# TWO-PARAMETER QUANTUM GROUPS AND DRINFEL'D DOUBLES

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Abstract. We investigate two-parameter quantum groups corresponding to the general linear and special linear Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$. We show that these quantum groups can be realized as Drinfel'd doubles of certain Hopf subalgebras with respect to Hopf pairings. Using the Hopf pairing, we construct a corresponding $R$-matrix and a quantum Casimir element. We discuss isomorphisms among these quantum groups and connections with multiparameter quantum groups.

## Introduction

In this work we study two two-parameter quantum groups $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ and $U = U_{r,s}(\mathfrak{sl}_n)$ corresponding to the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$. Our Hopf algebra $\widetilde{U}$ is isomorphic as an algebra to Takeuchi's $U_{r,s^{-1}}$ (see [T]), but as a Hopf algebra, it has the opposite coproduct. (A different presentation of $U_{r,s^{-1}}$ was obtained by Kulish [K] (see also [Ji]).) As an algebra, $\widetilde{U}$ has generators $e_j,\ f_j, (1 \le j < n)$, and $a_i^{\pm 1},\ b_i^{\pm 1}\quad (1 \le i \le n)$, and defining relations given in (R1)-(R7) below. The elements $e_j,\ f_j,\ \omega_j^{\pm 1}, (\omega_j')^{\pm 1}$ $(1 \le j < n)$, where $\omega_j = a_j b_{j+1}$ and $\omega_j' = a_{j+1} b_j$, generate the subalgebra $U = U_{r,s}(\mathfrak{sl}_n)$.

We show that both $\widetilde{U}$ and $U$ may be realized as Drinfel'd doubles of certain Hopf subalgebras with respect to suitable Hopf pairings. Using the Hopf pairing, we construct an $R$-matrix for $\widetilde{U}$ (which also works for $U$). For $\widetilde{U}$-modules $M$ and $M'$ in category $\mathcal{O}$ (defined in Section 4), there is an isomorphism $R_{M',M} : M' \otimes M \to M \otimes M'$. Moreover, the $R$-matrix satisfies the quantum Yang-Baxter equation and the hexagon identities. In [BW2], the $R$-matrix will be used to establish an analogue of Schur-Weyl duality in this setting: $\widetilde{U}$ has a natural $n$-dimensional module $V$, and the centralizer algebra $\mathrm{End}_{\widetilde{U}}(V^{\otimes k})$ is generated by a certain Hecke algebra $H_k(r,s)$. We construct a quantum Casimir element, which will play an essential role in [BW2] in proving that finite-dimensional modules in category $\mathcal{O}$ are completely reducible.

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Jing's work [Ji], which treats the special case of $\mathfrak{gl}_2$, adopts exactly the opposite approach to the one of this paper – it derives an analogue of the algebra $\widetilde{U}$ from one particular solution $R$ of the quantum Yang-Baxter equation. Similarly, Chin and Musson [ChM] and Dobrev and Parashar [DP] study multiparameter quantum universal enveloping algebras defined as duals of quantum function algebras arising from $R$-matrices. In Section 6, we relate the two-parameter quantum groups considered here with certain special cases of these multiparameter quantum groups. Moreover, we determine conditions for isomorphisms among the two-parameter quantum groups. In particular, the standard one-parameter quantum group $U_q(\mathfrak{sl}_2)$ of [Ja] is isomorphic to a quotient of $U_{r,s}(\mathfrak{sl}_2)$ by the ideal generated by $\omega_1' - \omega_1^{-1}$ whenever $q$ is a square root of $rs^{-1}$. However, for $n \geq 3$, no such isomorphism exists (see Proposition 6.1).

Our motivation to study these two-parameter quantum groups came from our work [BW1] on down-up algebras. Down-up algebras were introduced in [BR] as a generalization of the algebra generated by the down and up operators on posets. They are unital associative algebras $A(\alpha, \beta, \gamma)$ over a field $\mathbb{K}$ having generators $d, u$ which satisfy the defining relations

$$\begin{aligned} d^2u &= \alpha dud + \beta ud^2 + \gamma d \\ du^2 &= \alpha udu + \beta u^2 d + \gamma u, \end{aligned}$$

where $\alpha, \beta, \gamma$ are fixed but arbitrary scalars in $\mathbb{K}$. If $\gamma \neq 0$, then the down-up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$. Thus, there are basically two different cases: $\gamma = 0$ and $\gamma = 1$. Examples of down-up algebras include the universal enveloping algebras of $\mathfrak{sl}_2$, of the Heisenberg Lie algebra, and of the Lie superalgebra $\mathfrak{osp}(1,2)$, which are $A(2,-1,1)$, $A(2,-1,0)$, and $A(0,-1,1)$, respectively, and many of Witten's deformations of $U(\mathfrak{sl}_2)$ (see [B]). Down-up algebras exhibit many striking features including a Poincaré-Birkhoff-Witt type basis and a well-behaved representation theory ([BR], [KMP], [CaM], [Jor], [KK1], [KK2], [Ku], [BL, Sec. 4]). They are Noetherian domains whenever $\beta \neq 0$.

Essential to the structure of $A(\alpha, \beta, \gamma)$ are the roots of the equation

$$0 = t^2 - \alpha t - \beta = (t - r)(t - s),$$

Thus, $\alpha = r + s$ and $\beta = -rs$. When $rs \neq 0$ and $\gamma = 0$, the down-up algebra $A(\alpha, \beta, 0) = A(r+s, -rs, 0)$ can be extended by automorphisms to give a Hopf algebra $B(r+s, -rs, 0)$ (see [BW1]). This Hopf algebra is isomorphic to a subalgebra of $U_{r,s}(\mathfrak{sl}_3)$ when $r$ and $s$ are not roots of unity and to a quotient of a subalgebra when they are. It seemed natural to expect that there is a Drinfel'd double (quantum double) of the subalgebra, which yields a quantum group that depends on the two parameters $r$ and $s$. In fact, that quantum group is $U_{r,s}(\mathfrak{sl}_3)$. That result is a very special case of our theorem showing that $U$ and $\widetilde{U}$ are Drinfel'd doubles.

Throughout we will be working over a field $\mathbb{K}$, which is required to be algebraically closed from Section 3 to the end of the paper.

## §1. Preliminaries

Assume $\Phi$ is a finite root system of type $A_{n-1}$ with $\Pi$ a base of simple roots. We regard $\Phi$ as a subset of a Euclidean space $E = \mathbb{R}^n$ with an inner product $\langle\, ,\, \rangle$. We let $\epsilon_1, \ldots, \epsilon_n$ denote an orthonormal basis of $E$, and suppose $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1, \ldots, n-1\}$ and $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\}$.

Fix nonzero elements $r, s$ in a field $\mathbb{K}$. Here we assume $r \ne s$.

Let $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ be the unital associative algebra over $\mathbb{K}$ generated by elements $e_j,\ f_j, (1 \le j < n)$, and $a_i^{\pm 1},\ b_i^{\pm 1}\quad (1 \le i \le n)$, which satisfy the following relations.

(R1) The $a_i^{\pm 1},\ b_j^{\pm 1}$ all commute with one another and $a_i a_i^{-1} = b_j b_j^{-1} = 1$,

(R2) $a_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} e_j a_i \quad \text{and} \quad a_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} f_j a_i,$

(R3) $b_i e_j = s^{\langle \epsilon_i, \alpha_j \rangle} e_j b_i \quad \text{and} \quad b_i f_j = s^{-\langle \epsilon_i, \alpha_j \rangle} f_j b_i,$

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(a_i b_{i+1} - a_{i+1} b_i),$

(R5) $[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if} \quad |i-j| > 1,$

(R6) $e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + rs e_{i+1} e_i^2 = 0,$
$e_i e_{i+1}^2 - (r+s) e_{i+1} e_i e_{i+1} + rs e_{i+1}^2 e_i = 0,$

(R7) $f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0,$
$f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_{i+1}^2 f_i = 0.$

The relations in (R6) are just the two defining relations of the down-up algebra $A(r+s, -rs, 0)$, while those in (R7) are the defining relations of $A(r^{-1} + s^{-1}, -r^{-1}s^{-1}, 0)$. In fact these two down-up algebras are isomorphic via the map that takes $d$ to $u'$ and $u$ to $d'$ (assuming $d', u'$ are the generators of the latter) (see [BR]).

We will be interested in the subalgebra $U = U_{r,s}(\mathfrak{sl}_n)$ of $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ generated by the elements $e_j, f_j,\ \omega_j$, and $\omega_j'\quad (1 \le j < n)$, where

$$\omega_j = a_j b_{j+1} \quad \text{and} \quad \omega_j' = a_{j+1} b_j. \tag{1.1}$$

These elements satisfy (R5)-(R7) along with the following relations:

(R1') The $\omega_i^{\pm 1},\ \omega_j^{\pm 1}$ all commute with one another and $\omega_i \omega_i^{-1} = \omega_j'(\omega_j')^{-1} = 1$,

(R2') $\omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i \quad \text{and} \quad \omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i,$

(R3') $\omega_i' e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega_i' \quad \text{and} \quad \omega_i' f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega_i',$

(R4') $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(\omega_i - \omega_i').$

When $r = q$ and $s = q^{-1}$, the algebra $U_{r,s}(\mathfrak{gl}_n)$ modulo the ideal generated by the elements $b_i - a_i^{-1}$, $1 \le i \le n$, is just the quantum general linear group $U_q(\mathfrak{gl}_n)$, and $U_{r,s}(\mathfrak{sl}_n)$ modulo the ideal generated by the elements $\omega'_j - \omega_j^{-1}$, $1 \le j < n$, is $U_q(\mathfrak{sl}_n)$.

Let $Q = \mathbb{Z}\Phi$ denote the root lattice and set $Q^+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\ge 0}\alpha_i$. Then for any $\zeta = \sum_{i=1}^{n-1} \zeta_i\alpha_i \in Q$, we adopt the shorthand

$$\omega_\zeta = \omega_1^{\zeta_1} \cdots \omega_{n-1}^{\zeta_{n-1}}, \qquad \omega'_\zeta = (\omega'_1)^{\zeta_1} \cdots (\omega'_{n-1})^{\zeta_{n-1}} \tag{1.2}$$

The following lemma is straightforward to check.

**Lemma 1.3.** *Suppose that* $\zeta = \sum_{i=1}^{n-1} \zeta_i\alpha_i \in Q$. *Then*

$$\begin{aligned} w_\zeta e_i &= r^{-\langle \epsilon_{i+1},\zeta\rangle} s^{-\langle \epsilon_i,\zeta\rangle} e_i w_\zeta \qquad & w_\zeta f_i &= r^{\langle \epsilon_{i+1},\zeta\rangle} s^{\langle \epsilon_i,\zeta\rangle} f_i w_\zeta \\ w'_\zeta e_i &= r^{-\langle \epsilon_i,\zeta\rangle} s^{-\langle \epsilon_{i+1},\zeta\rangle} e_i w'_\zeta \qquad & w'_\zeta f_i &= r^{\langle \epsilon_i,\zeta\rangle} s^{\langle \epsilon_{i+1},\zeta\rangle} f_i w'_\zeta. \end{aligned}$$

The algebras $\widetilde{U}$ and $U$ are Hopf algebras, where the $a_i^{\pm 1}, b_i^{\pm 1}$ are group-like elements, and the remaining coproducts are determined by

$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i. \tag{1.4}$$

This forces the counit and antipode maps to be

$$\begin{aligned} &\varepsilon(a_i) = \varepsilon(b_i) = 1, \quad S(a_i) = a_i^{-1}, \quad S(b_i) = b_i^{-1} \\ &\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad S(e_i) = -\omega_i^{-1}e_i, \quad S(f_i) = -f_i(\omega'_i)^{-1}. \end{aligned} \tag{1.5}$$

## §2. Drinfel'd doubles

A *Hopf pairing* of two Hopf algebras $H$ and $H'$ is a bilinear form on $H' \times H$ satisfying the following properties (see [DT] or [Jo, 3.2.1]):

(2.1)(i) $(1, h) = \varepsilon_H(h), \quad (h', 1) = \varepsilon_{H'}(h')$

(ii) $(h', hk) = (\Delta_{H'}(h'), h \otimes k) = \sum (h'_{(1)}, h)(h'_{(2)}, k)$

(iii) $(h'k', h) = (h' \otimes k', \Delta_H(h)) = \sum (h', h_{(1)})(k', h_{(2)})$

for all $h, k \in H$ and $h', k' \in H'$, where $\varepsilon_H$ and $\varepsilon_{H'}$ denote the counits of $H$ and $H'$, respectively, and $\Delta_H$ and $\Delta_{H'}$ are their coproducts. It is a consequence of the defining properties that a Hopf pairing satisfies

$$(S_{H'}(h'), h) = (h', S_H(h))$$

for all $h \in H$ and $h' \in H'$, where $S_H$ and $S_{H'}$ denote the respective antipodes of $H$ and $H'$.

