SKEW DERIVATIONS AND DEFORMATIONS OF A FAMILY OF GROUP CROSSED PRODUCTS

SARAH WITHERSPOON

Abstract. We obtain deformations of a crossed product of a polynomial algebra with a group, under some conditions, from universal deformation formulas. We show that the resulting deformations are nontrivial by a comparison with Hochschild cohomology. The universal deformation formulas arise from actions of Hopf algebras generated by automorphisms and skew derivations, and are universal in the sense that they apply to deform all algebras with such Hopf algebra actions.

1. Introduction

Deformations of a polynomial algebra, such as the Weyl algebra or functions on quantum affine space, may be expressed by formulas involving derivations of the polynomial algebra. These formulas are power series in an indeterminate with coefficients in the universal enveloping algebra of the Lie algebra of derivations. There are generalizations of such deformations to other types of algebras, such as functions on a manifold or orbifold, that are of current interest.

In this note we give a new generalization of the formulas themselves, and apply it to crossed products of polynomial algebras with groups of linear automorphisms. These group crossed products are of interest in geometry due to their relationship with corresponding orbifolds. Particular deformations of such crossed products, called graded Hecke algebras, were defined by Drinfel'd [7]. These deformations have been studied by many authors, for example for crossed products with real reflection groups, see [18], with complex reflection groups, see [21], and with symplectic reflection groups, see [8]. For these crossed product algebras, the universal enveloping algebra of the Lie algebra of derivations does not capture all the known deformations. Instead we derive a deformation formula from the action of a Hopf algebra under some hypotheses, recovering more of these known deformations as well as some new ones.

We use the theory developed by Giaquinto and Zhang of a universal deformation formula based on a bialgebra B [11], extending earlier such formulas based on universal enveloping algebras of Lie algebras. Such a formula is *universal* in the sense that it applies to *any* B-module algebra to yield a formal deformation. Known examples include formulas based on universal enveloping algebras of Lie algebras (see examples

Date: October 31, 2005.

Partially supported by NSF grants #DMS–0422506 and #DMS-0443476. The author thanks the Mathematical Sciences Research Institute for its hospitality during much of the writing of this paper.

and references in [11]) and a formula based on a small noncocommutative bialgebra [4]. In Section 3 we generalize the formula in [4]. Our universal deformation formula is based on a bialgebra generated by skew-primitive and group-like elements, and depends on a parameter q. The bialgebra and formula were discovered in the generic case by Giaquinto and Zhang, but were not published [12]. We state their formula and modify it to include the case where q is a root of unity (Theorem 3.3). The case $q = -1$ is [4, Lemma 6.2], and as a first new example, we show that the smallest Taft algebra may be deformed by applying this earlier formula (Example 3.4). This is one of a series of algebras defined by quiver and relations whose deformations were given by Cibils [5, 6], and we recover one of his deformations. We end Section 3 with more details on the structure of the bialgebras involved, briefly reviewing Kharchenko's construction of a Hopf algebra of automorphisms and skew derivations of an algebra [16, 17].

We apply our universal deformation formula in Section 4 to deform some crossed products of polynomial algebras with groups (Corollary 4.8), generalizing [4, Example 6.3] in which the group was $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We use Hochschild cohomology to prove that the resulting deformations are nontrivial, by showing that their associated infinitesimals are not coboundaries. We apply the results of [25] to show how our deformation formula leads, in special cases, to (twisted) graded Hecke algebras (Example 4.13). Due to restrictive hypotheses, our formula does not give rise to very many of the examples of (twisted) graded Hecke algebras in [7, 8, 18, 21, 25]. However it does give an infinite series of universal deformation formulas based on noncocommutative Hopf algebras, as well as algebras thereby undergoing nontrivial deformation. It also proves existence of more general deformations of certain crossed products than those that are the (twisted) graded Hecke algebras.

In Section 5, we give two small examples for which our universal deformation formula nearly provides a universal deformation of the crossed product algebra in the *other* sense of the word *universal*. That is all possible nonclassical deformations of the algebra are parametrized by the formula, where by nonclassical we mean those not arising from deformations of the underlying polynomial algebra itself.

For completeness, we include an appendix in which the Hochschild cohomology of the relevant crossed product algebra is computed. In the case of a trivial twisting two-cocycle associated to the group, this was done elegantly by Farinati [9], and by Ginzburg and Kaledin [13] in a more general geometric setting. Their results are easily generalized to crossed products with twisting cocycles, however in Section 4 we need some details from a more direct calculation. We give an algebraic computation similar to that in [1, 23] where the crossed product was taken with a Weyl algebra instead of a polynomial algebra (and the group is symplectic). We thank R.-O. Buchweitz for explaining this computation of Hochschild cohomology to us.

We also owe many thanks to A. Giaquinto and J. Zhang for sharing their unpublished universal deformation formula with us, and especially to A. Giaquinto from whom we learned algebraic deformation theory. We thank C. Cibils and J. Stasheff for many comments on earlier versions of this paper.

We will work over the complex numbers, although the definitions make sense more generally. Unless otherwise indicated, $\otimes = \otimes_{\mathbb{C}}$.

2. Definitions

Let S be a $\mathbb{C}$-algebra. Denote by $\mathrm{Aut}_{\mathbb{C}}(S)$ the group of all $\mathbb{C}$-algebra automorphisms of S that preserve the multiplicative identity. Let $g, h \in \mathrm{Aut}_{\mathbb{C}}(S)$. A *$g, h$-skew derivation* of S is a $\mathbb{C}$-linear function $D : S \to S$ such that

$$D(rs) = D(r)g(s) + h(r)D(s)$$

for all $r, s \in S$. If $g = h = 1$ (the identity automorphism), then D is simply a derivation of S.

We will be interested in skew derivations of a crossed product algebra which we define next. For more details on group crossed products, see [20]. Let G be any subgroup of $\mathrm{Aut}_{\mathbb{C}}(S)$. Let $\alpha : G \times G \to \mathbb{C}^{\times}$ be a *two-cocycle*, that is a function satisfying

$$\alpha(g, h)\alpha(gh, k) = \alpha(g, hk)\alpha(h, k) \tag{2.1}$$

for all $g, h, k \in G$. The *crossed product ring* $S\#_{\alpha}G$ is $S \otimes \mathbb{C}G$ as a vector space, with multiplication

$$(r \otimes g)(s \otimes h) = \alpha(g, h)r \cdot g(s) \otimes gh$$

for all $r, s \in S$ and $g, h \in G$. This product is associative as α is a two-cocycle. We say α is a *coboundary* if there is some function $\beta : G \to \mathbb{C}^{\times}$ such that $\alpha(g, h) = \beta(g)\beta(h)\beta(gh)^{-1}$ for all $g, h \in G$. The set of two-cocycles modulo coboundaries forms an abelian group under pointwise multiplication, that is $(\alpha\alpha')(g, h) = \alpha(g, h)\alpha'(g, h)$ for all $g, h \in G$. The crossed product algebras $S\#_{\alpha}G$ and $S\#_{\alpha'}G$ are isomorphic if $\alpha' = \alpha\beta$ for some coboundary β (that is α and α' are cohomologous).

We will abbreviate the element $r \otimes g$ of $S\#_{\alpha}G$ by $r\overline{g}$. We will assume that α is *normalized* so that $\alpha(1, g) = \alpha(g, 1) = 1$ for all $g \in G$. Thus $\overline{1}$ is the multiplicative identity of $S\#_{\alpha}G$, and it also follows from this and (2.1) that $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)$ for all $g \in G$. The action of G on S extends to an inner action on $S\#_{\alpha}G$, with $g(a) = \overline{g}a(\overline{g})^{-1}$ for all $g \in G$, $a \in S\#_{\alpha}G$, where $(\overline{g})^{-1} = \alpha^{-1}(g, g^{-1})\overline{g^{-1}} = \alpha^{-1}(g^{-1}, g)\overline{g^{-1}}$.

Now let t be an indeterminate. A *formal deformation* of a $\mathbb{C}$-algebra A (for example $A = S\#_{\alpha}G$) is an associative algebra $A[[t]] = \mathbb{C}[[t]] \otimes A$ over formal power series $\mathbb{C}[[t]]$ with multiplication

$$a * b = ab + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \cdots \tag{2.2}$$

for all $a, b \in A$, where ab denotes the product in A and the $\mu_i : A \otimes A \to A$ are $\mathbb{C}$-linear maps extended to be $\mathbb{C}[[t]]$-linear. Associativity of $A[[t]]$ implies that μ_1 is a *Hochschild two-cocycle*, that is

$$\mu_1(a \otimes b)c + \mu_1(ab \otimes c) = \mu_1(a \otimes bc) + a\mu_1(b \otimes c) \tag{2.3}$$

for all $a, b, c \in A$, as well as further conditions on the μ_i, $i \geq 1$. Thus a Hochschild two-cocycle μ_1 is the first step towards a formal deformation, and it is called the *infinitesimal* of the deformation. In general it is difficult to determine whether a given μ_1 lifts to a formal deformation of A. Hochschild cohomology is defined in the appendix; for more details on Hochschild cohomology and deformations of algebras, see [10].

