hochschild-Kostant-Rosenberg Theorem:

Corollary 10.6. *Let G be a Coxeter group or the infinite family $G(r,1,n)$. Then $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ is generated as an algebra in degrees 0 and 1.*

Note that the two partial orders $\leq$ and $\leq_l$ do not always agree, i.e., that for a complex reflection group G in general, the reflection length function may not give codimension of fixed point spaces:

Example 10.7. Let G be the complex reflection group $G(4,2,2)$, the subgroup of $G(4,1,2)$ consisting of those matrices with determinant ± 1. Let g be the diagonal matrix $\operatorname{diag}(i,i)$ where $i = \sqrt{-1}$ with determinant -1. Every reflection in G has determinant -1, and hence g can only be written as the product of an odd number of reflections. Then $\operatorname{codim} V^g = 2$ and yet g can not be written as the product of two reflections.

11. Cup product on the invariant subalgebra

In the above sections, we investigated the cup product on the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$. In this section, we describe the cup product on $\mathrm{HH}^{\bullet}(S(V)\#G)$, its G-invariant subalgebra, using standard techniques from group theory. Note that generators for the algebra $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ may not be invariant under G and hence do not generally yield generators for $\mathrm{HH}^{\bullet}(S(V)\#G)$.

The cup product on $\mathrm{HH}^{\bullet}(S(V)\#G)$ in cohomological degrees 0 and 1 is easy to describe, and follows from an observation of Farinati [13] (see also [24]):

Lemma 11.1. *The only group elements in G that contribute to the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V)\#G)$ are those which act on V with determinant 1.*

Note that if G embeds in $\mathrm{SL}(V)$, then every g-component is nonzero (as any element in the one-dimensional subspace $1 \otimes \bigwedge^{\dim V} V^* \otimes g$ of $C^\bullet$ is automatically invariant under the centralizer of g; see (11.3) below).

Lemma 11.1 together with Remark 4.8 immediately implies that the cup product on $\mathrm{HH}^\bullet(S(V)\#G)$ in cohomological degrees 0 and 1 is simply the exterior (wedge) product of forms when G acts faithfully (since reflections do not have determinant 1):

Proposition 11.2. *Assume G acts faithfully on V, that is, G embeds in $\mathrm{GL}(V)$. Then the cup product of elements in $\mathrm{HH}^0(S(V)\#G)$ and $\mathrm{HH}^1(S(V)\#G)$ is given by the cup product on the G-invariant subspaces of $\mathrm{HH}^0(S(V))$ and $\mathrm{HH}^1(S(V))$:*

$$\mathrm{HH}^0(S(V)\#G) = S(V)^G \quad \textit{and} \quad \mathrm{HH}^1(S(V)\#G) = (S(V) \otimes V^*)^G \ .$$

When G acts nonfaithfully, we similarly find that only the kernel K of G acting on V contributes to the cohomology $\mathrm{HH}^\bullet(S(V)\#G)$ in degrees 0 and 1:

$$\mathrm{HH}^0(S(V)\#G) = \Big(\bigoplus_{k\in K} S(V)\otimes k\Big)^G \text{ and } \mathrm{HH}^1(S(V)\#G) = \Big(\bigoplus_{k\in K} S(V)\otimes V^* \otimes k\Big)^G .$$

The cup product in higher degrees is not as transparent. We give a formula in terms of a fixed set $\mathcal{C}$ of representatives of the conjugacy classes of G. We extend the isomorphism of Theorem 3.2, $\mathrm{HH}^\bullet(S(V)\#G) \cong \mathrm{HH}^\bullet(S(V), S(V)\#G)^G$:

$$\mathrm{HH}^\bullet(S(V)\#G) \cong \bigoplus_{g\in\mathcal{C}} \mathrm{HH}^\bullet(S(V), S(V)\otimes g)^{Z(g)}, \tag{11.3}$$

where $Z(g)$ denotes the centralizer in G of g. The term $\mathrm{HH}^\bullet(S(V), S(V)\otimes g)$ is isomorphic to the g-component of $H^\bullet$ of Remark 4.8. The isomorphism identifies an element α of $\mathrm{HH}^\bullet(S(V), S(V)\otimes g)^{Z(g)}$ with the sum

$$\sum_{h\in[G/Z(g)]} {}^h\alpha,$$

where $[G/Z(g)]$ denotes a set of representatives of left cosets of $Z(g)$ in G. In the next proposition, we give a formula for the cup product of $\mathrm{HH}^\bullet(S(V)\#G)$ expressed in terms of the additive decomposition (11.3).