Assume $\widetilde{B}$ is the Hopf subalgebra of $\widetilde{U}$ generated by $e_j,\ \omega_j^{\pm 1}$ $(1 \leq j < n)$, and $a_n^{\pm 1}$. Let $(\widetilde{B}')^{\mathrm{coop}}$ be the Hopf algebra having the *opposite coproduct* to the Hopf subalgebra of $\widetilde{U}$ generated by $f_j,\ (\omega_j')^{\pm 1}$ $(1 \leq j < n)$, and $b_n^{\pm 1}$. Similarly $B$ is the Hopf subalgebra of $U$ generated by $e_j, \omega_j^{\pm 1}$ $(1 \leq j < n)$, and $(B')^{\mathrm{coop}}$ is generated by $f_j, (\omega_j')^{\pm 1}$ $(1 \leq j < n)$.

**Lemma 2.2.** *There are Hopf pairings of $\widetilde{B}$ and $\widetilde{B}'$, respectively of $B$ and $B'$.*

*Proof.* We begin by defining a bilinear form for $\widetilde{B}' \times \widetilde{B}$ first on the generators:

$$\begin{aligned}
&(f_i, e_j) = \frac{\delta_{i,j}}{s-r}, \\
(2.3)\qquad &(\omega_i', \omega_j) = r^{\langle \epsilon_j, \alpha_i \rangle} s^{\langle \epsilon_{j+1}, \alpha_i \rangle} = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle}, \\
&(b_n, a_n) = 1, \quad (b_n, \omega_j) = s^{-\langle \epsilon_n, \alpha_j \rangle}, \quad (\omega_i', a_n) = r^{\langle \epsilon_n, \alpha_i \rangle},
\end{aligned}$$

for $(1 \leq i, j < n)$. If $\omega_i'$ is replaced by $(\omega_i')^{-1}$ in the second or third line of (2.3), we replace the image under the bilinear form by its inverse, and similarly for $\omega_j$ and $\omega_j^{-1}$, $a_n$ and $a_n^{-1}$, $b_n$ and $b_n^{-1}$. On all other pairs of generators the form is 0.

In the second line of (2.3) we have applied the identity

$$(2.4)\qquad \langle \epsilon_j, \alpha_i \rangle = -\langle \epsilon_{i+1}, \alpha_j \rangle,$$

which is quite useful in subsequent calculations.

The pairings in (2.3) may be *extended* to a bilinear form on $\widetilde{B}' \times \widetilde{B}$ by requiring that (2.1)(i)–(iii) hold. We need only verify that the relations in $\widetilde{B}$ and $\widetilde{B}'$ are preserved, ensuring that the bilinear form is well-defined. It will then be a Hopf pairing by definition. Restricting the form to $B' \times B$ gives the desired Hopf pairing of $B$ and $B'$.

It is straightforward to check that the bilinear form preserves all the relations among the $\omega_i^{\pm 1}$, $a_n^{\pm 1}$ in $\widetilde{B}$ and the $(\omega_j')^{\pm 1}$, $b_n^{\pm 1}$ in $\widetilde{B}'$. We will verify that the form on $\widetilde{B}' \times \widetilde{B}$ preserves one of the remaining relations in $\widetilde{B}$, and leave the other verifications to the reader. For each $i$, $1 \leq i < n$, consider

$$(X, e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + rs e_{i+1} e_i^2),$$

where $X$ is any word in the generators of $\widetilde{B}'$. By definition, this is equal to

$$(2.5)\qquad (\Delta^2(X), e_i \otimes e_i \otimes e_{i+1} - (r+s) e_i \otimes e_{i+1} \otimes e_i + rs e_{i+1} \otimes e_i \otimes e_i).$$

In order for any one of these terms to be nonzero, $X$ must involve exactly two $f_i$ factors, one $f_{i+1}$ factor, and arbitrarily many $(\omega_j')^{\pm 1}$ and $b_n^{\pm 1}$ factors $(1 \leq j < n)$. First assume that $X = f_i^2 f_{i+1}$. Then $\Delta^2(X)$ is equal to

$$(\omega_i' \otimes \omega_i' \otimes f_i + \omega_i' \otimes f_i \otimes 1 + f_i \otimes 1 \otimes 1)^2 (\omega_{i+1}' \otimes \omega_{i+1}' \otimes f_{i+1} + \omega_{i+1}' \otimes f_{i+1} \otimes 1 + f_{i+1} \otimes 1 \otimes 1).$$

The relevant terms of $\Delta^2(X)$ are

$$f_i\omega_i'\omega_{i+1}'\otimes f_i\omega_{i+1}'\otimes f_{i+1}+\omega_i'f_i\omega_{i+1}'\otimes f_i\omega_{i+1}'\otimes f_{i+1}$$
$$+f_i\omega_i'\omega_{i+1}'\otimes\omega_i'f_{i+1}\otimes f_i+\omega_i'f_i\omega_{i+1}'\otimes\omega_i'f_{i+1}\otimes f_i$$
$$+(\omega_i')^2f_{i+1}\otimes f_i\omega_i'\otimes f_i+(\omega_i')^2f_{i+1}\otimes\omega_i'f_i\otimes f_i.$$

Therefore (2.5) becomes

$$\begin{aligned}
&(f_i\omega_i'\omega_{i+1}',e_i)(f_i\omega_{i+1}',e_i)(f_{i+1},e_{i+1})+(\omega_i'f_i\omega_{i+1}',e_i)(f_i\omega_{i+1}',e_i)(f_{i+1},e_{i+1})\\
&\quad-(r+s)(f_i\omega_i'\omega_{i+1}',e_i)(\omega_i'f_{i+1},e_{i+1})(f_i,e_i)-(r+s)(\omega_i'f_i\omega_{i+1}',e_i)(\omega_i'f_{i+1},e_{i+1})(f_i,e_i)\\
&\qquad+rs((\omega_i')^2f_{i+1},e_{i+1})(f_i\omega_i',e_i)(f_i,e_i)+rs((\omega_i')^2f_{i+1},e_{i+1})(\omega_i'f_i,e_i)(f_i,e_i)\\
&\quad=\frac{1}{(s-r)^3}(1+(\omega_i',\omega_i)-(r+s)(\omega_i',\omega_{i+1})-(r+s)(\omega_i',\omega_i)(\omega_i',\omega_{i+1})\\
&\qquad\qquad+rs(\omega_i',\omega_{i+1})^2+rs(\omega_i',\omega_{i+1})^2(\omega_i',\omega_i))\\
&\quad=\frac{1}{(s-r)^3}(1+rs^{-1}-(r+s)r^{-1}-(r+s)s^{-1}+r^{-1}s+1)\;=\;0.
\end{aligned}$$

If $X=f_if_{i+1}f_i$ or $X=f_{i+1}f_i^2$, then similar calculations show that (2.5) is equal to 0. Finally if $X$ is *any* word involving exactly two $f_i$ factors, one $f_{i+1}$ factor, and arbitrarily many factors of $(\omega_j')^{\pm1}$ $(1\le j<n)$ and $b_n^{\pm1}$, then (2.5) will just be a scalar multiple of one of the quantities we have already calculated, and therefore will equal 0. (For example, if $X=f_i^2\omega_j'f_{i+1}$, then (2.5) will be $(\omega_j',\omega_{i+1})$ times the corresponding quantity for $X=f_i^2f_{i+1}$.)

Analogous calculations show that the relations in $\widetilde{B}'$ are preserved. □

As there is a Hopf pairing between $\widetilde{B}$ and $\widetilde{B}'$, there is a skew-Hopf pairing between $\widetilde{B}$ and $(\widetilde{B}')^{\mathrm{coop}}$, where the latter is $\widetilde{B}'$ as an algebra, but with the opposite coproduct. Therefore, we may form the Drinfel'd double $D(\widetilde{B},(\widetilde{B}')^{\mathrm{coop}})$ as in [Jo, 3.2]. This is a Hopf algebra whose underlying coalgebra is $\widetilde{B}\otimes(\widetilde{B}')^{\mathrm{coop}}$ (that is, $\widetilde{B}\otimes(\widetilde{B}')^{\mathrm{coop}}$ as a vector space with the tensor product coalgebra structure). The algebra structure is given as follows: $\widetilde{B}$ and $\widetilde{B}'$ are identified as algebras with $\widetilde{B}\otimes 1$ and $1\otimes\widetilde{B}'$ respectively in $D(\widetilde{B},(\widetilde{B}')^{\mathrm{coop}})$. Letting $a\in\widetilde{B}$ and $b\in\widetilde{B}'$, we have $(a\otimes 1)(1\otimes b)=a\otimes b$ and

$$(1\otimes b)(a\otimes 1)=\sum(S^{\mathrm{coop}}(b_{(1)}),a_{(1)})(b_{(3)},a_{(3)})a_{(2)}\otimes b_{(2)},$$

where $S^{\mathrm{coop}}$ denotes the antipode for $(\widetilde{B}')^{\mathrm{coop}}$. (This expression looks different from [Jo, Lemma 3.2.2 (iii)] as we have written our bilinear form in the reverse order.) A similar construction applies to $B$ and $B'$.

**Theorem 2.7.** *$D(\widetilde{B},(\widetilde{B}')^{\mathrm{coop}})$ is isomorphic to $\widetilde{U}$, and $D(B,(B')^{\mathrm{coop}})$ is isomorphic to $U$.*

*Proof.* We will prove the first statement, and the second will follow by restricting to fewer generators. We will denote the image $e_i\otimes 1$ of $e_i$ in $D(\widetilde{B},(\widetilde{B}')^{\mathrm{coop}})$ by $\check{e}_i$, and similarly for $\omega_i, a_n, f_i, \omega_i', b_n$. Define a map $\varphi : D(\widetilde{B}, (\widetilde{B}')^{\text{coop}}) \to \widetilde{U}$ by

$$\varphi(\check{e}_i) = e_i, \qquad \varphi(\check{f}_i) = f_i,$$
$$\varphi(\check{\omega}_i^{\pm 1}) = \omega_i^{\pm 1}, \quad \varphi((\check{\omega}_i')^{\pm 1}) = (\omega_i')^{\pm 1}, \quad \varphi(\check{a}_n^{\pm 1}) = a_n^{\pm 1}, \quad \varphi(\check{b}_n^{\pm 1}) = b_n^{\pm 1}.$$

First notice that by definition, $\varphi$ preserves the coalgebra structures, the relations in $\widetilde{B}$, and the relations in $\widetilde{B}'$. Next we will verify that the mixed relations in $D(\widetilde{B}, (\widetilde{B}')^{\text{coop}})$ correspond to those in $\widetilde{U}$.