One way in which to obtain a formal deformation of A is through the action of a bialgebra on A. A *bialgebra* over $\mathbb{C}$ is an associative $\mathbb{C}$-algebra B with $\mathbb{C}$-algebra maps $\Delta : B \to B \otimes B$ and $\varepsilon : B \to \mathbb{C}$ such that $(\Delta \otimes \mathrm{id}) \circ \Delta = \Delta \circ (\mathrm{id} \otimes \Delta)$ and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$. We will use the standard notation $\Delta(b) = \sum b_1 \otimes b_2$ for all $b \in B$, where the subscripts are merely place-holders. A *Hopf algebra* is a bialgebra H with a $\mathbb{C}$-linear map $S : H \to H$ such that $\sum (Sh_1)h_2 = \varepsilon(h) = \sum h_1(Sh_2)$ for all $h \in H$. For details on bialgebras and Hopf algebras, see [19].

A *universal deformation formula* (or *UDF*) based on a bialgebra B is an element $F \in (B \otimes B)[[t]]$ of the form $F = 1 \otimes 1 + tF_1 + t^2F_2 + \cdots$ with each $F_i \in B \otimes B$, satisfying

$$(\varepsilon \otimes \mathrm{id})(F) = 1 \otimes 1 = (\mathrm{id} \otimes \varepsilon)(F) \tag{2.4}$$
$$\text{and} \quad [(\Delta \otimes \mathrm{id})(F)](F \otimes 1) = [(\mathrm{id} \otimes \Delta)(F)](1 \otimes F),$$

where id denotes the identity map. These relations (2.4) are similar to some of the defining relations for an R-matrix, and in fact both the inverse R^{-1} and the transpose R_{21} of an R-matrix satisfy (2.4) (see [2, 11]).

Suppose A is a $\mathbb{C}$-algebra which is also a left B-module. Then A is a left *B-module algebra* if

$$h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1 = \varepsilon(h)1 \tag{2.5}$$

for all $a, b \in A$ and $h \in B$. This may be extended to a $\mathbb{C}[[t]]$-linear action of $B[[t]]$ by extending the scalars for A to $\mathbb{C}[[t]]$. Let $m : A \otimes A \to A$ denote multiplication in A, extended to be $\mathbb{C}[[t]]$-linear. The following proposition combines Theorem 1.3 and Definition 1.13 of [11], and we sketch a proof for completeness.

Proposition 2.6 (Giaquinto-Zhang). *Let B be a bialgebra, A a left B-module algebra, and F a universal deformation formula based on B. There is a formal deformation of A given by $a * b = (m \circ F)(a \otimes b)$ for all $a, b \in A$.*

Proof. The format of F as a power series in t implies that $a * b = (m \circ F)(a \otimes b)$ takes the form (2.2). Associativity of $*$ follows from the second relation in (2.4) and the first relation in (2.5):

$$\begin{aligned} m \circ F \circ (m \otimes \mathrm{id}) \circ (F \otimes 1) &= m \circ (m \otimes \mathrm{id}) \circ [(\Delta \otimes \mathrm{id})(F)] \circ (F \otimes 1) \\ &= m \circ (\mathrm{id} \otimes m) \circ [(\mathrm{id} \otimes \Delta)(F)] \circ (1 \otimes F) \\ &= m \circ F \circ (\mathrm{id} \otimes m) \circ (1 \otimes F) \end{aligned}$$

as functions from $A[[t]] \otimes_{\mathbb{C}[[t]]} A[[t]] \otimes_{\mathbb{C}[[t]]} A[[t]]$ to $A[[t]]$. Note that the first relation in (2.4) and the second relation in (2.5) imply that 1_A remains the multiplicative identity under $*$. □

We will need the following notation. Let $q \in \mathbb{C}^\times$. For every integer $i \geq 1$, let $(i)_q = 1+q+q^2+\cdots q^{i-1}$, and set $(0)_q = 0$. Let $(i)_q! = (i)_q(i-1)_q \cdots (1)_q$ and $(0)_q! = 1$. The *q-binomial coefficients* are

$$\binom{k}{i}_q = \frac{(k)_q!}{(i)_q!(k-i)_q!}$$

for any two integers $k \geq i \geq 0$. The well-known q-binomial formula states that

$$(y+z)^k = \sum_{i=0}^{k} \binom{k}{i}_q y^i z^{k-i} \tag{2.7}$$

in any $\mathbb{C}$-algebra in which y, z are elements such that $zy = qyz$.

Let $S = \mathbb{C}[x_1, \ldots, x_n]$ and $q \in \mathbb{C}^\times$. As in [15, Exer. IV.9.4], define the linear maps *q-differentiation* $\partial_{i,q} : S \to S$ by

$$\partial_{i,q}(x_1^{k_1} \cdots x_n^{k_n}) = (k_i)_q x_1^{k_1} \cdots x_{i-1}^{k_{i-1}} x_i^{k_i - 1} x_{i+1}^{k_{i+1}} \cdots x_n^{k_n}. \tag{2.8}$$

If $q = 1$, these are the usual partial differentiation operators. If q is an ℓth root of unity, then $\partial_{i,q}^\ell = 0$ as $(k)_q = 0$ whenever k is a multiple of ℓ. In general, $\partial_{i,q}$ is a skew derivation on S, specifically

$$\partial_{i,q}(rs) = \partial_{i,q}(r)\tau_{i,q}(s) + r\partial_{i,q}(s)$$

for all $r, s \in S$, where $\tau_{i,q}$ is the automorphism of S defined by $\tau_{i,q}(x_1^{k_1} \cdots x_n^{k_n}) = q^{k_i} x_1^{k_1} \cdots x_n^{k_n}$. Under some conditions, these skew derivations may be extended to a crossed product of S with a group of linear automorphisms, as we will see.

3. A universal deformation formula

Let $q \in \mathbb{C}^\times$ and let H be the algebra generated by $D_1, D_2, \sigma^{\pm 1}$, subject to the relations

$$D_1 D_2 = D_2 D_1, \quad q\sigma D_i = D_i \sigma \ (i = 1, 2), \quad \sigma\sigma^{-1} = 1 = \sigma^{-1}\sigma.$$

It is straightforward to check that H is a Hopf algebra with

$$\begin{array}{rclrclrcl} \Delta(D_1) & = & D_1 \otimes \sigma + 1 \otimes D_1, & \varepsilon(D_1) & = & 0, & S(D_1) & = & -D_1\sigma^{-1}, \\ \Delta(D_2) & = & D_2 \otimes 1 + \sigma \otimes D_2, & \varepsilon(D_2) & = & 0, & S(D_2) & = & -\sigma^{-1}D_2, \\ \Delta(\sigma) & = & \sigma \otimes \sigma, & \varepsilon(\sigma) & = & 1, & S(\sigma) & = & \sigma^{-1}. \end{array}$$

If q is a primitive ℓth root of unity ($\ell \geq 2$), the ideal I generated by D_1^ℓ and D_2^ℓ is a Hopf ideal, that is $\Delta(I) \subseteq I \otimes H + H \otimes I$, $\varepsilon(I) = 0$ and $S(I) \subseteq I$. Checking the condition on Δ involves the q-binomial formula (2.7) and the observation that $\binom{\ell}{i}_q = 0$ whenever $1 \leq i \leq \ell - 1$. Thus the quotient H/I is also a Hopf algebra. Let

$$H_q = \begin{cases} H/I, & \text{if } q \text{ is a primitive } \ell\text{th root of unity } (\ell \geq 2) \\ H, & \text{if } q = 1 \text{ or is not a root of unity.} \end{cases} \tag{3.1}$$

We will need the following lemma to obtain a universal deformation formula based on H_q. If $q = 1$ or is not a root of unity, we define the q-exponential function by

$$\exp_q(y) = \sum_{i=0}^{\infty} \frac{1}{(i)_q!} y^i$$

for any element y of a $\mathbb{C}$-algebra in which this sum is defined. In the proof of Theorem 3.3 below, the $\mathbb{C}$-algebra will be $(H_q \otimes H_q \otimes H_q)[[t]]$. If $q \neq 1$ is a root of unity, this formula makes no sense as some denominators will be zero. We modify the formula as follows in this case. Suppose q is a primitive ℓth root of unity for $\ell \geq 2$. Then we define

$$\exp_q(y) = \sum_{i=0}^{\ell-1} \frac{1}{(i)_q!} y^i$$

for any element y of a $\mathbb{C}$-algebra.

Lemma 3.2. *Suppose $\ell \geq 2$, q is a primitive ℓth root of unity, and y, z are elements of a $\mathbb{C}$-algebra such that $zy = qyz$ and $y^i z^{\ell-i} = 0$ for $0 \leq i \leq \ell$. Then*

$$\exp_q(y + z) = \exp_q(y) \exp_q(z).$$

Proof. By the assumed relations and the q-binomial formula (2.7), each side of the desired equation may be written in the form

$$\sum_{i,j=0}^{\ell-1} \frac{1}{(i)_q!(j)_q!} y^i z^j,$$

and thus they are equal. □

We note that if q is not a primitive root of 1, it is a standard result that $\exp_q(y+z) = \exp_q(y) \exp_q(z)$ whenever $zy = qyz$ and the relevant sums are defined (see for example [15, Prop. IV.2.4]). In the root of unity case, the additional hypothesis stated in the above lemma is required.