If A is any algebra with an action of G by automorphisms, and $J < L$ are subgroups of G, we define the *transfer* map $T_J^L : A^J \to A^L$ by

$$T_J^L(a) = \sum_{h\in[L/J]} {}^h a,$$

where $[L/J]$ is a set of representatives of the cosets L/J. To prove the following theorem, we use the theory of Green functors applied to this setting of a group action on an algebra.

Theorem 11.4. *The cup product on* $\mathrm{HH}^{\bullet}(S(V)\#G)$ *induces the following product* $\dot{\smile}$ *on* $\bigoplus_{g\in\mathcal{C}} \mathrm{HH}^{\bullet}(S(V), S(V)\otimes g)^{Z(g)}$ *under the isomorphism (11.3):*
For $\alpha \in \mathrm{HH}^{\bullet}(S(V), S(V)\otimes g)^{Z(g)}$ *and* $\beta \in \mathrm{HH}^{\bullet}(S(V), S(V)\otimes h)^{Z(h)}$,

$$\alpha \mathbin{\dot{\smile}} \beta \;=\; \sum_{x\in D} T^{Z(k)}_{\,{}^{y}Z(g)\,\cap\, {}^{yx}Z(h)}({}^{y}\alpha \smile {}^{yx}\beta),$$

where D *is a set of representatives of the double cosets* $Z(g)\backslash G/Z(h)$, *and* $k = k(x)$ *in* $\mathcal{C}$ *and* $y = y(x)$ *are chosen so that* ${}^{y}g \cdot {}^{yx}h = k$. *The product* ${}^{y}\alpha \smile {}^{yx}\beta$ *is the cup product in* $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)^{{}^{y}Z(g)\cap {}^{yx}Z(h)}$, *to which we apply the transfer map* $T^{Z(k)}_{\,{}^{y}Z(g)\,\cap\,{}^{yx}Z(h)}$ *to obtain an element in the* k*-component* $\mathrm{HH}^{\bullet}(S(V), S(V)\otimes k)^{Z(k)}$.

Proof. Let $A = \mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ (considered as a G-algebra). We obtain a standard Green functor by sending a subgroup L of G to the invariant subring A^{L}. The restriction maps (of the functor) are the inclusions, and the transfer maps are as defined above. The component $\mathrm{HH}^{\bullet}(S(V), S(V)\otimes g)^{Z(g)}$ is contained in $A^{Z(g)}$. In the decomposition (11.3), $\alpha \in \mathrm{HH}^{\bullet}(S(V), S(V)\otimes g)^{Z(g)}$ on the right side is identified with $T^{G}_{Z(g)}(\alpha)$ on the left side, and similarly for β. The formula in the proposition is the standard one for the product of $T^{G}_{Z(g)}(\alpha)$ and $T^{G}_{Z(h)}(\beta)$ given by the Mackey formula (e.g., see [29, Prop. 1.10], due to Green). $\square$

The product formula in Theorem 11.4 is more than a theoretical observation, but useful in computations, as the next example shows:

Example 11.5. Let $V = \mathbb{C}^6$ and $G = \mathrm{Sym}_3$. Let $v_1, w_1, v_2, w_2, v_3, w_3$ be the standard orthonormal basis of $\mathbb{C}^6$, and let G act on V via the permutation representation on $\{v_1, v_2, v_3\}$ and on $\{w_1, w_2, w_3\}$. Choose $1, (12), (123)$ as representatives of the conjugacy classes of Sym_3. By (11.3), $\mathrm{HH}^{\bullet}(S(V)\#\,\mathrm{Sym}_3)$ is isomorphic to

$$\mathrm{HH}^{\bullet}(S(V))^{\mathrm{Sym}_3} \oplus \mathrm{HH}^{\bullet}(S(V), S(V)\otimes (12))^{\langle (12)\rangle} \oplus \mathrm{HH}^{\bullet}(S(V), S(V)\otimes (123))^{\langle (123)\rangle}.$$