To calculate $\check{f}_j \check{e}_i$, we use

$$\begin{aligned} &\Delta^2(e_i) = e_i \otimes 1 \otimes 1 + \omega_i \otimes e_i \otimes 1 + \omega_i \otimes \omega_i \otimes e_i, \\ &(\Delta^{\text{coop}})^2(f_j) = 1 \otimes 1 \otimes f_j + 1 \otimes f_j \otimes \omega_j' + f_j \otimes \omega_j' \otimes \omega_j', \\ &\text{and} \quad S^{\text{coop}}(f_j) = -f_j(\omega_j')^{-1}, \end{aligned}$$

so that

$$\begin{aligned} \check{f}_j \check{e}_i &= (-f_j(\omega_j')^{-1}, e_i)(\omega_j', 1)\check{\omega}_j' + (1, \omega_i)(\omega_j', 1)\check{e}_i \check{f}_j + (1, \omega_i)(f_j, e_i)\check{\omega}_i \\ &= -\frac{\delta_{i,j}}{s-r}\check{\omega}_j' + \check{e}_i \check{f}_j + \frac{\delta_{i,j}}{s-r}\check{\omega}_i. \end{aligned}$$

That is, $[\check{e}_i, \check{f}_j] = \delta_{i,j}(s-r)^{-1}(\check{\omega}_i - \check{\omega}_i')$. Applying $\varphi$ gives the desired relation in $\widetilde{U}$.

We leave verification of the remaining relations to the reader. As $\widetilde{U}$ is generated by $e_i, f_i, \omega_i^{\pm 1}, (\omega_i')^{\pm 1}$ $(1 \le i < n)$, $a_n$ and $b_n$, the map $\varphi$ is surjective, and there is an obvious inverse map. □

## §3. Weight modules

Let $\Lambda = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, which is the weight lattice of $\mathfrak{gl}_n$. Corresponding to any $\lambda \in \Lambda$ is an algebra homomorphism $\hat{\lambda}$ from the subalgebra $\widetilde{U}^0$ of $\widetilde{U}$ generated by the elements $a_i^{\pm 1}, b_i^{\pm 1}$ $(1 \le i \le n)$ to $\mathbb{K}$ given by

$$\hat{\lambda}(a_i) = r^{\langle \epsilon_i, \lambda \rangle} \qquad \text{and} \qquad \hat{\lambda}(b_i) = s^{\langle \epsilon_i, \lambda \rangle}. \tag{3.1}$$

The restriction $\hat{\lambda} : U^0 \to \mathbb{K}$ to the subalgebra $U^0$ of $U$ generated by $\omega_j^{\pm 1}, (\omega_j')^{\pm 1}$ $(1 \le j < n)$ satisfies

$$\hat{\lambda}(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle} \qquad \text{and} \qquad \hat{\lambda}(\omega_j') = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle}. \tag{3.2}$$

Similarly for $U = U_{r,s}(\mathfrak{sl}_n)$, we let $\Lambda_{\mathfrak{sl}} = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_{n-1}$, the weight lattice of $\mathfrak{sl}_n$, where $\varpi_i$ is the fundamental weight

$$\varpi_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{n}\sum_{j=1}^{n} \epsilon_j.$$

If we fix $n$th roots $r^{1/n}$ and $s^{1/n}$ of $r$ and $s$, respectively, then (3.2) can be used to define an algebra homomorphism $\hat{\lambda} : U^0 \to \mathbb{K}$ for any $\lambda \in \Lambda_{\mathfrak{sl}}$.

Let $M$ be a module for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ of dimension $d < \infty$. If $\mathbb{K}$ is algebraically closed (*which will be our assumption throughout the remainder of this work*), then

$$M = \bigoplus_{\chi} M_\chi$$

where each $\chi : \widetilde{U}^0 \to \mathbb{K}$ is an algebra homomorphism, and $M_\chi$ is the generalized eigenspace given by

$$M_\chi = \{m \in M \mid (a_i - \chi(a_i)\,1)^d m = 0 = (b_i - \chi(b_i)\,1)^d m, \quad \text{for all } i\}. \tag{3.3}$$

When $M_\chi \neq 0$ we say that $\chi$ is a *weight* and $M_\chi$ is the corresponding *weight space.* (If $M$ decomposes into genuine eigenspaces relative to $\widetilde{U}^0$ (resp. $U^0$), then we say that $\widetilde{U}^0$ (resp. $U^0$) *acts semisimply on* $M$.)

From relations (R2) and (R3) we deduce that

$$\begin{aligned} e_j M_\chi &\subseteq M_{\chi\cdot\widehat{\alpha_j}} \\ f_j M_\chi &\subseteq M_{\chi\cdot(\widehat{-\alpha_j})} \end{aligned} \tag{3.4}$$

where $\widehat{\alpha_j}$ is as in (3.1), and $\chi \cdot \psi$ is the homomorphism with values $(\chi\cdot\psi)(a_i) = \chi(a_i)\psi(a_i)$ and $(\chi\cdot\psi)(b_i) = \chi(b_i)\psi(b_i)$. In fact, if $(a_i - \chi(a_i)\,1)^k m = 0$, then $(a_i - \chi(a_i) r^{\langle \epsilon_i, \alpha_j\rangle}\,1)^k e_j m = 0$, and similarly for $b_i$ and for $f_j$. This can be used to show that the sum of the eigenspaces is a submodule of $M$, and so if $M$ is simple, this sum must be $M$ itself. Thus, in (3.3), we may replace the power $d$ by 1 whenever $M$ is simple, and $\widetilde{U}^0$ must act semisimply in this case. We also can see from (3.4) that for each simple $M$ there is a homomorphism $\chi$ so that all the weights of $M$ are of the form $\chi \cdot \hat{\zeta}$, where $\zeta \in Q$.

*When all the weights of a module $M$ are of the form $\hat{\lambda}$, where $\lambda \in \Lambda$, then for brevity we say that $M$ has weights in $\Lambda$. Rather than writing $M_{\hat{\lambda}}$ for the weight space, we simplify the notation by writing $M_\lambda$.*

Note then (3.4) can be rewritten as $e_j M_\lambda \subseteq M_{\lambda+\alpha_j}$ and $f_j M_\lambda \subseteq M_{\lambda-\alpha_j}$. Any simple $\widetilde{U}$-module having one weight in $\Lambda$ has all its weights in $\Lambda$.

We would like to argue that when $rs^{-1}$ is not a root of unity, the elements $e_j$ and $f_j$ act nilpotently on any finite-dimensional module. For this we require the following result.

**Proposition 3.5.** *Suppose $\hat{\zeta} = \hat{\eta}$ as functions on $\widetilde{U}^0$ (resp., $U^0$), where $\zeta, \eta \in \Lambda$ (resp., $\Lambda_{\mathfrak{sl}}$). If $rs^{-1}$ is not a root of unity, then $\zeta = \eta$.*

*Proof.* The result for $\widetilde{U}^0$ is a straightforward consequence of (3.1), as $rs^{-1}$ is not a root of 1. We will prove the statement for $U^0$. We may assume $\zeta = \sum_{i=1}^{n-1} \zeta_i \alpha_i$ and $\eta = \sum_{i=1}^{n-1} \eta_i \alpha_i$, where $n\zeta_i, n\eta_i \in \mathbb{Z}$. The condition that $\hat{\zeta} = \hat{\eta}$ gives the equations

$$\begin{aligned}\hat{\zeta}(\omega_i) &= r^{\langle \epsilon_i, \zeta\rangle} s^{\langle \epsilon_{i+1}, \zeta\rangle} = r^{\zeta_i - \zeta_{i-1}} s^{\zeta_{i+1}-\zeta_i} \\ &= \hat{\eta}(\omega_i) = r^{\eta_i - \eta_{i-1}} s^{\eta_{i+1}-\eta_i} \\ \hat{\zeta}(\omega_i') &= r^{\langle \epsilon_{i+1}, \zeta\rangle} s^{\langle \epsilon_i, \zeta\rangle} = r^{\zeta_{i+1}-\zeta_i} s^{\zeta_i - \zeta_{i-1}} \\ &= \hat{\eta}(\omega_i') = r^{\eta_{i+1}-\eta_i} s^{\eta_i - \eta_{i-1}},\end{aligned}$$

where $\zeta_0 = \zeta_n = 0 = \eta_0 = \eta_n$. Letting $\mu_i = \zeta_i - \eta_i$, we may rewrite the above equations as

$$r^{\mu_i - \mu_{i-1}} s^{\mu_{i+1}-\mu_i} = 1 \tag{3.6}$$

$$r^{\mu_{i+1} - \mu_i} s^{\mu_i - \mu_{i-1}} = 1. \tag{3.7}$$

Combining these we have

$$r^{\mu_{i+2}-\mu_{i+1}-\mu_i+\mu_{i-1}} = 1 \tag{3.8}$$

$$s^{\mu_{i+2}-\mu_{i+1}-\mu_i+\mu_{i-1}} = 1 \tag{3.9}$$

for $i = 1, \dots, n-2$. Since we are assuming that $rs^{-1}$ is not a root of unity, not both $r$ and $s$ are roots of unity, so from these relations we see that

$$\mu_{i+2} - \mu_{i+1} - \mu_i + \mu_{i-1} = 0 \tag{3.10}$$

for $i = 1, \dots, n-2$. We claim that the solution to the system of equations given by (3.10) satisfies

$$\mu_{2k} = k\mu_2 \qquad \text{and} \qquad \mu_{2k+1} = k\mu_2 + \mu_1. \tag{3.11}$$

This is true for $\mu_3$ as $\mu_0 = 0$. Moreover, $\mu_4 = \mu_3 + \mu_2 - \mu_1 = 2\mu_2$. An easy induction proves the rest. Now $\mu_n = 0$, and using that fact in (3.10) we have

$$\mu_{n-1} = -\mu_{n-2} + \mu_{n-3}. \tag{3.12}$$

If $n = 2m$ for some $m$, then (3.11) and (3.12) give $\mu_2 = 0$. From (3.6), we have $(rs^{-1})^{\mu_1} = 1$, and because $rs^{-1}$ is not a root of unity, this says $\mu_1 = 0$. The relations in (3.11) then show $\mu_i = \zeta_i - \eta_i = 0$ for all $i$. Hence $\zeta = \eta$ when $n$ is even.

Now if instead $n = 2m+1$, then (3.11) and (3.12) show that $\mu_1 = -m\mu_2$. The equations in (3.6) and (3.7) imply

$$r^{-m\mu_2} s^{(m+1)\mu_2} = 1 \qquad \text{and} \qquad r^{(m+1)\mu_2} s^{-m\mu_2} = 1, \tag{3.13}$$

and hence that $(rs)^{\mu_2} = 1$. Then from (3.13) we see that $s^{(2m+1)\mu_2} = 1 = r^{(2m+1)\mu_2}$. As not both $r$ and $s$ are roots of unity, $\mu_2 = 0$. From this, the desired conclusion $\zeta = \eta$ follows. □

**Corollary 3.14.** *Let $M$ be a finite-dimensional module for $U_{r,s}(\mathfrak{sl}_n)$ or for $U_{r,s}(\mathfrak{gl}_n)$. If $rs^{-1}$ is not a root of unity, then the elements $e_i, f_i$ $(1 \le i < n)$ act nilpotently on $M$.*

*Proof.* Because $M$ is a direct sum of its weight spaces, it suffices to argue that $e_i$ and $f_i$ act nilpotently on each $M_\chi$. As the weights $\widehat{k\alpha_i}$ for $k = 1, 2, \dots$ are distinct by Proposition 3.5, and $e_i^k M_\chi \subseteq M_{\chi \cdot (\widehat{k\alpha_i})}$, it must be that some power of $e_i$ maps $M_\chi$ to 0. A similar argument applies to show that $f_i$ is nilpotent also. □

## §4. $R$-matrix and quantum Casimir operator

Let $\mathcal{O}$ denote the category of modules $M$ for $\widetilde{U} = U_{r,s}(\mathfrak{gl}_n)$ which satisfy the conditions:

- ($\mathcal{O}1$) $\widetilde{U}^0$ acts semisimply on $M$, and the set $\mathrm{wt}(M)$ of weights of $M$ belongs to $\Lambda$: $M = \bigoplus_{\lambda \in \mathrm{wt}(M)} M_\lambda$, where $M_\lambda = \{m \in M \mid a_i.m = r^{\langle \epsilon_i, \lambda \rangle}, \quad b_i.m = s^{\langle \epsilon_i, \lambda \rangle}$ for all $i\}$;
- ($\mathcal{O}2$) $\dim_{\mathbb{K}} M_\lambda < \infty$ for all $\lambda \in \mathrm{wt}(M)$;
- ($\mathcal{O}3$) $\mathrm{wt}(M) \subseteq \bigcup_{\mu \in F} (\mu - Q^+)$ for some finite set $F \subset \Lambda$.