Theorem 3.3. *Let $q \in \mathbb{C}^\times$ and let H_q be the Hopf algebra defined in (3.1). Then $\exp_q(tD_1 \otimes D_2)$ is a universal deformation formula based on H_q.*

Proof. In case q is not a root of unity, this is an unpublished result of Giaquinto and Zhang [12]. Their proof may be adapted to the case q is a root of unity by using Lemma 3.2 as follows. (The proof in case q is not a root of unity is essentially the same.) Note that the hypotheses of Lemma 3.2 hold for the pairs $y = tD_1 \otimes \sigma \otimes D_2$, $z = t \otimes D_1 \otimes D_2$ and $y = tD_1 \otimes \sigma \otimes D_2$, $z = tD_1 \otimes D_2 \otimes 1$ as $D_2^\ell = 0$ and $D_1^\ell = 0$, respectively. As Δ is an algebra homomorphism and D_1 commutes with D_2, we thus have

$$\begin{aligned}
&(\Delta\otimes\mathrm{id})(\exp_q(tD_1\otimes D_2))[\exp_q(tD_1\otimes D_2)\otimes 1]\\
&\quad= \exp_q((\Delta\otimes\mathrm{id})(tD_1\otimes D_2))\exp_q(tD_1\otimes D_2\otimes 1)\\
&\quad= \exp_q(tD_1\otimes\sigma\otimes D_2 + t\otimes D_1\otimes D_2)\exp_q(tD_1\otimes D_2\otimes 1)\\
&\quad= \exp_q(tD_1\otimes\sigma\otimes D_2)\exp_q(t\otimes D_1\otimes D_2)\exp_q(tD_1\otimes D_2\otimes 1)\\
&\quad= \exp_q(tD_1\otimes\sigma\otimes D_2)\exp_q(tD_1\otimes D_2\otimes 1)\exp_q(t\otimes D_1\otimes D_2)\\
&\quad= \exp_q(tD_1\otimes\sigma\otimes D_2 + tD_1\otimes D_2\otimes 1)\exp_q(t\otimes D_1\otimes D_2)\\
&\quad= \exp_q((\mathrm{id}\otimes\Delta)(tD_1\otimes D_2))(1\otimes\exp_q(tD_1\otimes D_2))\\
&\quad= (\mathrm{id}\otimes\Delta)(\exp_q(tD_1\otimes D_2))(1\otimes\exp_q(tD_1\otimes D_2)).
\end{aligned}$$

The remaining relation in (2.4) holds as $\varepsilon(D_1)=\varepsilon(D_2)=0$. Thus $\exp_q(tD_1\otimes D_2)$ is a universal deformation formula. □

By Proposition 2.6 and Theorem 3.3, we need only find an H_q-module algebra A, and $m\circ\exp_q(tD_1\otimes D_2)$ will provide a formal deformation of A. Our first such example is next; a large family of examples is given in Section 4.

Example 3.4. (A Taft algebra.) Let A be the algebra defined by generators and relations as follows, where the indices are read modulo 2:

$$\begin{aligned}
A=\mathbb{C}\langle s_0,s_1,\gamma_0,\gamma_1 \quad | \quad & s_0+s_1=1,\ s_i^2=s_i,\ s_is_{i+1}=0, \gamma_i^2=0,\ \gamma_i\gamma_{i+1}=0,\\
& s_i\gamma_i=0,\ s_{i+1}\gamma_i=\gamma_i,\ \gamma_is_i=\gamma_i,\ \gamma_is_{i+1}=0\rangle.
\end{aligned}$$

This is an algebra defined by a quiver and relations as in [5, Thm. 5.1(b)]; the quiver is Gabriel's quiver consisting of two arrows in opposite directions between two vertices. The algebra A is isomorphic to $\mathbb{C}\langle x,g \mid gx=-xg,\ x^2=0,\ g^2=1\rangle$ via the map $x\mapsto\gamma_0-\gamma_1,\ g\mapsto s_0-s_1$. It has the structure of a Hopf algebra first discovered by Sweedler, and is one of a series of Hopf algebras constructed by Taft [19, Example 1.5.6]. In [5, 6], Cibils gave deformations of more general classes of algebras defined by quivers and relations, and in this special case one of his deformations may be obtained by applying a universal deformation formula. Specifically, let $q=-1$ and

$$H_{-1}=\mathbb{C}\langle D_1,D_2,\sigma^{\pm1}\mid D_1D_2=D_2D_1,\ -\sigma D_i=D_i\sigma,\ D_i^2=0,\ \sigma\sigma^{-1}=1=\sigma^{-1}\sigma\rangle$$

as above. Define

$$\sigma(\gamma_i)=-\gamma_{i+1},\ \sigma(s_i)=s_{i+1},$$
$$D_1(\gamma_i)=s_{i+1},\ D_2(\gamma_i)=s_i,\ D_i(s_j)=0.$$

It may be checked that the relations of H_{-1} are preserved on the generators of A, making the vector space $V=\mathrm{Span}_{\mathbb{C}}\{s_0,s_1,\gamma_0,\gamma_1\}$ into an H_{-1}-module. Therefore the tensor algebra $T(V)$ is an H_{-1}-module algebra, where the action of H_{-1} is extended to $T(V)$ by (2.5). As A is a quotient of $T(V)$, it remains to check that the relations of A are preserved by the generators of H, a straightforward computation. (In fact, A is also a module algebra for the finite dimensional quotient $H_{-1}/(\sigma^2-1)$.) By Proposition 2.6

and Theorem 3.3, $\exp_{-1}(tD_1 \otimes D_2) = 1 + tD_1 \otimes D_2$ yields a formal deformation of A. The deformation is

$$A_t = \mathbb{C}\langle s_0, s_1, \gamma_0, \gamma_1 \quad | \quad s_0 + s_1 = 1,\ s_i^2 = s_i,\ s_i s_{i+1} = 0,\ \gamma_i^2 = 0,\ s_i\gamma_i = 0,$$
$$s_{i+1}\gamma_i = \gamma_i,\ \gamma_i s_i = \gamma_i, \gamma_i s_{i+1} = 0,\ \gamma_i\gamma_{i+1} = ts_{i+1}\rangle,$$

which is precisely that given in [5, Thm. 5.1(b)]. This deformation is nontrivial since if we specialize to $t \neq 0$, A_t is isomorphic to the 2×2 matrix algebra, and thus is not isomorphic to A. We do not know whether any of Cibils' other deformations are given by universal deformation formulas.

More generally, suppose that A is any H_q-module algebra. Due to (2.5) and the nature of the coproducts of D_1, D_2, the following general lemma implies that $\mu_1 = m \circ (D_1 \otimes D_2)$ is a Hochschild two-cocycle on A, that is it satisfies (2.3). This generalizes the well-known fact that the cup product of derivations is a Hochschild two-cocycle. The lemma is proved by direct computation, with no assumption made on the relations among D_1, D_2, σ. If the relations of H_q do hold however, then Theorem 3.3 gives an alternative proof that $\mu_1 = m \circ (D_1 \otimes D_2)$ is a Hochschild two-cocycle.

Lemma 3.5. *Let A be an algebra over a field, with multiplication $m : A \otimes A \to A$. Let σ be an automorphism of A, D_1 a $\sigma, 1$-skew derivation and D_2 a $1, \sigma$-skew derivation of A. Then $\mu_1 = m \circ (D_1 \otimes D_2)$ is a Hochschild two-cocycle.*

We end this section with a construction due to Kharchenko [16, §6.5.5] of a Hopf algebra of automorphisms and skew derivations of an algebra A. The Hopf algebras H_q are related to some of Kharchenko's Hopf algebras, and it may be useful to consider his general construction in questions regarding deformations of algebras.

Let K be a subgroup of $\mathrm{Aut}_{\mathbb{C}}(A)$. For each $k \in K$, let L_k be a vector subspace of $\mathrm{End}_{\mathbb{C}}(A)$ consisting of $1, k$-skew derivations of A. Let $L = \oplus_{k \in K} L_k$, and assume K acts on L in such a way that $m(L_k) = L_{mkm^{-1}}$ for all $k, m \in K$. Thus K acts by automorphisms on the tensor algebra $T(L)$, and we let $H = T(L)\#K$. The coproducts

$$\Delta(k) = k \otimes k \quad \text{and} \quad \Delta(D) = D \otimes 1 + k \otimes D$$

for all $k \in K$ and $D \in L_k$ extend, by requiring Δ to be an algebra homomorphism, to a coproduct Δ on H. Similarly, the counit ε and antipode S defined as follows on generators extend to H: $\varepsilon(k) = 1$, $\varepsilon(D) = 0$, $S(k) = k^{-1}$ and $S(D) = -k^{-1}D$ for all $k \in K$ and $D \in L_k$. Thus H is a Hopf algebra.

We obtain the Hopf algebras H_q by this construction in the following way: If A is an H_q-module algebra, let K be the group generated by the action of σ on A, $L_\sigma = \mathrm{Span}_{\mathbb{C}}\{D_2\}$, $L_{\sigma^{-1}} = \mathrm{Span}_{\mathbb{C}}\{D_1\sigma^{-1}\}$, and $L_\tau = \{0\}$ if $t \neq \sigma^{\pm 1}$. If σ has infinite order as an automorphism of A, then H_q is a quotient of Kharchenko's Hopf algebra defined by this data. Otherwise we must take a quotient of H_q, in which σ has the correct order, to obtain a quotient of Kharchenko's Hopf algebra. See [17] for further details on this construction.