These summands are as follows, considering that the actions of (12) and of (123) on the latter two summands, respectively, are trivial:

$$\begin{aligned}
\mathrm{HH}^{\bullet}(S(V))^{\mathrm{Sym}_3} &= (S(V)\otimes {\textstyle\bigwedge}^{\bullet} V^{*})^{\mathrm{Sym}_3}\\
\mathrm{HH}^{\bullet}(S(V), S(V)\otimes (12))^{\langle (12)\rangle} &= S(V^{(12)})\otimes {\textstyle\bigwedge}^{\bullet-2}(V^{(12)})^{*}\otimes (v_1^{*}-v_2^{*})\wedge (w_1^{*}-w_2^{*})\otimes (12)\\
\mathrm{HH}^{\bullet}(S(V), S(V)\otimes (123))^{\langle (123)\rangle} &= S(V^{(123)})\otimes {\textstyle\bigwedge}^{\bullet-4}(V^{(123)})^{*}\otimes\\
&\quad\ (v_1^{*}-v_2^{*})\wedge (w_1^{*}-w_2^{*})\wedge (v_2^{*}-v_3^{*})\wedge (w_2^{*}-w_3^{*})\otimes (123)
\end{aligned}$$

where the fixed point spaces are $V^{(12)} = \mathrm{Span}_{\mathbb{C}}\{v_1+v_2,\ w_1+w_2,\ v_3,\ w_3\}$ and $V^{(123)} = \mathrm{Span}_{\mathbb{C}}\{v_1+v_2+v_3,\ w_1+w_2+w_3\}$. The product of an element of $\mathrm{HH}^{\bullet}(S(V))^{\mathrm{Sym}_3}$ with an element in any of the three components is given simply by the componentwise product on the exterior algebras and the symmetric algebras. The product of any element of the (123)-component with any element of the (12)-component is 0 as the exterior product is of linearly dependent elements. The product of any two elements of the (123)-component is 0 by degree considerations.

It remains to determine products of pairs of elements from the (12)-component. In the notation of Theorem 11.4, $g = (12) = h$, and we may take the set D of representatives of the double cosets $\langle(12)\rangle\backslash \operatorname{Sym}_3 /\langle(12)\rangle$ to be $\{1, (123)\}$. If $x = 1$, we have $(12)\cdot(12) = 1$, so we take $y = 1$; in any corresponding product $\alpha \smile \beta$, the exterior part is a product of linearly dependent elements. Thus the product corresponding to this choice of x must be 0. Now consider the case when $x = (123)$. We have $(12)\cdot {}^{(123)}(12) = (12)(123)(12)(132) = (123)$, and so we may take $y = 1$.

We now have, for any pair $\alpha, \beta \in \mathrm{HH}^{\bullet}(S(V), S(V)\otimes(12))$, the corresponding cup product in $\mathrm{HH}^{\bullet}(S(V)\#\operatorname{Sym}_3)$ given in terms of the decomposition (11.3):

$$\alpha \mathbin{\dot\smile} \beta = T_{\{1\}}^{\langle\,(123)\,\rangle}(\alpha \smile {}^{(123)}\beta).$$

For example,

$$\mathrm{vol}^{\perp}_{(12)} \mathbin{\dot\smile} \mathrm{vol}^{\perp}_{(12)} = T_{\{1\}}^{\langle\,(123)\,\rangle}(\mathrm{vol}^{\perp}_{(12)} \smile \mathrm{vol}^{\perp}_{(23)}) = 3\,\mathrm{vol}^{\perp}_{(123)}\,.$$

The efficient formula of Theorem 11.4 saves time by reducing nine product calculations to one in this case.

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