The morphisms in $\mathcal{O}$ are $\widetilde{U}$-module homomorphisms. In defining category $\mathcal{O}$ for the algebra $U = U_{r,s}(\mathfrak{sl}_n)$, we replace $\Lambda$ by the weight lattice $\Lambda_{\mathfrak{sl}}$ of $\mathfrak{sl}_n$.

Category $\mathcal{O}$ is closed under tensor product. For any two modules $M$ and $M'$ in $\mathcal{O}$, we construct a $\widetilde{U}$-module isomorphism $R_{M',M} : M' \otimes M \to M \otimes M'$, by the method used by Jantzen [Ja, Chap. 7] for the quantum groups $U_q(\mathfrak{g})$. These isomorphisms work equally well for $U$-modules.

The map $R_{M',M}$ is the composite of three linear transformations $P$, $\widetilde{f}$, $\Theta$, which we now describe:

(i) $P = P_{M',M} : M' \otimes M \to M \otimes M', \quad P(m' \otimes m) = m \otimes m'.$

(ii) $\widetilde{f} = \widetilde{f}_{M,M'} : M \otimes M' \to M' \otimes M$ is such that $\widetilde{f}(m \otimes m') = f(\lambda, \mu)(m \otimes m')$ when $m \in M_\lambda$ and $m' \in M'_\mu$, where the map $f : \Lambda \times \Lambda \to \mathbb{K}^\#$ is defined as follows. (The map $\widetilde{f}$ will be well-defined on modules in category $\mathcal{O}$ by Proposition 3.5.)

Suppose that $\alpha_n = \epsilon_n$ so that $\alpha_i + \alpha_{i+1} + \cdots + \alpha_n = \epsilon_i$ for $i = 1, \ldots, n$. Let $\Lambda = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$, (the weight lattice of $\mathfrak{gl}_n$). If $\lambda = \sum_{i=1}^n \lambda_i \alpha_i$ is in the weight lattice $\Lambda$, we define

$$
\begin{aligned}
\omega_\lambda &= \omega_1^{\lambda_1} \cdots \omega_{n-1}^{\lambda_{n-1}} a_n^{\lambda_n} \\
\omega'_\lambda &= (\omega'_1)^{\lambda_1} \cdots (\omega'_{n-1})^{\lambda_{n-1}} b_n^{\lambda_n},
\end{aligned}
\tag{4.1}
$$

which agrees with (1.2) in case $\lambda \in Q$. If also $\mu = \sum_{i=1}^n \mu_i \alpha_i$ is in $\Lambda$, we define

$$f(\lambda, \mu) = (\omega'_\mu, \omega_\lambda)^{-1}. \tag{4.2}$$

The values of this bilinear form are given by (2.1) and (2.3). It may be checked that for all $\lambda, \mu, \nu \in \Lambda$ and $1 \le i, j < n$, the following hold:

$$
\begin{aligned}
f(\lambda + \mu, \nu) &= f(\lambda, \nu) f(\mu, \nu) \\
f(\lambda, \mu + \nu) &= f(\lambda, \mu) f(\lambda, \nu) \\
f(\alpha_j, \mu) &= r^{-\langle \epsilon_j, \mu \rangle} s^{-\langle \epsilon_{j+1}, \mu \rangle} \\
f(\lambda, \alpha_i) &= r^{\langle \epsilon_{i+1}, \lambda \rangle} s^{\langle \epsilon_i, \lambda \rangle}.
\end{aligned}
\tag{4.3}
$$

If $\lambda, \mu \in Q$, then $\lambda_n = \mu_n = 0$ and

$$f(\lambda,\mu)=\prod_{i=1}^{n-1}(\hat{\lambda}(\omega_i'))^{\mu_i}=\prod_{i,j=1}^{n-1}(r^{\langle\epsilon_{i+1},\alpha_j\rangle}s^{\langle\epsilon_i,\alpha_j\rangle})^{\mu_i\lambda_j}=\prod_{j=1}^{n-1}(\hat{\mu}(\omega_j))^{-\lambda_j}.$$

For $U=U_{r,s}(\mathfrak{sl}_n)$, where $\Lambda$ is replaced by $\Lambda_{\mathfrak{sl}}=(1/n)Q$, this may be taken as the definition of $f$ once we have fixed roots $r^{1/n^2}$ and $s^{1/n^2}$ of $r$ and $s$ in $\mathbb{K}$.

We will need to compute

$$f(\epsilon_i,\epsilon_j)=f(\alpha_i+\cdots+\alpha_n,\alpha_j+\cdots+\alpha_n).$$

Supposing first that $1\le i,j<n$, by (4.3) we have

$$\begin{aligned}
f(\epsilon_i,\epsilon_j)&=(\omega'_{\alpha_j+\cdots+\alpha_n},a_n)^{-1}\prod_{k=i}^{n-1}f(\alpha_k,\alpha_j+\cdots+\alpha_n)\\
&=r\prod_{k=i}^{n-1}r^{-\langle\epsilon_k,\alpha_j+\cdots+\alpha_n\rangle}s^{-\langle\epsilon_{k+1},\alpha_j+\cdots+\alpha_n\rangle}\\
&=r\prod_{k=i}^{n-1}r^{-\langle\epsilon_k,\epsilon_j\rangle}s^{-\langle\epsilon_{k+1},\epsilon_j\rangle}\\
&=\begin{cases}s^{-1} & \text{if } i<j\\ 1 & \text{if } i=j\\ r & \text{if } i>j.\end{cases}
\end{aligned}$$

Now if $1\le i<n$, then

$$f(\epsilon_i,\epsilon_n)=f(\alpha_i+\cdots+\alpha_n,\alpha_n)=(b_n,\omega_i)^{-1}\cdots(b_n,\omega_{n-1})^{-1}(b_n,a_n)^{-1}=s^{-1}$$
$$f(\epsilon_n,\epsilon_i)=f(\alpha_n,\alpha_i+\cdots+\alpha_n)=(\omega_i,a_n)^{-1}\cdots(\omega_{n-1},a_n)^{-1}(b_n,a_n)^{-1}=r \qquad \text{and}$$
$$f(\epsilon_n,\epsilon_n)=(b_n,a_n)=1.$$

As a result, the following holds:

**Lemma 4.4.** *For all $1\le i,j\le n$, we have*

$$f(\epsilon_i,\epsilon_j)=\begin{cases}s^{-1} & \textit{if } i<j\\ 1 & \textit{if } i=j\\ r & \textit{if } i>j.\end{cases}$$

(iii) Now we turn our attention to the construction of our final mapping. Observe it is a consequence of (R2) and (R3) that the subalgebra $U^+$ of $\widetilde{U}$ (or of $U=U_{r,s}(\mathfrak{sl}_n)$) generated by 1 and $e_i$ $(1\le i<n)$ has the following decomposition

$$U^+=\bigoplus_{\zeta\in Q^+}U^+_\zeta$$

where

$$U_\zeta^+ = \{z \in U^+ \mid a_i z = r^{\langle \epsilon_i, \zeta\rangle} z a_i, \quad b_i z = s^{\langle \epsilon_i, \zeta\rangle} z b_i, \quad (1 \le i < n)\}.$$

The weight space $U_\zeta^+$ is spanned by all the monomials $e_{i_1} \ \cdots \ e_{i_\ell}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_\ell} = \zeta$.

Similarly, the subalgebra $U^-$ generated by 1 and the $f_i$'s has a decomposition $U^- = \bigoplus_{\zeta \in Q^+} U_{-\zeta}^-$, and the spaces $U_\zeta^+$ and $U_{-\zeta}^-$ are nondegenerately paired.

Since $\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i$, we have

$$\Delta(x) \in \sum_{0 \le \nu \le \zeta} U_{\zeta-\nu}^+ \omega_\nu \otimes U_\nu^+$$

for all $x \in U_\zeta^+$. (In writing this, we are using the standard partial order on $Q$ in which $\nu \le \zeta$ if $\zeta - \nu \in Q^+$.) For each $i$, there are elements $p_i(x)$ and $p_i'(x) \in U_{\zeta-\alpha_i}^+$ such that

$$\begin{aligned}\Delta(x) &= x \otimes 1 + \sum_{i=1}^{n-1} p_i(x)\omega_i \otimes e_i + \text{ the rest} \\ \Delta(x) &= \omega_\zeta \otimes x + \sum_{i=1}^{n-1} e_i \omega_{\zeta-\alpha_i} \otimes p_i'(x) + \text{ the rest,}\end{aligned} \tag{4.5}$$

where in each case "the rest" refers to terms involving products of more than one $e_j$ in the second factor (respectively, in the first factor). (Compare the expressions in Lemma 5.2 below.)

**Lemma 4.6.** *(Compare [Ja, Lemma 6.14, 6.17].) For all $x \in U_\zeta^+$, $x' \in U_{\zeta'}^+$, and $y \in U^-$, the following hold:*

(i) $p_i(xx') = r^{\langle \epsilon_i, \zeta'\rangle} s^{\langle \epsilon_{i+1}, \zeta'\rangle} p_i(x)x' + x p_i(x')$.

(ii) $p_i'(xx') = p_i'(x)x' + r^{-\langle \epsilon_{i+1}, \zeta\rangle} s^{-\langle \epsilon_i, \zeta\rangle} x p_i'(x)$.

(iii) $(f_i y, x) = (f_i, e_i)(y, p_i'(x)) = (s-r)^{-1}(y, p_i'(x))$.

(iv) $(y f_i, x) = (f_i, e_i)(y, p_i(x)) = (s-r)^{-1}(y, p_i(x))$.

(v) $f_i x - x f_i = (s-r)^{-1}\big(p_i(x)\omega_i - \omega_i' p_i'(x)\big)$.

*Proof.* The proofs of (i) and (ii) amount to equating the expressions for $\Delta(xx') = \Delta(x)\Delta(x')$. We demonstrate the second:

$$\Delta(xx') = \omega_{\zeta+\zeta'} \otimes xx' + \sum_{i=1}^{n-1} e_i\omega_{\zeta+\zeta'-\alpha_i} \otimes p_i'(xx') + \text{ the rest}$$

$$\begin{aligned}
\Delta(x)\Delta(x') &= \Big(\omega_\zeta \otimes x + \sum_{i=1}^{n-1} e_i\omega_{\zeta-\alpha_i} \otimes p_i'(x) + \text{ the rest}\Big)\times \\
&\qquad \Big(\omega_{\zeta'} \otimes x' + \sum_{i=1}^{n-1} e_i\omega_{\zeta'-\alpha_i} \otimes p_i'(x') + \text{ the rest}\Big) \\
&= \omega_{\zeta+\zeta'} \otimes xx' \\
&\quad + \sum_{i=1}^{n-1} \Big(e_i\omega_{\zeta+\zeta'-\alpha_i} \otimes p_i'(x)x' + \omega_\zeta e_i\omega_{\zeta'-\alpha_i} \otimes xp_i'(x')\Big) + \text{ the rest} \\
&= \omega_{\zeta+\zeta'} \otimes xx' \\
&\quad + \sum_{i=1}^{n-1} e_i\omega_{\zeta+\zeta'-\alpha_i} \otimes \Big(p_i'(x)x' + r^{-\langle\epsilon_{i+1},\zeta\rangle}s^{-\langle\epsilon_i,\zeta\rangle}xp_i'(x')\Big) + \text{ the rest},
\end{aligned}$$

(using Lemma 1.3). Equating terms gives (ii).