4. Deformations of group crossed products

In this section we give a large family of group crossed products to which the formula of Theorem 3.3 applies to yield nontrivial deformations.

Let G be a group with a representation on a $\mathbb{C}$-vector space V of dimension n, so that G acts by automorphisms on the symmetric algebra $S(V)$. We will identify $S(V)$ with polynomials in the variables $x_1, \ldots, x_n$. In this section, we will be interested in formal deformations of a crossed product $S(V)\#_\alpha G$ for which the infinitesimal μ_1 satisfies $\mu_1(V \otimes V) \subset S(V)\overline{g}$ for some $g \in G$. Not all elements $g \in G$ correspond to such noncoboundary infinitesimals μ_1. In case G is finite, examination of Hochschild cohomology (see Corollary 6.5 and subsequent comments) shows that we may assume such an element g has determinant 1 as an operator on V, and $\mathrm{codim}(V^g) = 0$ or 2, where $V^g = \{v \in V \mid g(v) = v\}$, the subspace of V invariant under g. In this section, we will make this assumption, and in addition will assume that g is central in G. Again if the order of g is finite, g acts diagonally with respect to some basis of V, and without loss of generality this is $x_1, \ldots, x_n$. Specifically, we will assume that

$$g(x_1) = qx_1,\ g(x_2) = q^{-1}x_2,\ g(x_3) = x_3, \cdots,\ g(x_n) = x_n \tag{4.1}$$

for some $q \in \mathbb{C}^\times$. In order to include some infinite groups, we will not assume that q is a root of unity. To obtain explicit formulas, we will further need to make a more restrictive assumption:

$$G \text{ preserves the subspaces } \mathbb{C}x_1, \mathbb{C}x_2 \text{ of } V. \tag{4.2}$$

If $q \neq \pm 1$, this is automatically the case by the assumed centrality of g. Under the assumption (4.2), we may abuse notation and define the functions $x_i : G \to \mathbb{C}^\times$ $(i = 1, 2)$ by

$$h(x_i) = x_i(h)x_i$$

for each $h \in G$. Let D_1, D_2 and $\sigma : S(V)\#_\alpha G \to S(V)\#_\alpha G$ be the linear functions defined on a basis $\{x_1^{k_1} \cdots x_n^{k_n}\overline{h} \mid k_i \in \mathbb{Z}^{\geq 0},\ h \in G\}$ as follows:

$$D_1(x_1^{k_1} \cdots x_n^{k_n}\overline{h}) = x_1(h^{-1})\partial_{1,q}(x_1^{k_1} \cdots x_n^{k_n})\overline{h}, \tag{4.3}$$

$$D_2(x_1^{k_1} \cdots x_n^{k_n}\overline{h}) = q^{k_1}\partial_{2,q^{-1}}(x_1^{k_1} \cdots x_n^{k_n})s\overline{g} \cdot \overline{h}, \tag{4.4}$$

$$\sigma(x_1^{k_1} \cdots x_n^{k_n}\overline{h}) = x_1(h^{-1})q^{k_1}x_1^{k_1} \cdots x_n^{k_n}\overline{h}, \tag{4.5}$$

where $\partial_{1,q}$, $\partial_{2,q^{-1}}$ are defined in (2.8) and $s \in \mathbb{C}[x_3, \ldots, x_n]$ satisfies

$$h(s) = x_1(h)x_2(h)\alpha(g,h)\alpha^{-1}(h,g)s \tag{4.6}$$

for all $h \in G$, that is s is a semi-invariant of G. (Our condition on the polynomial s is informed by knowledge of Hochschild cohomology; see Corollary 6.5 and the computation (4.11) below.) Calculations using (4.1)–(4.6) and centrality of g show that σ is an automorphism and D_1, D_2 are skew derivations with respect to σ, specifically

$$D_1(ab) = D_1(a)\sigma(b) + aD_1(b) \quad \text{and} \quad D_2(ab) = D_2(a)b + \sigma(a)D_2(b)$$

for all $a, b \in S(V)\#_\alpha G$. A direct calculation shows that $\mu_1 = m \circ (D_1 \otimes D_2)$ is a Hochschild two-cocycle on $S(V)\#_\alpha G$, that is μ_1 satisfies (2.3). This is also a consequence of Lemma 3.5, or of Theorem 3.3 in combination with Theorem 4.7 below.

Taking D_1, D_2 to be the skew derivations defined in (4.3), (4.4), the corresponding Hochschild two-cocycle μ_1 takes $V \otimes V$ to $S(V)\overline{g}$, the g-component of $S(V)\#_\alpha G$. If g were *not* central in G, an associated Hochschild two-cocycle would necessarily involve all components of $S(V)\#_\alpha G$ corresponding to the elements of the conjugacy class of g (see Corollary 6.5). We do not know if the explicit formulas of this section can be generalized to noncentral g.

Let H_q be the Hopf algebra defined in (3.1).

Theorem 4.7. *Let g be a central element of G such that (4.1) and (4.2) hold. Then $S(V)\#_\alpha G$ is an H_q-module algebra under the action defined in (4.3)–(4.5).*

Proof. The relations among the generators in H_q may be checked to be preserved under the action, so that $S(V)\#_\alpha G$ is an H_q-module. In particular, in case q is a primitive ℓth root of unity, $D_1^\ell = 0 = D_2^\ell$ as $(k)_q = 0$ whenever k is a multiple of ℓ. As stated earlier, σ is an automorphism of $S(V)$ and D_1 and D_2 are skew derivations. Clearly $D_1(1) = 0 = D_2(1)$ as $(0)_q = 0$. Therefore (2.5) holds, so $S(V)\#_\alpha G$ is an H_q-module algebra. □

Combining Proposition 2.6 and Theorems 3.3 and 4.7, we now have the following corollary. In case G is finite, the deformations in the corollary are shown to be nontrivial in the remainder of this section. We expect that the same is also true in case G is infinite.

Corollary 4.8. *Let g be a central element of G such that (4.1) and (4.2) hold. Then $\exp_q(tD_1 \otimes D_2)$ yields a formal deformation of $S(V)\#_\alpha G$.*

We point out that if $g = 1$ then $\exp_q(tD_1 \otimes D_2)$ restricts to a classical formula on $S(V)$, namely

$$\sum_{i=0}^{\infty} \frac{t^i}{i!} \left(\frac{\partial}{\partial x_1}\right)^i \otimes \left(s\frac{\partial}{\partial x_2}\right)^i,$$

where $s \in \mathbb{C}[x_3, \ldots, x_n]$ satisfies $h(s) = x_1(h)x_2(h)s$ for all $h \in G$. Taking G to be the identity group, $n = 2$, and $s = 1$, this formula applied to $\mathbb{C}[x_1, x_2]$ yields the Weyl algebra on two generators.

In the special case $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $V = \mathbb{C}^3$ with a particular diagonal action of G, the formula and deformation of Corollary 4.8 were obtained in [4, §6]. The deformations in that case are nontrivial since their corresponding Hochschild two-cocycles are not coboundaries, a consequence of the computations in [4]. Similarly, we now show that the same is true in the more general setting of a finite group G, based on a computation of the Hochschild cohomology of $S(V)\#_\alpha G$. The Hochschild cohomology was computed by Farinati, Ginzburg and Kaledin in the case $\alpha = 1$ [9, 13]. The addition of a nontrivial cocycle α poses no difficulties, however we need to use some of the details from an explicit algebraic computation. These we provide in the appendix.

There is a chain map from the bar complex (6.1) for $A = S(V)$ to the Koszul complex $K(\{x_i \otimes 1 - 1 \otimes x_i\}_{i=1}^n)$,

$$\begin{array}{ccccccccccc}
\cdots & \to & S(V)^{\otimes 4} & \xrightarrow{\delta_2} & S(V)^{\otimes 3} & \xrightarrow{\delta_1} & S(V)^e & \xrightarrow{m} & S(V) & \to & 0 \\
 & & \downarrow \psi_2 & & \downarrow \psi_1 & & \| & & \| & & \\
\cdots & \to & \bigwedge^2(V) \otimes S(V)^e & \xrightarrow{d_2} & \bigwedge^1(V) \otimes S(V)^e & \xrightarrow{d_1} & S(V)^e & \xrightarrow{m} & S(V) & \to & 0
\end{array}$$

We will need an explicit formula for ψ_2 in particular. A straightforward computation shows that the following formulas work (cf. [4], in which slightly different formulas are given in the case $n = 3$):

$$\psi_1(1 \otimes x_1^{k_1} \cdots x_n^{k_n} \otimes 1) = \sum_{i=1}^{n} \sum_{p=1}^{k_i} e_i \otimes x_i^{k_i - p} x_{i+1}^{k_{i+1}} \cdots x_n^{k_n} \otimes x_1^{k_1} \cdots x_{i-1}^{k_{i-1}} x_i^{p-1}, \tag{4.9}$$