Now for (iii) we argue as follows using the second equation of (4.5):

$$\begin{aligned}
(f_iy, x) &= \sum (f_i, x_{(1)})(y, x_{(2)}) \\
&= (f_i, e_i\omega_{\zeta-\alpha_i})(y, p_i'(x)) \\
&= (1 \otimes f_i + f_i \otimes \omega_i',\ \omega_{\zeta-\alpha_i} \otimes e_i)(y, p_i'(x)) \\
&= (f_i, e_i)(y, p_i'(x)).
\end{aligned}$$

Let's begin the proof of (v) by observing that it is true if $x = 1 \in U_0^+$ since $p_i(1) = p_i'(1) = 0$ for all $i$. The relation in (v) also holds for $x = e_j$, because $\Delta(e_j) = e_j \otimes 1 + w_j \otimes e_j$ implies that $p_i(e_j) = \delta_{i,j} = p_i'(e_j)$. We suppose the result is true for $x \in U_\zeta^+$ and $x' \in U_{\zeta'}^+$ and prove it for $xx' \in U_{\zeta+\zeta'}^+$. Now

$$\begin{aligned}
f_ixx' - xx'f_i &= (f_ix - xf_i)x' - x(f_ix' - x'f_i) \\
&= \frac{1}{s-r}\Big(\big(p_i(x)\omega_i - \omega_i'p_i'(x)\big)x' + x\big(p_i(x')\omega_i - \omega_i'p_i'(x')\big)\Big) \\
&= \frac{1}{s-r}\Bigg(\Big(r^{\langle\epsilon_i,\zeta'\rangle}s^{\langle\epsilon_{i+1},\zeta'\rangle}p_i(x)x' + x\big(p_i(x')\big)\Big)\omega_i \\
&\qquad - \omega_i'\Big(p_i'(x)x' + r^{-\langle\epsilon_{i+1},\zeta\rangle}s^{-\langle\epsilon_i,\zeta\rangle}xp_i'(x')\Big)\Bigg) \\
&= \frac{1}{s-r}\left(p_i(xx')\omega_i - \omega_i'p(xx')\right). \quad \square
\end{aligned}$$

Assuming for $y \in U_{-\zeta}^-$ that $p_i(y)$ and $p_i'(y)$ are defined by

$$
\begin{aligned}
\Delta(y) &= y \otimes \omega'_\zeta + \sum_{i=1}^{n} p_i(y) \otimes f_i \omega'_{\zeta-\alpha_i} + \text{ the rest} \\
\Delta(y) &= 1 \otimes y + \sum_{i=1}^{n} f_i \otimes p'_i(y)\omega'_i + \text{ the rest,}
\end{aligned}
\tag{4.7}
$$

the same type of argument produces this analogue of Lemma 4.6:

**Lemma 4.8.** *For all $y \in U^-_{-\zeta}$, $y' \in U^-_{-\zeta'}$, and $x \in U^+$, the following hold:*

(i) $p_i(yy') = p_i(y)y' + r^{\langle \epsilon_i, \zeta' \rangle} s^{\langle \epsilon_{i+1}, \zeta' \rangle} y p_i(y')$.

(ii) $p'_i(yy') = r^{-\langle \epsilon_{i+1}, \zeta' \rangle} s^{-\langle \epsilon_i, \zeta' \rangle} p'_i(y)y' + y p'_i(y)$.

(iii) $(y, e_i x) = (f_i, e_i)(p_i(y), x) = (s-r)^{-1}(p_i(y), (x))$.

(iv) $(y, x e_i) = (f_i, e_i)(p'_i(y), x) = (s-r)^{-1}(p'_i(y), x)$.

(v) $e_i y - y e_i = (r-s)^{-1}\big(w_i p_i(y) - p'_i(y)\omega'_i\big)$.

Because the spaces $U^+_\zeta$ and $U^-_{-\zeta}$ are nondegenerately paired, we may select a basis $\{u^\zeta_k\}_{k=1}^{d_\zeta}$, ($d_\zeta = \dim_{\mathbb{K}} U^+_\zeta$), for $U^+_\zeta$ and a dual basis $\{v^\zeta_k\}_{k=1}^{d_\zeta}$ for $U^-_{-\zeta}$. Then for each $x \in U^+_\zeta$ and $y \in U^-_{-\zeta}$ we have

$$
x = \sum_{k=1}^{d_\zeta} (v^\zeta_k, x) u^\zeta_k \quad \text{and} \quad y = \sum_{k=1}^{d_\zeta} (y, u^\zeta_k) v^\zeta_k. \tag{4.9}
$$

For $\zeta \in Q^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i$, we define

$$
\Theta_\zeta = \sum_{k=1}^{d_\zeta} v^\zeta_k \otimes u^\zeta_k.
$$

Set $\Theta_\zeta = 0$ if $\zeta \notin Q^+$.

**Lemma 4.10.** (i) $(a_i \otimes a_i)\Theta_\zeta = \Theta_\zeta(a_i \otimes a_i)$;

(ii) $(b_i \otimes b_i)\Theta_\zeta = \Theta_\zeta(b_i \otimes b_i)$;

(iii) $(e_i \otimes 1)\Theta_\zeta + (\omega_i \otimes e_i)\Theta_{\zeta-\alpha_i} = \Theta_\zeta(e_i \otimes 1) + \Theta_{\zeta-\alpha_i}(\omega'_i \otimes e_i)$;

(iv) $(1 \otimes f_i)\Theta_\zeta + (f_i \otimes \omega'_i)\Theta_{\zeta-\alpha_i} = \Theta_\zeta(1 \otimes f_i) + \Theta_{\zeta-\alpha_i}(f_i \otimes \omega_i)$;

*for $1 \leq i < n$.*

*Proof.* The first two are easy to check. We demonstrate (iv) and leave (iii) as an exercise. The calculation below will use (iii)-(v) of Lemma 4.6 and (4.9).

$$
\begin{aligned}
&(1 \otimes f_i)\Theta_\zeta - \Theta_\zeta(1 \otimes f_i) \\
&\qquad = \sum_k v_k^\zeta \otimes (f_i u_k^\zeta - u_k^\zeta f_i) \\
&\qquad = \frac{1}{s-r}\sum_k v_k^\zeta \otimes \Big(p_i(u_k^\zeta)\omega_i - \omega_i' p_i'(u_k^\zeta)\Big) \\
&\qquad = \frac{1}{s-r}\sum_k v_k^\zeta \otimes \left(\sum_j \big(v_j^{\zeta-\alpha_i},\, p_i(u_k^\zeta)\big) u_j^{\zeta-\alpha_j}\right)\omega_i \\
&\qquad\qquad - \frac{1}{s-r}\sum_k v_k^\zeta \otimes \omega_i' \left(\sum_j \big(v_j^{\zeta-\alpha_i},\, p_i'(u_k^\zeta)\big) u_j^{\zeta-\alpha_j}\right) \\
&\qquad = \sum_k v_k^\zeta \otimes \left(\sum_j (v_j^{\zeta-\alpha_i} f_i,\, u_k^\zeta) u_j^{\zeta-\alpha_i}\right)\omega_i \\
&\qquad\qquad - \sum_k v_k^\zeta \otimes \omega_i' \left(\sum_j \big(f_i v_j^{\zeta-\alpha_i},\, u_k^\zeta\big) u_j^{\zeta-\alpha_j}\right) \\
&\qquad = \sum_j \left(\sum_k (v_j^{\zeta-\alpha_i} f_i,\, u_k^\zeta) v_k^\zeta\right) \otimes u_j^{\zeta-\alpha_i}\omega_i \\
&\qquad\qquad - \sum_j \left(\sum_k (f_i v_j^{\zeta-\alpha_i},\, u_k^\zeta) v_k^\zeta\right) \otimes \omega_i' u_j^{\zeta-\alpha_j} \\
&\qquad\qquad = \sum_j v_j^{\zeta-\alpha_i} f_i \otimes u_j^{\zeta-\alpha_i}\omega_i \; - \; \sum_j f_i v_j^{\zeta-\alpha_i} \otimes \omega_i' u_j^{\zeta-\alpha_j} \\
&\qquad\qquad = \Theta_{\zeta-\alpha_i}(f_i \otimes \omega_i) - (f_i \otimes \omega_i')\Theta_{\zeta-\alpha_i}.
\end{aligned}
$$

Consequently, (iv) holds. □

We now define

$$\Theta = \sum_{\zeta \in Q^+} \Theta_\zeta$$

(which we can think of as living in the completion of $\widetilde{U} \otimes \widetilde{U}$ where infinite sums are allowed). For fixed $\widetilde{U}$-modules $M$ and $M'$ in $\mathcal{O}$, we may apply $\Theta$ to their tensor product:

$$\Theta = \Theta_{M,M'} : M \otimes M' \to M \otimes M'.$$

Note that $\Theta_\zeta : M_\lambda \otimes M'_\mu \to M_{\lambda-\zeta} \otimes M'_{\mu+\zeta}$ for all $\lambda, \mu \in \Lambda$, and because of condition ($\mathcal{O}3$), there are only finitely many $\zeta \in Q^+$ such that $M'_{\mu+\zeta} \neq 0$. Hence, this is a well-defined linear transformation on $M \otimes M'$.

We can choose countable bases of weight vectors for both $M$ and $M'$ and their tensor products as a basis for $M \otimes M'$. Then ordering this basis appropriately shows that each $\Theta_\zeta$ with $\zeta > 0$ has a strictly upper triangular matrix. Because $\Theta_0 = 1 \otimes 1$ acts as the identity transformation on $M \otimes M'$, $\Theta_{M,M'}$ is an invertible transformation.

**Theorem 4.11.** *Let $M$ and $M'$ be modules in $\mathcal{O}$. Then the map*

$$\Theta \circ \widetilde{f} \circ P : M' \otimes M \to M \otimes M'$$

*is an isomorphism of $\widetilde{U}$-modules.*

*Proof.* Since each of the maps is invertible, once we show that $\Theta \circ \widetilde{f} \circ P$ is a $\widetilde{U}$-module homomorphism, we will be done. The proof amounts to verifying that

$$\Delta(a)(\Theta \circ \widetilde{f} \circ P)(m' \otimes m) = (\Theta \circ \widetilde{f} \circ P)\Delta(a)(m' \otimes m) \tag{4.12}$$

holds for all $a \in \widetilde{U}$, $m \in M_\lambda$ and $m' \in M'_\mu$. Because $\Delta$ is an algebra homomorphism, it suffices to check (4.12) on the generators $e_i, f_i, a_i, b_i$. We will present the computation just for $a = e_i$. In this case, the right side of (4.12) becomes

$$\begin{aligned}
(\Theta \circ \widetilde{f} \circ P)\Delta(e_i)(m' \otimes m) &= (\Theta \circ \widetilde{f} \circ P)(e_i m' \otimes m + \omega_i m' \otimes e_i m) \\
&= (\Theta \circ \widetilde{f})(m \otimes e_i m' + e_i m \otimes \omega_i m') \\
&= f(\lambda, \mu + \alpha_i)\Theta(m \otimes e_i m') + f(\lambda + \alpha_i, \mu)\Theta(e_i m \otimes \omega_i m') \\
&= f(\lambda, \mu + \alpha_i)\Big(\sum_\zeta \Theta_\zeta\Big)(1 \otimes e_i)(m \otimes m') \\
&\qquad + f(\lambda + \alpha_i, \mu)\Big(\sum_\zeta \Theta_\zeta\Big)(e_i \otimes \omega_i)(m \otimes m')
\end{aligned}$$

Now let's compute the left side using (iii) of Lemma 4.10:

$$\begin{aligned}
\Delta(e_i)(\Theta \circ \widetilde{f} \circ P)(m' \otimes m) &= f(\lambda, \mu)\Delta(e_i)\Theta(m \otimes m') \\
&= f(\lambda, \mu)(e_i \otimes 1)\Big(\sum_\zeta \Theta_\zeta\Big)(m \otimes m') \\
&\qquad + f(\lambda, \mu)(\omega_i \otimes e_i)\Big(\sum_\zeta \Theta_{\zeta - \alpha_i}\Big)(m \otimes m') \\
&= f(\lambda, \mu)\Big(\sum_\zeta \Theta_\zeta\Big)(e_i m \otimes m') \\
&\qquad + f(\lambda, \mu)\Big(\sum_\zeta \Theta_{\zeta - \alpha_i}\Big)(\omega_i' m \otimes e_i m') \\
&= f(\lambda, \mu) r^{-\langle \epsilon_i, \mu \rangle} s^{-\langle \epsilon_{i+1}, \mu \rangle}\Big(\sum_\zeta \Theta_\zeta\Big)(e_i m \otimes \omega_i m') \\
&\qquad + f(\lambda, \mu) r^{\langle \epsilon_{i+1}, \lambda \rangle} s^{\langle \epsilon_i, \lambda \rangle}\Big(\sum_\zeta \Theta_{\zeta - \alpha_i}\Big)(m \otimes e_i m').
\end{aligned}$$

This expression can be seen to equal the previous one by (4.3). (Note in this computation we have made liberal use of the fact that $\sum_\zeta \Theta_\zeta = \sum_\zeta \Theta_{\zeta-\alpha_i}$ because of our convention that $\Theta_\eta = 0$ whenever $\eta \notin Q^+$.) $\square$

**Quantum Casimir operator.**

In this subsection, we construct a quantum Casimir operator which commutes with the action of $\widetilde{U}$ on any $\widetilde{U}$-module in $\mathcal{O}$.

Again letting $\{u_k^\zeta\}_{k=1}^{d_\zeta}$ and $\{v_k^\zeta\}_{k=1}^{d_\zeta}$ be dual bases for $U_\zeta^+$ and $U_{-\zeta}^-$ respectively ($d_\zeta = \dim_{\mathbb{K}} U_\zeta^+$), define

$$\Omega = \sum_{\zeta\in Q^+} \sum_{k=1}^{d_\zeta} S(v_k^\zeta) u_k^\zeta, \tag{4.15}$$

where $S$ denotes the antipode. Note that by (3.4), $\Omega$ preserves the weight spaces of any $M \in \mathcal{O}$.

**Lemma 4.16.** *For $M \in \mathcal{O}$, assume $m \in M_\lambda$. Then*

(i) $\Omega e_i.m = (rs^{-1})^{-\langle\alpha_i,\lambda+\alpha_i\rangle} e_i \Omega.m,$

(ii) $\Omega f_i.m = (rs^{-1})^{\langle\alpha_i,\lambda\rangle} f_i \Omega.m.$

*Proof.* Apply $\mathfrak{m} \circ (S \otimes 1)$ to Lemma 4.10 (iii), where $\mathfrak{m}$ is the multiplication map. As $S$ is an algebra anti-automorphism, this yields

$$\begin{aligned} -\sum_{k=1}^{d_\zeta} S(v_k^\zeta)\omega_i^{-1} e_i u_k^\zeta + \sum_{k=1}^{d_{\zeta-\alpha_i}} S(v_k^{\zeta-\alpha_i})\omega_i^{-1} e_i u_k^{\zeta-\alpha_i} \\ = -\sum_{k=1}^{d_\zeta} \omega_i^{-1} e_i S(v_k^\zeta) u_k^\zeta + \sum_{k=1}^{d_{\zeta-\alpha_i}} (\omega_i')^{-1} S(v_k^{\zeta-\alpha_i}) u_k^{\zeta-\alpha_i} e_i. \end{aligned}$$

Now act on $m \in M_\lambda$ with the result, and sum over all $\zeta \in Q^+$. The two sums on the left side cancel, while the right side produces

$$\omega_i^{-1} e_i \Omega.m = (\omega_i')^{-1} \Omega e_i.m.$$

By definition, $\Omega$ preserves weight spaces, and so $e_i\Omega.m,\ \Omega e_i.m \in M_{\lambda+\alpha_i}$. Therefore we have

$$r^{-\langle\epsilon_i,\lambda+\alpha_i\rangle} s^{-\langle\epsilon_{i+1},\lambda+\alpha_i\rangle} e_i \Omega.m = r^{-\langle\epsilon_{i+1},\lambda+\alpha_i\rangle} s^{-\langle\epsilon_i,\lambda+\alpha_i\rangle} \Omega e_i.m,$$

which is equivalent to (i).

The proof of (ii) is virtually identical, and uses Lemma 4.10 (iv). $\square$

Now we introduce a certain function $g : \Lambda \to \mathbb{K}^\#$ on the weight lattice $\Lambda = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$ of $\mathfrak{gl}_n$. If $\rho$ denotes the half sum of the positive roots, then $2\rho = \sum_{j=1}^{n}(n+1-2j)\epsilon_j \in \Lambda$. We fix roots $r^{1/2}$ and $s^{1/2}$ and for $\lambda \in \Lambda$, set

$$g(\lambda) = (rs^{-1})^{\frac{1}{2}\langle \lambda+2\rho,\lambda\rangle}. \tag{4.17}$$

Because $\langle \rho, \alpha_i\rangle = 1$ for all $i = 1, \dots, n-1$, it is straightforward to verify that

$$g(\lambda+\alpha_i) = (rs^{-1})^{\langle \alpha_i, \lambda+\alpha_i\rangle} g(\lambda) \tag{4.18}$$

for all $\lambda \in \Lambda,\ i \in \{1, \dots, n-1\}$.

For $M \in \mathcal{O}$, define the linear operator $\Xi : M \to M$ by

$$\Xi(m) = g(\lambda)m \tag{4.19}$$

for all $m \in M_\lambda,\ \lambda \in \Lambda$. Then $\Xi$ is well-defined by Proposition 3.5. (It is necessary to first fix roots $r^{1/2n}$ and $s^{1/2n}$ of $r$ and $s$ in $\mathbb{K}$ for $U = U_{r,s}(\mathfrak{sl}_n)$.)

**Theorem 4.20.** *The operator $\Omega\Xi : M \to M$ commutes with the action of $\widetilde{U}$ on any module $M \in \mathcal{O}$.*

*Proof.* As $\Omega\Xi$ preserves the weight spaces of $M$, it commutes with the action of $a_i, b_i$ $(1 \le i \le n)$. It remains to show that $\Omega\Xi$ commutes with $e_i, f_i$ $(1 \le i < n)$. Let $m \in M_\lambda$. By Lemma 4.16 (i) and (4.18), we have

$$\begin{aligned}\Omega\Xi(e_i.m) &= g(\lambda+\alpha_i)\Omega e_i.m \\ &= (rs^{-1})^{\langle \alpha_i,\lambda+\alpha_i\rangle} g(\lambda)\Omega e_i.m \\ &= g(\lambda)e_i\Omega.m = e_i\Omega\Xi(m).\end{aligned}$$

The calculation for $f_i$ is similar. □

## §5. Yang-Baxter equation and hexagon identities

For pairs $M$, $M'$ of $\widetilde{U}$-modules in category $\mathcal{O}$, we will show first that the maps $R_{M,M'} = \Theta \circ \widetilde{f} \circ P : M \otimes M' \to M' \otimes M$ satisfy the quantum Yang-Baxter equation. That is, given three $\widetilde{U}$-modules $M$, $M'$, $M''$ in $\mathcal{O}$, we have $R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}$ as maps from $M \otimes M' \otimes M''$ to $M'' \otimes M' \otimes M$. This abbreviated notation is standard, for example $R_{12}$ is an application of $R_{M,M'}$ to the first two of three factors and the identity map on the third factor.

We will need the following inner product relation. If $x \in U_\gamma^+$, $y \in U_{-\gamma}^-$, and $\zeta, \eta \in Q$, then

$$(y\omega_\zeta', x\omega_\eta) = (y,x)(\omega_\zeta', \omega_\eta). \tag{5.1}$$

To derive this, we apply (4.5) and (4.7), keeping in mind that we need to take the *opposite* coproduct in the first position (equivalently, reverse the order of the factors in the second position):

$$\begin{aligned}(y\omega'_\zeta, x\omega_\eta) &= (y \otimes \omega'_\zeta, \Delta(x)\Delta(\omega_\eta)) \\ &= (y, x\omega_\eta)(\omega'_\zeta, \omega_\eta) \\ &= (\Delta(y), \omega_\eta \otimes x)(\omega'_\zeta, \omega_\eta) \\ &= (y, x)(\omega'_\zeta, \omega_\eta).\end{aligned}$$

**Lemma 5.2.** *Let $x \in U^+_\gamma$ and $y \in U^-_{-\gamma}$. Then*

(i) $\Delta(x) = \sum_{0\le\zeta\le\gamma} \sum_{i,j} (v_i^{\gamma-\zeta} v_j^\zeta, x) u_i^{\gamma-\zeta}\omega_\zeta \otimes u_j^\zeta$,

(ii) $\Delta(y) = \sum_{0\le\zeta\le\gamma} \sum_{i,j} (y, u_i^{\gamma-\zeta} u_j^\zeta) v_j^\zeta \otimes v_i^{\gamma-\zeta}\omega'_\zeta$.

*Proof.* As $x \in U^+_\gamma$, we have $\Delta(x) \in \sum_{0\le\zeta\le\gamma} U^+_{\gamma-\zeta}\omega_\zeta \otimes U^+_\zeta$. Let $c^\zeta_{ij} \in \mathbb{K}$ be such that

$$\Delta(x) = \sum_{\zeta,i,j} c^\zeta_{ij} u_i^{\gamma-\zeta}\omega_\zeta \otimes u_j^\zeta.$$

Then for all $k, \ell$, and $\nu$, we see from (5.1) that

$$\begin{aligned}(v_k^{\gamma-\nu} v_\ell^\nu, x) &= (v_k^{\gamma-\nu} \otimes v_\ell^\nu, \Delta(x)) \\ &= \sum_{\zeta,i,j} c^\zeta_{i,j} (v_k^{\gamma-\nu}, u_i^{\gamma-\zeta}\omega_\zeta)(v_\ell^\nu, u_j^\zeta) = c^\nu_{k\ell},\end{aligned}$$

which proves (i). The argument for (ii) is similar. □

Letting $\Theta^{\rm op} = \sum_{\gamma\in Q^+} \sum_i u_i^\gamma \otimes v_i^\gamma$, $\Theta_{12} = \sum_{\gamma\in Q^+} \sum_i v_i^\gamma \otimes u_i^\gamma \otimes 1$, $\Theta^f_{ij} = \Theta_{ij} \circ \widetilde{f}_{ij}$, and defining the other expressions in a like manner, we have the following identities for operators on $M \otimes M' \otimes M''$.

**Lemma 5.3.** (i) $(\Delta \otimes 1)(\Theta^{\rm op}) \circ \widetilde{f}_{31} \circ \widetilde{f}_{32} = \Theta^f_{31} \circ \Theta^f_{32}$.

(ii) $\widetilde{f}_{31} \circ \widetilde{f}_{32} \circ \Theta_{12} = \Theta_{12} \circ \widetilde{f}_{31} \circ \widetilde{f}_{32}$.

*Proof.* Let $m \in M_\lambda$, $m' \in M'_\mu$, and $m'' \in M''_\nu$. Then by Lemma 5.2(i) and (4.9), the left side of (i) applied to $m \otimes m' \otimes m''$ is

$$\begin{aligned}(\Delta \otimes 1)(\Theta^{\rm op}) \circ \widetilde{f}_{31} \circ \widetilde{f}_{32} &= f(\nu,\mu)f(\nu,\lambda)(\Delta \otimes 1)\big(\sum_{\gamma,k} u_k^\gamma \otimes v_k^\gamma\big) \\ &= f(\nu,\mu)f(\nu,\lambda) \sum_{\gamma,k}\sum_{\zeta,i,j} (v_i^{\gamma-\zeta} v_j^\zeta, u_k^\gamma) u_i^{\gamma-\zeta}\omega_\zeta \otimes u_j^\zeta \otimes v_k^\gamma \\ &= f(\nu,\mu)f(\nu,\lambda) \sum_{\gamma,\zeta,i,j} u_i^{\gamma-\zeta}\omega_\zeta \otimes u_j^\zeta \otimes \big(\sum_k \big(v_i^{\gamma-\zeta} v_j^\zeta, u_k^\gamma\big) v_k^\gamma\big) \\ &= f(\nu,\mu)f(\nu,\lambda) \sum_{\gamma,\zeta,i,j} u_i^{\gamma-\zeta}\omega_\zeta \otimes u_j^\zeta \otimes v_i^{\gamma-\zeta} v_j^\zeta.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Theta^f_{31}\circ\Theta^f_{32}(m\otimes m'\otimes m'') &= f(\nu,\mu)\sum_{\eta,\zeta,i,j} f(\nu-\zeta,\lambda)u_i^\eta m\otimes u_j^\zeta m'\otimes v_i^\eta v_j^\zeta m''\\ &= f(\nu,\mu)f(\nu,\lambda)\sum_{\eta,\zeta,i,j} f(-\zeta,\lambda)u_i^\eta\otimes u_j^\zeta\otimes v_i^\eta v_j^\zeta(m\otimes m'\otimes m'').\end{aligned}$$

Changing variables in the first expression above to $\eta=\gamma-\zeta$, and noticing that $\omega_\zeta.m = f(-\zeta,\lambda)m$, we obtain the second expression, proving (i).