(4.10)

$$\psi_2(1 \otimes x_1^{k_1} \cdots x_n^{k_n} \otimes x_1^{m_1} \cdots x_n^{m_n} \otimes 1) =$$

$$\sum_{1 \le i < j \le n} \sum_{r=1}^{m_j} \sum_{p=1}^{k_i} e_i \wedge e_j \otimes x_i^{k_i - p} x_{i+1}^{k_{i+1}} \cdots x_{j-1}^{k_{j-1}} x_j^{k_j + m_j - r} x_{j+1}^{k_{j+1} + m_{j+1}} \cdots x_n^{k_n + m_n} \otimes$$
$$x_1^{k_1 + m_1} \cdots x_{i-1}^{k_{i-1} + m_{i-1}} x_i^{m_i + p - 1} x_{i+1}^{m_{i+1}} \cdots x_{j-1}^{m_{j-1}} x_j^{r-1}.$$

Now assume G is a finite group acting on V, and g is a central element of G satisfying (4.1) and (4.2) where q is a primitive ℓth root of unity, $\ell \geq 2$. Under these assumptions, by Proposition 6.4 and Corollary 6.5, $\mathrm{HH}^2(S(V)\#_\alpha G)$ contains as the g-component

$$\mathrm{HH}^2(S(V), S(V)\overline{g})^G \cong (\det(\mathrm{Span}_{\mathbb{C}}\{x_1, x_2\}^*) \otimes \mathbb{C}[x_3, \ldots x_n]\overline{g})^G.$$

Let $s \in \mathbb{C}[x_3, \ldots, x_n] - \{0\}$ satisfy (4.6), that is $h(s) = x_1(h)x_2(h)\alpha(g,h)\alpha^{-1}(h,g)s$ for all $h \in G$. Identify the dual function $(e_1 \wedge e_2)^*$ with a basis of the one-dimensional space $\det(\mathrm{Span}_{\mathbb{C}}\{x_1, x_2\}^*)$, where the notation e_i comes from the Koszul complex and is defined in the appendix. We first claim that $(e_1 \wedge e_2)^* \otimes s\overline{g}$ corresponds to a nonzero element of $\mathrm{HH}^2(S(V)\#_\alpha G)$ under the above isomorphism. We need only show that $(e_1 \wedge e_2)^* \otimes s\overline{g}$ is invariant under the action of G. Let $h \in G$. Then

$$\begin{aligned}
h((e_1 \wedge e_2)^* \otimes s\overline{g}) &= x_1(h^{-1})x_2(h^{-1})(e_1 \wedge e_2)^* \otimes (h(s))\overline{h} \cdot \overline{g} \cdot (\overline{h})^{-1} \\
&= \alpha(g,h)\alpha^{-1}(h,g)(e_1 \wedge e_2)^* \otimes s\overline{h} \cdot \overline{g} \cdot (\overline{h})^{-1} \\
&= \alpha(g,h)(e_1 \wedge e_2)^* \otimes s\overline{hg} \cdot (\overline{h})^{-1} \\
&= \alpha(g,h)\alpha^{-1}(h,h^{-1})\alpha(hg,h^{-1})(e_1 \wedge e_2)^* \otimes s\overline{hgh^{-1}} \\
&= (e_1 \wedge e_2)^* \otimes s\overline{g}
\end{aligned} \tag{4.11}$$

by an application of the two-cocycle identity (2.1) to the triple g, h, h^{-1}, since $g \in C(G)$.

Next we show that the nonzero element $(e_1 \wedge e_2)^* \otimes s\overline{g}$ of $\mathrm{HH}^2(S(V)\#_\alpha G)$ may be identified with a Hochschild two-cocycle μ_1 of the form $m \circ (D_1 \otimes D_2)$ where D_1, D_2 are defined in (4.3), (4.4). This will prove that the formal deformations of $S(V)\#_\alpha G$ given by Corollary 4.8 are nontrivial in case G is finite. We will need [4, Thm. 5.4], which will be applied to a Koszul resolution:

Proposition 4.12 (Caldararu-Giaquinto-Witherspoon)**.** *Let $A = S(V)\#_\alpha G$. Let $f : P_n \to A$ be a function representing an element of $\mathrm{HH}^n(S(V), A)^G \cong \mathrm{HH}^n(A)$ expressed in terms of any $S(V)^e$-projective resolution $P_{\bullet}$ of $S(V)$ carrying an action of G. The corresponding function $\widetilde{f} \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, A)$ on the bar complex (6.1) is given by*

$$\widetilde{f}(p_1\overline{\sigma_1} \otimes \cdots \otimes p_n\overline{\sigma_n}) = ((f \circ \psi_n)(1 \otimes p_1 \otimes \sigma_1(p_2) \otimes \cdots \otimes (\sigma_1 \cdots \sigma_{n-1})(p_n) \otimes 1))\overline{\sigma_1} \cdots \overline{\sigma_n}.$$

In particular, if $n = 2$, we obtain the infinitesimal deformation $\mu_1 : A \otimes A \to A$,

$$\mu_1(p_1\overline{\sigma_1} \otimes p_2\overline{\sigma_2}) = ((f \circ \psi_2)(1 \otimes p_1 \otimes \sigma_1(p_2) \otimes 1))\overline{\sigma_1} \cdot \overline{\sigma_2}.$$

As a consequence of the proposition, the element $(e_1 \wedge e_2)^* \otimes s\overline{g}$ of $\mathrm{HH}^2(S(V)\#_\alpha G)$ may be identified with the function $\mu_1 : A \otimes A \to A$ where $\mu_1(x_1^{k_1} \cdots x_n^{k_n}\overline{h} \otimes x_1^{m_1} \cdots x_n^{m_n}\overline{k})$ is $\psi_2(1 \otimes x_1^{k_1} \cdots x_n^{k_n} \otimes h(x_1^{m_1} \cdots x_n^{m_n}) \otimes 1)$ followed by application of the function representing $(e_1 \wedge e_2)^* \otimes s\overline{g}$ at the chain level, and right multiplication by $\overline{h} \cdot \overline{k}$. By our hypotheses, we have

$$\begin{aligned}&\psi_2(1 \otimes x_1^{k_1} \cdots x_n^{k_n} \otimes h(x_1^{m_1} \cdots x_n^{m_n}) \otimes 1)\\ &\qquad = x_1(h)^{m_1} x_2(h)^{m_2} \psi_2(1 \otimes x_1^{k_1} \cdots x_n^{k_n} \otimes x_1^{m_1} x_2^{m_2} h(x_3^{m_3} \cdots x_n^{m_n})).\end{aligned}$$

By (4.10), the resulting coefficient of $e_1 \wedge e_2$ is

$$x_1(h)^{m_1} x_2(h)^{m_2} \sum_{r=1}^{m_2} \sum_{p=1}^{k_1} x_1^{k_1-p} x_2^{k_2+m_2-r} x_3^{k_3} \cdots x_n^{k_n} \cdot h(x_3^{m_3} \cdots x_n^{m_n}) \otimes x_1^{m_1+p-1} x_2^{r-1}.$$

Applying $(e_1 \wedge e_2)^* \otimes s\overline{g}$ and multiplying by $\overline{h} \cdot \overline{k}$, we obtain

$$\begin{aligned}&x_1(h)^{m_1} x_2(h)^{m_2} \sum_{r=1}^{m_2} \sum_{p=1}^{k_1} x_1^{k_1-p} x_2^{k_2+m_2-r} x_3^{k_3} \cdots x_n^{k_n} h(x_3^{m_3} \cdots x_n^{m_n}) s\overline{g} x_1^{m_1+p-1} x_2^{r-1} \overline{h} \cdot \overline{k}\\ &= x_1(h)^{m_1} x_2(h)^{m_2} \sum_{r=1}^{m_2} \sum_{p=1}^{k_1} q^{m_1+p-r} x_1^{k_1+m_1-1} x_2^{k_2+m_2-1} x_3^{k_3} \cdots x_n^{k_n} h(x_3^{m_3} \cdots x_n^{m_n}) s\overline{g} \cdot \overline{h} \cdot \overline{k}\\ &= q^{m_1} x_1(h)^{m_1} x_2(h)^{m_2} (k_1)_q (m_2)_{q^{-1}} x_1^{k_1+m_1-1} x_2^{k_2+m_2-1} x_3^{k_3} \cdots x_n^{k_n} h(x_3^{m_3} \cdots x_n^{m_n}) s\overline{g} \cdot \overline{h} \cdot \overline{k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(m \circ (D_1 \otimes D_2))(x_1^{k_1} \cdots x_n^{k_n}\overline{h} \otimes x_1^{m_1} \cdots x_n^{m_n}\overline{k})\\ &= x_1(h^{-1})(k_1)_q x_1^{k_1-1} x_2^{k_2} \cdots x_n^{k_n}\overline{h} \cdot q^{m_1}(m_2)_{q^{-1}} x_1^{m_1} x_2^{m_2-1} x_3^{m_3} \cdots x_n^{m_n} s\overline{g} \cdot \overline{k}\\ &= q^{m_1} x_1(h)^{m_1} x_2(h)^{m_2} (k_1)_q (m_2)_{q^{-1}} x_1^{k_1+m_1-1} x_2^{k_2+m_2-1} x_3^{k_3} \cdots x_n^{k_n} h(x_3^{m_3} \cdots x_n^{m_n}) s\overline{g} \cdot \overline{h} \cdot \overline{k}.\end{aligned}$$

Therefore $m\circ(D_1\otimes D_2)$ is the Hochschild two-cocycle represented by $(e_1\wedge e_2)^*\otimes s\overline{g}$ and so is not a coboundary. This implies that the formal deformations given by Corollary 4.8 are nontrivial in case G is finite.