Identity (ii) results from a simple calculation using (4.3). □

We are now ready to verify the quantum Yang-Baxter equation.

**Theorem 5.4.** *(Compare [Ja, §7.6].) $R_{12}\circ R_{23}\circ R_{12} = R_{23}\circ R_{12}\circ R_{23}$ as maps from $M\otimes M'\otimes M''$ to $M''\otimes M'\otimes M$.*

*Proof.* Note that $P_\sigma\circ\Theta^f_{ij} = \Theta^f_{\sigma(i)\sigma(j)}\circ P_\sigma$ for all permutations $\sigma$, and that the $\widetilde{f}_{ij}$ commute with one another. Applying Lemma 5.3 and Theorem 4.11, we have

$$\begin{aligned}R_{12}\circ R_{23}\circ R_{12} &= P_{12}\circ P_{23}\circ\Theta^f_{31}\circ\Theta^f_{32}\circ R_{12}\\ &= P_{12}\circ P_{23}\circ(\Delta\otimes 1)(\Theta^{\rm op})\circ\widetilde{f}_{31}\circ\widetilde{f}_{32}\circ\Theta_{12}\circ\widetilde{f}_{12}\circ P_{12}\\ &= P_{12}\circ P_{23}\circ(\Delta\otimes 1)(\Theta^{\rm op})\circ\Theta_{12}\circ\widetilde{f}_{31}\circ\widetilde{f}_{32}\circ\widetilde{f}_{12}\circ P_{12}\\ &= P_{12}\circ P_{23}\circ(\Delta\otimes 1)(\Theta^{\rm op})\circ\Theta_{12}\circ\widetilde{f}_{12}\circ P_{12}\circ\widetilde{f}_{32}\circ\widetilde{f}_{31}\\ &= P_{12}\circ P_{23}\circ(\Delta\otimes 1)(\Theta^{\rm op})\circ R_{12}\circ\widetilde{f}_{32}\circ\widetilde{f}_{31}\\ &= P_{12}\circ P_{23}\circ R_{12}\circ(\Delta\otimes 1)(\Theta^{\rm op})\circ\widetilde{f}_{32}\circ\widetilde{f}_{31}\\ &= P_{12}\circ P_{23}\circ\Theta^f_{12}\circ P_{12}\circ\Theta^f_{31}\circ\Theta^f_{32}\\ &= \Theta^f_{23}\circ P_{12}\circ P_{23}\circ P_{12}\circ\Theta^f_{31}\circ\Theta^f_{32}\\ &= \Theta^f_{23}\circ P_{23}\circ P_{12}\circ P_{23}\circ\Theta^f_{31}\circ\Theta^f_{32}\\ &= \Theta^f_{23}\circ P_{23}\circ\Theta^f_{12}\circ P_{12}\circ\Theta^f_{23}\circ P_{23}\\ &= R_{23}\circ R_{12}\circ R_{23}.\quad\square\end{aligned}$$

Next we will verify the hexagon identities. For this we require two additional lemmas regarding operators on $M\otimes M'\otimes M''$.

**Lemma 5.5.** $(\Delta\otimes 1)(\Theta_\gamma) = \sum_{0\le\zeta\le\gamma}(\Theta_{\gamma-\zeta})_{23}(\Theta_\zeta)_{13}(1\otimes\omega'_\zeta\otimes 1)$, *and* $(1\otimes\Delta)(\Theta_\gamma) = \sum_{0\le\zeta\le\gamma}(\Theta_{\gamma-\zeta})_{12}(\Theta_\zeta)_{13}(1\otimes\omega_\zeta\otimes 1)$.

*Proof.* By the definition of $\Theta_\gamma$, Lemma 5.2(ii), and (4.9), we have

$$\begin{aligned}(\Delta\otimes 1)(\Theta_\gamma) &= \sum_k \Delta(v_k^\gamma)\otimes u_k^\gamma\\ &= \sum_k\sum_{\zeta,i,j}(v_k^\gamma, u_i^{\gamma-\zeta}u_j^\zeta)v_j^\zeta\otimes v_i^{\gamma-\zeta}\omega'_\zeta\otimes u_k^\gamma\\ &= \sum_{\zeta,i,j} v_j^\zeta\otimes v_i^{\gamma-\zeta}\omega'_\zeta\otimes(\sum_k (v_k^\gamma, u_i^{\gamma-\zeta}u_j^\zeta)u_k^\gamma)\\ &= \sum_{\zeta,i,j} v_j^\zeta\otimes v_i^{\gamma-\zeta}\omega'_\zeta\otimes u_i^{\gamma-\zeta}u_j^\zeta\\ &= \sum_{0\le\zeta\le\gamma}(\Theta_{\gamma-\zeta})_{23}(\Theta_\zeta)_{13}(1\otimes\omega'_\zeta\otimes 1).\end{aligned}$$

The second identity may be checked in just the same way. □

**Lemma 5.6.** *$\widetilde{f}_{12}\circ(\Theta_\eta)_{13} = (\Theta_\eta)_{13}\circ(1\otimes\omega_\eta\otimes 1)\circ\widetilde{f}_{12}$, and $\widetilde{f}_{23}\circ(\Theta_\eta)_{13} = (\Theta_\eta)_{13}\circ(1\otimes\omega'_\eta\otimes 1)\circ\widetilde{f}_{23}$.*

*Proof.* Let $m\in M_\lambda$, $m'\in M'_\mu$, $m''\in M''_\nu$. Then

$$\begin{aligned}\widetilde{f}_{12}\circ(\Theta_\eta)_{13}(m\otimes m'\otimes m'') &= f(\lambda-\eta,\mu)\sum_i v_i^\eta m\otimes m'\otimes u_i^\eta m''\\ &= f(\lambda,\mu)f(-\eta,\mu)\sum_i v_i^\eta m\otimes m'\otimes u_i^\eta m''\\ &= f(\lambda,\mu)\sum_i v_i^\eta m\otimes\omega_\eta m'\otimes u_i^\eta m''\\ &= (\Theta_\eta)_{13}\circ(1\otimes\omega_\eta\otimes 1)\circ\widetilde{f}_{12}(m\otimes m'\otimes m'').\end{aligned}$$

The second identity can be shown using $\omega'_\eta m' = f(\mu,\eta)m'$. □

We continue with our assumption that $M, M'$, and $M''$ are $\widetilde{U}$-modules in $\mathcal{O}$. To verify the hexagon identities, let $\widetilde{f}'$ denote the transformation on $M''\otimes M\otimes M'$ taking $m''\otimes m\otimes m'\in M''_\nu\otimes M_\lambda\otimes M'_\mu$ to $f(\nu,\lambda+\mu)m''\otimes m\otimes m'$. Let $\widetilde{f}''$ be the transformation on $M'\otimes M''\otimes M$ taking $m'\otimes m''\otimes m\in M'_\mu\otimes M''_\nu\otimes M_\lambda$ to $f(\mu+\nu,\lambda)m'\otimes m''\otimes m$. As in [Ja, Thm. 3.18], the hexagon identities are equivalent to $R_{12}\circ R_{23} = (1\otimes\Delta)(\Theta)\circ\widetilde{f}'\circ P_{12}\circ P_{23}$ as maps from $M\otimes(M'\otimes M'')\to(M''\otimes M)\otimes M'$, and $R_{23}\circ R_{12} = (\Delta\otimes 1)(\Theta)\circ\widetilde{f}''\circ P_{23}\circ P_{12}$ as maps from $(M\otimes M')\otimes M''$ to $M'\otimes(M''\otimes M)$.

**Theorem 5.7.** *The hexagon identities hold, that is,*

(i) $R_{12}\circ R_{23} = (1\otimes\Delta)(\Theta)\circ\widetilde{f}'\circ P_{12}\circ P_{23}$, *and*

(ii) $R_{23}\circ R_{12} = (\Delta\otimes 1)(\Theta)\circ\widetilde{f}''\circ P_{23}\circ P_{12}$.

*Proof.* Let $m \otimes m' \otimes m'' \in M_\lambda \otimes M'_\mu \otimes M''_\nu$. By Lemma 5.5, the right side of (i) applied to $m \otimes m' \otimes m''$ gives

$$(1 \otimes \Delta)(\Theta) \circ \widetilde{f}' \circ P_{12} \circ P_{23}(m \otimes m' \otimes m'')$$
$$= f(\nu, \lambda + \mu) \sum_{\gamma \in Q^+} \sum_{0 \le \zeta \le \gamma} (\Theta_{\gamma - \zeta})_{12} (\Theta_\zeta)_{13} (1 \otimes \omega_\zeta \otimes 1)(m'' \otimes m \otimes m').$$

On the other hand, by Lemma 5.6, the left side of (i) can be seen to equal

$$\Theta_{12} \circ \widetilde{f}_{12} \circ P_{12} \circ \Theta_{23} \circ \widetilde{f}_{23} \circ P_{23}(m \otimes m' \otimes m'')$$
$$= \Theta_{12} \circ \widetilde{f}_{12} \circ \Theta_{13} \circ \widetilde{f}_{13}(m'' \otimes m \otimes m')$$
$$= \sum_{\eta \in Q^+} \Theta_{12} \circ (\Theta_\eta)_{13} \circ (1 \otimes \omega_\eta \otimes 1) \circ \widetilde{f}_{12} \circ \widetilde{f}_{13}(m'' \otimes m \otimes m')$$
$$= f(\nu, \lambda) f(\nu, \mu) \sum_{\eta \in Q^+} \Theta_{12} \circ (\Theta_\eta)_{13} \circ (1 \otimes \omega_\eta \otimes 1)(m'' \otimes m \otimes m').$$

Then because $f(\nu, \lambda + \mu) = f(\nu, \lambda) f(\nu, \mu)$, a change of variables shows that this is equal to the right side of (i). The proof of (ii) is similar. □

**Remark 5.8.** $\mathcal{O}$ is a braided monoidal category with braiding $R = R_{M',M}$ for each pair of modules $M', M$ in $\mathcal{O}$.

## §6. Isomorphisms among quantum groups

We will now investigate isomorphisms among the two-parameter quantum groups, and their connections with multiparameter quantum groups. The case $n = 2$ is special, and the two parameters collapse to one in the following sense. Let $r, r', s, s' \in \mathbb{K}^\#$ and $r \neq s$, $r' \neq s'$. If $rs^{-1} = r'(s')^{-1}$, there is an isomorphism of Hopf algebras

$$\phi : U_{r,s}(\mathfrak{sl}_2) \to U_{r',s'}(\mathfrak{sl}_2)$$

given by $\phi(\omega^{\pm 1}) = \check{\omega}^{\pm 1}$, $\phi((\omega')^{\pm 1}) = (\check{\omega}')^{\pm 1}$, $\phi(e) = \check{e}$, $\phi(f) = r^{-1} r' \check{f}$, where "ˇ" denotes generators of $U_{r',s'}(\mathfrak{sl}_2)$. (When dealing with $\mathfrak{sl}_2$, we omit the subscript "1" on the generators.) The proof is a simple check that the relations and coproducts are preserved. In particular, if $q$ is a square root of $rs^{-1}$, then $U_{r,s}(\mathfrak{sl}_2) \cong U_{q,q^{-1}}(\mathfrak{sl}_2)$ as Hopf algebras. Therefore the one-parameter quantum group $U_q(\mathfrak{sl}_2)$ is isomorphic to the quotient of $U_{r,s}(\mathfrak{sl}_2)$ by the ideal generated by $\omega' - \omega^{-1}$.