Example 4.13. (Twisted graded Hecke algebras.) Let G be a finite subgroup of $GL(V)$, and $g\in G$ a central element satisfying (4.1) and (4.2). Suppose $s=1$ satisfies (4.6), that is $1=x_1(h)x_2(h)\alpha(g,h)\alpha^{-1}(h,g)$ for all $h\in G$. We may rewrite this condition as $\det(h|_{(V^g)^\perp})=\alpha(h,g)\alpha^{-1}(g,h)$. Then $\mu_1=m\circ(D_1\otimes D_2)$ is a bilinear form on $S(V)\#_\alpha G$ of degree -2, where $S(V)\#_\alpha G$ is a graded algebra in which elements of V have degree 1 and elements of G have degree 0. More generally, in the formula $\exp_q(tD_1\otimes D_2)$, the bilinear form $\mu_i=\frac{1}{(i)_q!}m\circ(D_1^i\otimes D_2^i)$ has degree $-2i$. By [25, Thm 3.2], the resulting formal deformation of $S(V)\#_\alpha G$ becomes a (twisted) graded Hecke algebra [25] when the scalars are restricted to $\mathbb{C}[t]$. In this case, that means the associated deformation of $S(V)\#_\alpha G$ over $\mathbb{C}[t]$ is isomorphic to

$$T(V)\#_\alpha G[t]/(vw-wv-a_g(v,w)t\overline{g}),$$

the quotient by the ideal generated by all elements $vw-wv-a_g(v,w)t\overline{g}$, for $v,w\in V$, where $a_g(v,w)=\mu_1(v,w)-\mu_1(w,v)$. This (twisted) graded Hecke algebra is special in that only one such function a_g is nonzero. In the next section, we give some examples for which there is an analogous deformation with more than one group element g having a_g nonzero.

5. Universal Deformations

In this section we give examples for which some of the universal deformation formulas from the last section, corresponding to different group elements, may be combined into larger formulas. The first example generalizes [4, Lemma 6.2].

Example 5.1. Let $n\geq 3$ and $\ell\geq 2$ be integers, q a primitive ℓth root of unity, $G=(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$ and $V=\mathbb{C}^n$. Identify G with the subgroup of $\mathrm{SL}(V)$ generated by the diagonal matrices

$$\begin{aligned} g_1 &= \mathrm{diag}(q,q^{-1},1,\ldots,1),\\ g_2 &= \mathrm{diag}(1,q,q^{-1},1,\ldots,1),\\ &\vdots\\ g_{n-1} &= \mathrm{diag}(1,\ldots,1,q,q^{-1}), \end{aligned}$$

with respect to a basis $x_1,\ldots,x_n$ of V. Let $g_n=g_1^{-1}\cdots g_{n-1}^{-1}=\mathrm{diag}(q^{-1},1,\ldots,1,q)$. Let $\alpha:G\times G\to\mathbb{C}^\times$ be the following two-cocycle:

$$\alpha(g_1^{i_1}\cdots g_{n-1}^{i_{n-1}},\ g_1^{j_1}\cdots g_{n-1}^{j_{n-1}})=q^{-\sum_{1\leq k\leq n-2} i_k j_{k+1}}.$$

It may be checked directly that α satisfies the two-cocycle condition (2.1). Note that α is not a coboundary: By their definition, two-coboundaries for abelian groups are symmetric, but α is clearly not symmetric. In case $n = 3$, $\ell = 2$, α is cohomologous to the nontrivial cocycle given in [4, Example 3.4], as in that case there is a unique nontrivial two-cocycle up to coboundary. Note that

$$\alpha(g_{i+1}, g_i) = q\alpha(g_i, g_{i+1}) \quad \text{for } 1 \leq i \leq n \tag{5.2}$$

(where $g_{n+1} = g_1$ by definition). Direct calculations also show that

$$h(x_i x_{i+1}) = \frac{\alpha(h, g_i)}{\alpha(g_i, h)} x_i x_{i+1} \quad \text{for } 1 \leq i \leq n, \tag{5.3}$$

for all $h \in G$ (where $x_{n+1} = x_1$ by definition).

Let $H_i = H_q$ $(1 \leq i \leq n)$ be the Hopf algebra defined in (3.1), acting on $S(V)\#_\alpha G$ via the formulas (4.3)–(4.5), where we replace g by g_i and x_1, x_2 by x_i, x_{i+1}. Applying (4.6) and (5.3), it may be checked that the polynomial s arising in the action of H_i must be in $S(V^{g_i}) \cap (S(V))^G = \mathbb{C}[x_1^\ell, \ldots, x_{i-1}^\ell, x_{i+2}^\ell, \ldots, x_n^\ell]$, where if $i = n$ we leave out x_1 and x_n. Using this fact, equation (5.2), and the identity $(i + \ell)_{q^{-1}} = (i)_{q^{-1}}$ for all integers i, it may be checked directly that the images of the H_i $(1 \leq i \leq n)$ in $\mathrm{End}_{\mathbb{C}}(S(V)\#_\alpha G)$ mutually commute. Thus there is a corresponding algebra homomorphism from the Hopf algebra $H_1 \otimes \cdots \otimes H_n$ to $\mathrm{End}_{\mathbb{C}}(S(V)\#_\alpha G)$, and $S(V)\#_\alpha G$ is a module algebra for $H_1 \otimes \cdots \otimes H_n$. A product of universal deformation formulas is again a universal deformation formula, based on the tensor product of the bialgebras. Thus by Theorem 3.3,

$$\exp_q(tD_1^{g_1} \otimes D_2^{g_1}) \cdots \exp_q(tD_1^{g_n} \otimes D_2^{g_n}) \tag{5.4}$$

(where superscripts indicate the Hopf subalgebra from which the operators originate) is a universal deformation formula based on $H_1 \otimes \cdots \otimes H_n$. By Proposition 2.6, this formula applies to yield a formal deformation of $S(V)\#_\alpha G$.

In case $s = 1$, restricting this deformation to one over $\mathbb{C}[t]$ results in the twisted graded Hecke algebra

$$T(V)\#_\alpha G[t] \Big/ \left(vw - wv - \sum_{i=1}^{n} a_{g_i}(v, w) t \overline{g}_i \right)$$

where $a_{g_i}(v, w) = D_1^{g_i}(v) D_2^{g_i}(w) - D_1^{g_i}(w) D_2^{g_i}(v)$. The scalar coefficients of the a_{g_i} may be varied independently to obtain a vector space of dimension n parametrizing the possible twisted graded Hecke algebras realizable by the formula (5.4) and scalar modifications. It is shown in [25, Example 2.16] that these are in fact *all* the twisted graded Hecke algebras for this choice of G and α in case $\ell \neq 2$.

In case $\ell = 2$ and $n = 3$, the elements g_1, g_2, g_3 are precisely the nonidentity elements of G. The formal deformation of $S(V)\#_\alpha G$ arising from the formula (5.4) is nearly the universal deformation, as is justified by considering the Hochschild cohomology of $S(V)\#_\alpha G$ (see [4, Example 4.7] or the more general Corollary 6.5). That is, every Hochschild two-cocycle μ_1 with image in $S(V)\#_\alpha(G - \{1\})$ is an infinitesimal of the formal deformation resulting from (5.4) with appropriate choices of the polynomials s

in (4.4). (Classical deformations corresponding to the choice $g = 1$ involve derivations that do not commute with the actions of the H_i, and so we do not include these in the formula.) If $n > 3$ or $\ell > 2$, there are nonidentity group elements other than $g_1, \ldots, g_n$, and the actions of the corresponding Hopf algebras may no longer commute (but see the next example below). If α is not taken to be the cocycle we have chosen, the images of the H_i in $\mathrm{End}_{\mathbb{C}}(S(V)\#_\alpha G)$ again may no longer commute, and we do not know whether there is a universal deformation formula more complicated than (5.4) involving these operators.