If $n \geq 3$, there is no such isomorphism, as the following proposition shows.

**Proposition 6.1.** *Let $n \geq 3$, and assume there is an isomorphism of Hopf algebras*

$$\phi : U_{r,s}(\mathfrak{sl}_n) \to U_{q,q^{-1}}(\mathfrak{sl}_n)$$

*for some $q$. Then $r = q$ and $s = q^{-1}$.*

*Proof.* Let $\phi$ be an isomorphism as hypothesized, and assume

$$\pi : U_{q,q^{-1}}(\mathfrak{sl}_n) \to U_q(\mathfrak{sl}_n)$$

is the surjection onto the standard one-parameter quantum group of [Ja] given by $\pi(e_i) = E_i$, $\pi(f_i) = F_i$, $\pi(\omega_i^{\pm 1}) = K_i^{\pm 1}$, $\pi((\omega_i')^{\pm 1}) = K_i^{\mp 1}$. For $1 \le i \le n-1$, we have

$$\begin{aligned} \Delta(\pi\phi(e_i)) = \pi\phi(\Delta(e_i)) &= \pi\phi(e_i \otimes 1 + \omega_i \otimes e_i) \\ &= \pi\phi(e_i) \otimes 1 + \pi\phi(\omega_i) \otimes \pi\phi(e_i). \end{aligned} \tag{6.2}$$

Note that $\pi\phi(\omega_i)$ is necessarily a group-like element. Therefore $\pi\phi(e_i)$ is a skew-primitive element in $U_q(\mathfrak{sl}_n)$. By Theorem 5.4.1, Lemma 5.5.5, and the subsequent comments in [M], the set of group-like elements in $U_q(\mathfrak{sl}_n)$ is the group $G$ generated by $K_j$ $(1 \le j \le n-1)$, and the skew-primitive elements together with the group-like elements span the subspace

$$\sum_{j=1}^{n-1} (\mathbb{K}E_j + \mathbb{K}F_j) + \mathbb{K}G.$$

Therefore

$$\pi\phi(e_i) = \sum_{j=1}^{n-1} \alpha_j^i E_j + \beta_j^i F_j + \sum_{g \in G} \gamma_g^i g$$

for some scalars $\alpha_j^i, \beta_j^i, \gamma_g^i$ $(1 \le i, j \le n-1,\ g \in G)$. Consequently

$$\Delta(\pi\phi(e_i)) = \sum_{j=1}^{n-1} \alpha_j^i (E_j \otimes 1 + K_j \otimes E_j) + \beta_j^i (1 \otimes F_j + F_j \otimes K_j^{-1}) + \sum_{g \in G} \gamma_g^i g \otimes g.$$

By (6.2), this must be equal to

$$\sum_{j=1}^{n-1} (\alpha_j^i E_j \otimes 1 + \beta_j^i F_j \otimes 1 + \pi\phi(\omega_i) \otimes \alpha_j^i E_j + \pi\phi(\omega_i) \otimes \beta_j^i F_j) + \sum_g (\gamma_g^i g \otimes 1 + \pi\phi(\omega_i) \otimes \gamma_g^i g).$$

Comparing these two expressions, and noting that $\pi\phi(\omega_i) \ne 1$ as $\phi$ is an isomorphism (or by the comparison of expressions), we see first that

$$\sum_{j=1}^{n-1} \alpha_j^i E_j + \gamma_1^i 1 = \sum_{j=1}^{n-1} (\alpha_j^i E_j + \beta_j^i F_j) + \sum_{g \in G} \gamma_g^i g + \gamma_1^i \pi\phi(\omega_i).$$

Therefore $\gamma_g^i = 0$ for all $g$ except $g \in \{1, \pi\phi(\omega_i)\}$, $\gamma_{\pi\phi(\omega_i)}^i = -\gamma_1^i$, and all $\beta_j^i = 0$. We now have

$$\pi\phi(e_i) = \sum_{1 \le j \le n-1} \alpha_j^i E_j + \gamma_1^i (1 - \pi\phi(\omega_i)).$$

A further comparison of coproducts yields

$$\begin{aligned} &\sum_{j=1}^{n-1} \alpha_j^i \Big(E_j \otimes 1 + K_j \otimes E_j\Big) + \gamma_1^i \Big(1 \otimes 1 - \pi\phi(\omega_i) \otimes \pi\phi(\omega_i)\Big) \\ &\quad = \sum_{j=1}^{n-1} \alpha_j^i \Big(E_j \otimes 1 + \pi\phi(\omega_i) \otimes E_j\Big) \\ &\qquad + \gamma_1^i \Big(1 \otimes 1 - \pi\phi(\omega_i) \otimes 1\Big) + \gamma_1^i \Big(\pi\phi(\omega_i) \otimes 1 - \pi\phi(\omega_i) \otimes \pi\phi(\omega_i)\Big), \end{aligned}$$

which implies that

$$\alpha^i_j(K_j - \pi\phi(\omega_i)) = 0, \quad \text{for all} \quad 1 \le j \le n-1.$$

Thus all $\alpha^i_j = 0$ except possibly one, and some $\alpha^i_{j_i}$ must be nonzero as $\pi\phi$ is surjective. Therefore $\pi\phi(\omega_i) = K_{j_i}$.

Next we will apply $\pi\phi$ to relation (R2') and use the relations in $U_q(\mathfrak{sl}_n)$:

$$\begin{aligned} \pi\phi(\omega_i e_i) &= \pi\phi(rs^{-1} e_i \omega_i) \\ K_{j_i}(\alpha^i_{j_i} E_{j_i} + \gamma^i_1(1 - K_{j_i})) &= rs^{-1}(\alpha^i_{j_i} E_{j_i} + \gamma^i_1(1 - K_{j_i}))K_{j_i} \\ \alpha^i_{j_i} q^2 E_{j_i} K_{j_i} + \gamma^i_1 K_{j_i} - \gamma^i_1 K^2_{j_i} &= \alpha^i_{j_i} rs^{-1} E_{j_i} K_{j_i} + \gamma^i_1 rs^{-1} K_{j_i} - \gamma^i_1 rs^{-1} K^2_{j_i}. \end{aligned}$$

This forces $rs^{-1} = q^2$, and $\gamma^i_1 = 0$ as $r \neq s$. Applying relation (R2') again, we have

$$\begin{aligned} \pi\phi(\omega_i e_{i+1}) &= \pi\phi(s e_{i+1} \omega_i) \\ K_{j_i} \alpha^{i+1}_{j_{i+1}} E_{j_{i+1}} &= \alpha^{i+1}_{j_{i+1}} s E_{j_{i+1}} K_{j_i} \\ \alpha^{i+1}_{j_{i+1}} q^{\langle \alpha_{j_i}, \alpha_{j_{i+1}} \rangle} E_{j_{i+1}} K_{j_i} &= \alpha^{i+1}_{j_{i+1}} s E_{j_{i+1}} K_{j_i}. \end{aligned}$$

Thus $s = q^{\langle \alpha_{j_i}, \alpha_{j_{i+1}} \rangle}$ must hold. By relations (R5) and (R6), we see that $|j_i - j_{i+1}| = 1$, so that in fact $s = q^{-1}$. Combining this with $rs^{-1} = q^2$ we now have $r = q$ as well. □

**Multiparameter quantum groups.**

Deformations of $GL_n$ involving $1 + \binom{n}{2}$ parameters were constructed independently by several authors (see [AST], [R], [S]). The dual version is a multiparameter universal enveloping algebra, which was studied by Chin and Musson [ChM] and Dobrev and Parashar [DP]. We will show that our two-parameter quantum groups are essentially special cases of these multiparameter quantum groups, as should be expected. We adopt the notation of Chin and Musson.

The $1 + \binom{n}{2}$ parameters are denoted $\lambda$ and $p_{i,j}$ $(1 \le i < j \le n)$ in [ChM]. Set $\lambda = rs^{-1}$ and $p_{i,j} = s^{-1}$ for all $i < j$. Let $\widehat{U}$ be the Hopf algebra generated by $E_i, F_i$ $(1 \le i < n)$ and $K_i^{\pm 1}, L_i^{\pm 1}$ $(1 \le i \le n)$ with relations given by

(ChM0) The $K_i^{\pm 1}, L_j^{\pm 1}$ all commute with one another and $K_i K_i^{-1} = L_i L_i^{-1}$.

(ChM1) $K_j E_i = r^{-\delta_{i,j}} s^{-\delta_{i,j-1}} E_i K_j$ and $K_j F_i = r^{\delta_{i,j}} s^{\delta_{i,j-1}} F_i K_j$.

(ChM2) $L_j E_i = r^{\delta_{i,j-1}} s^{\delta_{i,j}} E_i L_j$ and $L_j F_i = r^{-\delta_{i,j-1}} s^{-\delta_{i,j}} F_i L_j$.

(ChM3) $E_i F_i - r^{-1} s F_i E_i = (r^{-1}s - 1)(L_{i+1} K_{i+1} L_i^{-1} K_i^{-1} - 1)$.

(ChM4) $E_i F_j = r^{\delta_{i,j+1}} s^{-\delta_{i,j-1}} F_j E_i$ if $i \neq j$.

(ChM5) $\mathrm{ad}_\ell(E_i)^{1-\langle \alpha_i, \alpha_j \rangle}(E_j) = 0$ and $\mathrm{ad}_\ell(F_i)^{1-\langle \alpha_i, \alpha_j \rangle}(F_j) = 0$ if $i \neq j$.

These relations are given in [ChM, Thm. 4.8] (for more general $\lambda$, $p_{i,j}$) as relations for a Hopf algebra that is defined as a subalgebra of the finite dual $A^0$ of a multiparameter quantum function algebra $A$. In addition, Chin and Musson give one more set of conditions: (ChM6) those relations among the $K_i, L_j$ which determine the structure of the group they generate *as a subgroup of the group of units of* $A^0$. This results in a Hopf algebra $\overline{U}$ (denoted $U$ in their paper). Thus the multiparameter Hopf algebra $\overline{U}$ of Chin and Musson is the quotient of our $\widehat{U}$ by their relations (ChM6) (in case $\lambda = rs^{-1}$ and $p_{i,j} = s^{-1}$).

The Hopf structure of $\widehat{U}$ is defined by requiring $K_i, L_i$ to be group-like elements and

$$\Delta(E_i) = E_i \otimes 1 + L_{i+1}L_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_{i+1}K_i^{-1} \otimes F_i.$$

**Proposition 6.3.** *There is a Hopf algebra morphism* $\phi : \widehat{U} \to U_{r,s}(\mathfrak{gl}_n)$ *given by*

$$\begin{aligned}
\phi(L_i) &= a_1 \cdots a_{i-1} b_{i+1}^{-1} \cdots b_n^{-1}, \\
\phi(K_i) &= b_1^{-1} \cdots b_{i-1}^{-1} a_{i+1} \cdots a_n, \\
\phi(E_i) &= -s^{-1}(r-s)^2 e_i, \\
\phi(F_i) &= (\omega_i')^{-1} f_i.
\end{aligned}$$

The proof is just a check that the relations of $\widehat{U}$ are preserved by $\phi$, and that the coproducts of the generators correspond to the coproducts of their images.

**Remark 6.4.** Note that the kernel and cokernel of $\phi$ are generated by group-like elements. We have $\omega_i = \phi(L_i^{-1}L_{i+1})$ and $\omega_i' = \phi(K_i K_{j+1}^{-1})$, which implies that $U_{r,s}(\mathfrak{sl}_n)$ is contained in the image of $\phi$. In case $n = 2$, it is easy to see that $\phi$ is an isomorphism. If $n = 3$, straightforward calculations show that the cokernel of $\phi$ is precisely $\mathbb{K}\langle a_1^{\pm 1}\rangle$.

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