Example 5.5. Let G be a group acting on a vector space V of dimension n, g a central element of G, and assume (4.1) and (4.2) hold with q a primitive ℓth root of unity, $\ell > 2$. Thus g corresponds to $\mathrm{diag}(q, q^{-1}, 1, \ldots, 1)$ and g^{-1} corresponds to $\mathrm{diag}(q^{-1}, q, 1, \ldots, 1)$. Assume further that $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)$, as is true in the last example for $g = g_i$. (In case G is finite, this assumption imposes no loss of generality, as any two-cocyle is cohomologous to one satisfying this assumption [14, Thm. 3.6.2]). Consider the images of H_q and $H_{q^{-1}}$ in $\mathrm{End}_{\mathbb{C}}(S(V)\#_\alpha G)$, where we let $D_1^{g^{-1}}$ involve q^{-1}-differentiation with respect to x_1 and $D_2^{g^{-1}}$ involve q-differentiation with respect to x_2 in (4.3) and (4.4). Multiplying and dividing the left side of the equation below by $q^{i-2}q^{i-1}$ yields the right side:

$$\frac{(i)_q(i-1)_{q^{-1}}}{(i)_{q^{-1}}(i-1)_q} = q.$$

Using this identity and the assumption $\alpha(g, g^{-1}) = \alpha(g^{-1}, g)$, we find that the following relations hold in $\mathrm{End}_{\mathbb{C}}(S(V)\#_\alpha G)$ among the images of the generators of H_q and $H_{q^{-1}}$:

$$\begin{aligned}
D_1^{g^{-1}} D_1^g = q D_1^g D_1^{g^{-1}} \quad &, \quad D_2^{g^{-1}} D_2^g = q^{-1} D_2^g D_2^{g^{-1}}, \\
D_1^g D_2^{g^{-1}} = D_2^{g^{-1}} D_1^g \quad &, \quad D_1^{g^{-1}} D_2^g = D_2^g D_1^{g^{-1}}, \\
\sigma^g \sigma^{g^{-1}} &= \sigma^{g^{-1}} \sigma^g, \\
\sigma^g D_1^{g^{-1}} = q^{-1} D_1^{g^{-1}} \sigma^g \quad &, \quad \sigma^g D_2^{g^{-1}} = q D_2^{g^{-1}} \sigma^g, \\
\sigma^{g^{-1}} D_1^g = q D_1^g \sigma^{g^{-1}} \quad &, \quad \sigma^{g^{-1}} D_2^g = q^{-1} D_2^g \sigma^{g^{-1}}.
\end{aligned}$$

These relations are preserved by Δ, ε and S, and so the algebra generated by H_q and $H_{q^{-1}}$, subject to the above relations, is a Hopf algebra. The proof of Theorem 3.3 may be modified to show that $\exp_q(tD_1^g \otimes D_2^g) \exp_{q^{-1}}(tD_1^{g^{-1}} \otimes D_2^{g^{-1}})$ is a universal deformation formula. The key idea is to move factors corresponding to g^{-1} past factors corresponding to g, so that the proof of Theorem 3.3 may be applied separately for each of g, g^{-1}. The relations above imply that indeed the appropriate factors commute. In case $G = \mathbb{Z}/3\mathbb{Z}$, this formula will nearly result in a universal deformation (again having infinitesimal with image in $S(V)\#_\alpha(G - \{1\})$).

6. Appendix: A computation of Hochschild cohomology

The *Hochschild cohomology* of a $\mathbb{C}$-algebra A is $\mathrm{HH}^{\bullet}(A) := \mathrm{Ext}^{\bullet}_{A^e}(A, A)$, where $A^e = A \otimes A^{op}$ acts on A by left and right multiplication. More generally, if M is an A-bimodule (equivalently, an A^e-module), we may define $\mathrm{HH}^{\bullet}(A, M) := \mathrm{Ext}^{\bullet}_{A^e}(A, M)$, so that $\mathrm{HH}^{\bullet}(A) = \mathrm{HH}^{\bullet}(A, A)$. These Ext groups may be expressed via the A^e-free resolution of A:

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{m} A \to 0, \tag{6.1}$$

where m is multiplication and

$$\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}.$$

Applying $\mathrm{Hom}_{A^e}(-, M)$ and dropping the term $\mathrm{Hom}_{A^e}(A, M)$, we obtain

$$0 \to \mathrm{Hom}_{A^e}(A^e, M) \xrightarrow{\delta_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \xrightarrow{\delta_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \xrightarrow{\delta_3^*} \cdots$$

Then $\mathrm{HH}^i(A, M) = \mathrm{Ker}(\delta_{i+1}^*)/\mathrm{Im}(\delta_i^*)$ and $\mathrm{HH}^{\bullet}(A, M) = \oplus_{i \geq 0} \mathrm{HH}^i(A, M)$. Noting that $\mathrm{Hom}_{A^e}(A^{\otimes(i+2)}, A) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes i}, A)$, a straightforward calculation shows that $\mathrm{HH}^2(A)$ may be identified with the space of $\mathbb{C}$-linear functions $\mu_1 : A \otimes A \to A$ satisfying the Hochschild two-cocycle condition (2.3), modulo coboundaries. See [24] for more details on Hochschild cohomology.

Let G be a finite subgroup of $\mathrm{GL}(V)$. We will compute $\mathrm{HH}^{\bullet}(S(V)\#_{\alpha}G)$, using techniques similar to those in [1, 23], where the crossed product was taken with a Weyl algebra rather than a polynomial algebra. We will use a result of Ştefan on Hopf Galois extensions [22, Cor. 3.4]. It implies that there is an action of G on $\mathrm{HH}^{\bullet}(S(V), S(V)\#_{\alpha}G)$ for which

$$\mathrm{HH}^{\bullet}(S(V)\#_{\alpha}G) \cong \mathrm{HH}^{\bullet}(S(V), S(V)\#_{\alpha}G)^G, \tag{6.2}$$

where the superscript G denotes the subspace of G-invariant elements. (A more explicit proof of this result, useful in this context, is given in [4, §5]). A Koszul complex may then be used to compute $\mathrm{HH}^{\bullet}(S(V), S(V)\#_{\alpha}G)$. This is done in a more general geometric setting by Ginzburg and Kaledin [13] in the case G is symplectic and α is trivial, although they note that their techniques apply to any finite group G. An elegant algebraic computation is given by Farinati [9] for an arbitrary finite group G, and trivial α. The additive structure of $\mathrm{HH}^{\bullet}(S(V), S(V)\#_{\alpha}G)$, before taking G-invariants, is independent of α since the $S(V)$-bimodule structure of $S(V)\#_{\alpha}G$ does not involve α. Thus the techniques of either [9] or [13] apply here. For completeness, we give an explicit algebraic computation whose details are needed in Section 4.

Note that $S(V)\#_{\alpha}G = \oplus_{g \in G} S(V)\overline{g}$, where $S(V)\overline{g} = \{s\overline{g} \mid s \in S(V)\}$, as an $S(V)$-bimodule. Thus there is an additive decomposition of Hochschild cohomology,

$$\mathrm{HH}^{\bullet}(S(V), S(V)\#_{\alpha}G) \cong \bigoplus_{g \in G} \mathrm{HH}^{\bullet}(S(V), S(V)\overline{g}). \tag{6.3}$$

We will determine each summand $\mathrm{HH}^{\bullet}(S(V), S(V)\overline{g})$, noting again that α plays no role here as we need only the $S(V)$-module structure of each $S(V)\overline{g}$. If $g = 1$, we have $\mathrm{HH}^{\bullet}(S(V), S(V)\overline{1}) = \mathrm{HH}^{\bullet}(S(V))$, and the Hochschild-Kostant-Rosenberg Theorem states that

$$\mathrm{HH}^{\bullet}(S(V)) \cong \textstyle\bigwedge^{\bullet}_{S(V)}(S(V)^n) \cong \bigwedge^{\bullet}(V^*) \otimes S(V),$$

where $n = \dim V$. (See [24, Exer. 9.1.3 and Thm. 9.4.7].) Letting $x_1, \ldots, x_n$ be a basis of V, this may be computed directly from the $S(V)^e$-projective Koszul resolution $K(\{x_i \otimes 1 - 1 \otimes x_i\}_{i=1}^n) \cong \bigwedge^{\bullet}(V) \otimes S(V)^e$ of $S(V)$. (See [24, §4.5] for details on Koszul complexes.) The differential $d_m : \bigwedge^m(V) \otimes S(V)^e$ is given by

$$d_m(e_{i_1} \wedge \cdots \wedge e_{i_m} \otimes 1 \otimes 1) = \sum_{k=1}^{m} (-1)^{k+1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_m} \otimes (x_{i_k} \otimes 1 - 1 \otimes x_{i_k}),$$

where we use the standard notation e_{i_j} for the element x_{i_j} in $\bigwedge^1(V)$. After application of $\mathrm{Hom}_{S(V)^e}(-, S(V))$, all chain maps become 0 as $S(V)$ is commutative. We will identify $\mathrm{HH}^{\bullet}(S(V))$ with $\bigwedge^{\bullet}(V^*) \otimes S(V)$, as the group action is clear in that notation: It is diagonal on the factors, with the standard actions on $S(V)$ and on $\bigwedge^{\bullet}(V^*)$. In case of an element g not necessarily equal to 1, we have the following.

Proposition 6.4. *For each $g \in G$,*

$$\mathrm{HH}^{\bullet}(S(V), S(V)\overline{g}) \cong \textstyle\bigwedge^{\bullet - \operatorname{codim} V^g}((V^g)^*) \otimes S(V^g).$$

In particular, the lowest degree j for which $\mathrm{HH}^j(S(V), S(V)\overline{g}) \neq 0$ is $j = \operatorname{codim} V^g$.

Proof. Fix $g \in G$. As the order of g is finite, we may assume without loss of generality that the action of g is diagonal with respect to the basis $x_1, \ldots, x_n$ of V. Thus there are scalars λ_i with $g \cdot x_i = \lambda_i x_i$ $(i = 1, \ldots, n)$. We will further assume, for notational convenience, that the basis is ordered so that $\lambda_i = 1$ for $i = 1, \ldots, r$ and $\lambda_i \neq 1$ for $i = r+1, \ldots, n$. (Set $r = 0$ if $V^g = 0$ and $r = n$ if $V^g = V$.) We may also assume that $\mathrm{Span}_{\mathbb{C}}\{x_{r+1}, \ldots, x_n\} = (V^g)^{\perp}$ where the orthogonal complement is taken with respect to some nondegenerate G-invariant Hermitian form on V.

Consider the complex $\mathrm{Hom}_{S(V)^e}(K(\{x_i \otimes 1 - 1 \otimes x_i\}_{i=1}^n), S(V)\overline{g})$, which we may identify with $\mathrm{Hom}_{S(V)^e}(\bigwedge^{\bullet}(V) \otimes S(V)^e, S(V)\overline{g}) \cong \bigwedge^{\bullet}(V^*) \otimes S(V)\overline{g}$. Additively, this is the same as $\bigwedge^{\bullet}(V^*) \otimes S(V)$, but the factor $\overline{g}$ affects the differentials, which we will determine next. They are *not* necessarily all zero (in contrast to the case $g = 1$). If $s \in S(V)$, we have

$$(x_i \otimes 1 - 1 \otimes x_i) \cdot s\overline{g} = x_i s\overline{g} - s\overline{g} x_i = (x_i - g \cdot x_i) s\overline{g}.$$

If $i = 1, \ldots, r$, this element is 0. If $i = r+1, \ldots, n$, the factor $x_i - g \cdot x_i = (1 - \lambda_i) x_i$ is a nonzero scalar multiple of x_i. Thus $\mathrm{Hom}_{S(V)^e}(K(\{x_i \otimes 1 - 1 \otimes x_i\}_{i=1}^n), S(V)\overline{g})$ is equivalent to the dual Koszul complex $\overline{K}(0, \ldots, 0, x_{r+1}, \ldots, x_n)$ for $S(V)$, where the bar denotes the reverse order. This is the tensor product (over $S(V)$) of two complexes for $S(V)$: $\overline{K}(0, \ldots, 0)$ and $\overline{K}(x_{r+1}, \ldots, x_n)$. The second complex is exact other than in degree $n - r$ (as the corresponding Koszul complex is exact other than in degree 0), where it has cohomology $S(V)/(x_{r+1},\dots,x_n)S(V) \cong S(V^g)$ (see [24, Cor. 4.5.4]). We will identify this with $\det(((V^g)^{\perp})^*) \otimes S(V^g)$, where $\det(((V^g)^{\perp})^*)$ is the one-dimensional space $\bigwedge^{\operatorname{codim} V^g}(((V^g)^{\perp})^*)$, to account for the degree shift and the action of G. The spectral sequence of the double complex $\overline{K}(0,\dots,0)\otimes_{S(V)}\overline{K}(x_{r+1},\dots,x_n)$ thus collapses at E_2 with $E_2^{pq}=0$ for $q\neq n-r$, and $E_2^{p,\,n-r} = \mathrm{H}^p(\overline{K}(0,\dots,0))\otimes_{S(V)} S(V^g)$ by freeness of the terms of the chain complex over $S(V)$. This follows from [3, Thm. 3.4.2], which also implies that the cohomology is precisely $E_2^{p,\,n-r}$. Now $\mathrm{H}^p(\overline{K}(0,\dots,0)) \cong \bigwedge^p_{S(V)}(S(V)^{\dim V^g}) \cong \bigwedge^p((V^g)^*)\otimes S(V)$, and as this is tensored with the cohomology of $\overline{K}(x_{r+1},\dots,x_n)$, namely $S(V)/(x_{r+1},\dots,x_n)S(V)\cong S(V^g)$ in degree $n-r$, we obtain the stated result. □

We will identify the cohomology $\bigwedge^{\bullet-\operatorname{codim} V^g}((V^g)^*)\otimes S(V^g)$ of the theorem with $\bigwedge^{\bullet-\operatorname{codim} V^g}((V^g)^*)\otimes \det(((V^g)^{\perp})^*)\otimes S(V^g)\overline{g}$. The action of G is nontrivial on the one-dimensional factor $\det(((V^g)^{\perp})^*)$, as may be seen by considering the action of G on the corresponding cochain complex, and the action on $\overline{g}$ is by conjugation by $\overline{h}$ ($h\in G$).

The following corollary is immediate from (6.2), (6.3) and Proposition 6.4, after making the above identifications.

Corollary 6.5. *There is an additive decomposition of Hochschild cohomology,*

$$\mathrm{HH}^{\bullet}(S(V)\#_{\alpha}G) \cong \left(\bigoplus_{g\in G}\bigwedge^{\bullet-\operatorname{codim} V^g}((V^g)^*)\otimes\det(((V^g)^{\perp})^*)\otimes S(V^g)\overline{g}\right)^G .$$

Compare the above corollary with [9, Thm. 3.6], or with the formula just above (6.4) in [13].

As a consequence of Corollary 6.5, we obtain a necessary condition for there to exist a Hochschild two-cocycle μ_1, with image in the g-component $S(V)\overline{g}$, that is not a coboundary. Due to the degree shift $2-\operatorname{codim} V^g$, such an element g must satisfy $\operatorname{codim} V^g\in\{0,1,2\}$. We claim that the determinant of g on V must be 1. If $\det(g)\neq 1$, the action of g itself on the one-dimensional space $\det(((V^g)^{\perp})^*)$ is nontrivial, whereas its actions on $\bigwedge^{2-\operatorname{codim} V^g}((V^g)^*)$ and on $S(V^g)\overline{g}$ are trivial. Consequently there can be no such G-invariant elements. Therefore $\det(g)=1$, which also now implies $\operatorname{codim} V^g\in\{0,2\}$. (See also [9, Ex. 3.10].)

References

[1] J. Alev, M. A. Farinati, T. Lambre, and A. L. Solotar, "Homologie des invariants d'une algèbre de Weyl sous l'action d'un groupe fini," J. Algebra 232 (2000), 564–577.

[2] G. Benkart and S. Witherspoon, "Quantum group actions, twisting elements, and deformations of algebras," `math.QA/0503310`.

[3] D. J. Benson, *Representations and Cohomology II: Cohomology of Groups and Modules*, Cambridge Univ. Press, Cambridge, 1991.

[4] A. Căldăraru, A. Giaquinto, and S. Witherspoon, "Algebraic deformations arising from orbifolds with discrete torsion," J. Pure Appl. Algebra 187 (2004), no. 1–3, 51–70.

[5] C. Cibils, "Rigidity of truncated quiver algebras," Adv. Math. 79 (1990), 18–42.

[6] C. Cibils, "Rigid monomial algebras," Math. Ann. 289 (1991), 95–109.
[7] V. G. Drinfel'd, "Degenerate affine Hecke algebras and Yangians," Funct. Anal. Appl. 20 (1986), 58–60.
[8] P. Etingof and V. Ginzburg, "Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphisms," Invent. Math. 147 (2002), no. 2, 243–348.
[9] M. Farinati, "Hochschild duality, localization, and smash products," J. Algebra 284 (2005), no. 1, 415–434.
[10] M. Gerstenhaber and S. D. Schack, "Algebras, bialgebras, quantum groups, and algebraic deformations," in *Deformation Theory and Quantum Groups with Applications to Mathematical Physics*, Contemp. Math. 134 (1992), 51–92.
[11] A. Giaquinto and J. Zhang, "Bialgebra actions, twists, and universal deformation formulas," J. Pure Appl. Algebra 128 (1998), 133–151.
[12] A. Giaquinto and J. Zhang, private communication.
[13] V. Ginzburg and D. Kaledin, "Poisson deformations of symplectic quotient singularities," Adv. Math. 186 (2004), no. 1, 1–57.
[14] G. Karpilovsky, *Projective representations of finite groups*, Marcel-Dekker, 1985.
[15] C. Kassel, *Quantum Groups*, Springer-Verlag, 1995.
[16] V. K. Kharchenko, *Automorphisms and Derivations of Associative Rings*, Kluwer, 1991.
[17] V. K. Kharchenko, "Skew primitive elements in Hopf algebras and related identities," J. Algebra 238 (2001), 534–559.
[18] G. Lusztig, "Affine Hecke algebras and their graded version," J. Amer. Math. Soc. 2 (1989), 599–635.
[19] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Math., Number 82, Amer. Math. Soc., 1993.
[20] D. S. Passman, *Infinite Crossed Products*, Academic Press, New York, 1989.
[21] A. Ram and A. V. Shepler, "Classification of graded Hecke algebras for complex reflection groups," Comment. Math. Helv. 78 (2003), no. 2, 308–334.
[22] D. Ştefan, "Hochschild cohomology on Hopf Galois extensions," J. Pure Appl. Algebra 103 (1995), 221–233.
[23] M. Suarez-Alvarez, "Algebra structure on the Hochschild cohomology of the ring of invariants of a Weyl algebra under a finite group," J. Algebra 248 (2002), 291–306.
[24] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.
[25] S. Witherspoon, "Twisted graded Hecke algebras," `math.RT/0506152`.

Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: `sjw@math.tamu.edu`