Asymptotic Symmetries for Conformal Scalar Curvature Equations with Singularity

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Abstract

We give conditions on a positive Hölder continuous function $K(x)$ such that every C^2 positive solution $u(x)$ of the conformal scalar curvature equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$$

in a punctured neighborhood of the origin in $\mathbf{R}^n$ either has a removable singularity at the origin or satisfies

$$u(x) = u_0(|x|)(1 + \mathcal{O}(|x|^\beta)) \qquad \text{as} \qquad |x| \to 0^+$$

for some positive singular solution $u_0(|x|)$ of

$$\Delta u_0 + K(0)u_0^{\frac{n+2}{n-2}} = 0 \qquad \text{in} \quad \mathbf{R}^n \setminus \{0\}$$

where $\beta \in (0,1)$ is the Hölder exponent of K.

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1 Introduction

In this paper we study the conformal scalar curvature equation

$$\Delta u + K(x)u^{n^*} = 0 \qquad \text{in} \quad \mathbf{B}^n \setminus \{0\},\ n \geq 3, \tag{1.1}$$

where $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$ and $n^* = (n+2)/(n-2)$. Specifically, we give conditions on a Hölder continuous function $K : \mathbf{B}^n \to (0,\infty)$ such that every C^2 positive solution $u(x)$ of (1.1) with a non-removable singularity at the origin satisfies

$$u(x) = u_0(|x|)(1 + \mathcal{O}(|x|^\beta)) \qquad \text{as} \qquad |x| \to 0^+ \tag{1.2}$$

for some C^2 positive solution $u_0(|x|)$ of

$$\begin{cases} \Delta u_0 + K(0)u_0^{n^*} = 0 & \text{in} \quad \mathbf{R}^n \setminus \{0\} \\ u_0(|x|) \to \infty & \text{as} \quad |x| \to 0^+ \end{cases} \tag{1.3}$$

where $\beta \in (0,1)$ is the Hölder exponent of K. All C^2 positive solutions $u_0(|x|)$ of (1.3) have been described by Fowler [6]. We also prove that the convergence rate in (1.2) is optimal.

It is well known that a solution u of (1.1) defines a conformally flat metric $g_{ij} = u^{4/(n-2)}\delta_{ij}$ with scalar curvature $\frac{4(n-1)}{n-2}K$. The work of Schoen and Yau [13, 14, 15] on conformally flat manifolds and the Yamabe problem has highlighted the importance of studying singular solutions of (1.1) in neighborhoods of their singular sets. Solutions of (1.1) with an isolated singularity at the origin are worthy of study because they are the simplest examples of singular solutions.

When $K(x)$ is identically a positive constant in $\mathbf{B}^n$, Caffarelli, Gidas, and Spruck [1] proved that every C^2 positive solution $u(x)$ of (1.1) with a non-removable singularity at the origin satisfies

$$u(x) = u_0(|x|)(1 + o(1)) \qquad \text{as} \qquad |x| \to 0^+ \tag{1.4}$$

for some $u_0(|x|)$ as above. In particular, (1.4) implies that

$$u(x) = \mathcal{O}(|x|^{-(n-2)/2}) \qquad \text{as} \qquad |x| \to 0^+. \tag{1.5}$$

If $K : \mathbf{B}^n \to (0,\infty)$ is a continuous non-constant perturbation of a positive constant it is natural to conjecture that every C^2 positive solution $u(x)$ of (1.1) with a non-removable singularity at the origin should still satisfy (1.4) or at least satisfy (1.5). However this is not the case. Indeed, Taliaferro and Zhang [19] have shown that given any C^1 function $K_0 : \overline{\mathbf{B}^n} \to (0,\infty)$ and any large continuous function $\varphi : (0,1) \to (0,\infty)$ there exists a continuous function $K : \mathbf{B}^n \to (0,\infty)$ satisfying

$$|K(x) - K_0(x)| \le \frac{1}{\varphi(|x|)} \qquad \text{in} \qquad \mathbf{B}^n \setminus \{0\}$$

(and in fact equal to K_0 except on a set of arbitrarily small measure) such that (1.1) has a C^2 positive solution $u(x)$ which does not satisfy

$$u(x) = \mathcal{O}(\varphi(|x|)) \qquad \text{as} \qquad |x| \to 0^+. \tag{1.6}$$

Leung [8] showed that when $K_0(x) \equiv 1$ and $n \ge 5$, such a function K can be found which is *Lipschitz* continuous on $\mathbf{B}^n$.

Chen and Lin in a series of masterful papers [2], [4], and [11] have given conditions on a Hölder continuous function K such that every C^2 positive solution $u(x)$ of (1.1) with a non-removable singularity at the origin satisfies (1.4). They first prove the a priori bound (1.5) which immediately implies the spherical Harnack inequality

$$\max_{|x|=r} u \le C \min_{|x|=r} u, \qquad C \quad \text{independent of } r. \tag{1.7}$$

They then use the Pohozaev identity, which requires them to assume that K is at least C^1 in $\mathbf{B}^n \setminus \{0\}$, to conclude that

$$|x|^{(n-2)/2}u(x) \qquad \text{is bounded between positive constants for } |x| \text{ small and positive.} \tag{1.8}$$

However Chen informed us that their proof in [4] that (1.8) implies (1.4) needs considerably more detail, an outline of which he e-mailed to us. Using a method very different than his, we prove in Section 2 the following theorem, which gives conditions under which (1.8) implies the sharper result (1.2).

Theorem 1. *Let $n \ge 3$ be an integer and let $u : \mathbf{B}^n \setminus \{0\} \to \mathbf{R}$ be a C^2 positive function satisfying*

$$\frac{-\Delta u(x)}{u(x)^{n^*}} = 1 + \mathcal{O}(|x|^\beta) \quad \textit{as} \quad |x| \to 0^+ \tag{1.9}$$

for some constant $\beta \in (0,1)$. If u satisfies (1.8), then u satisfies (1.2) for some C^2 positive radial solution $u_0(|x|)$ of

$$\Delta u_0 + u_0^{n^*} = 0 \qquad \textit{in} \quad \mathbf{R}^n \setminus \{0\}. \tag{1.10}$$

When u is a C^2 positive solution of $-\Delta u/u^{n^*} \equiv 1$ in $\mathbf{B}^n \setminus \{0\}$ with a non-removable singularity at the origin, an asymptotic estimate even sharper than (1.2) was given by Korevaar, Mazzeo, Pacard, and Schoen [7].

Theorem 1 is optimal in two ways. First, the rate of convergence in (1.2) is optimal because

$$u_0(|x|) := \left(\frac{n-2}{2|x|}\right)^{\frac{n-2}{2}} \qquad \text{and} \qquad u(x) := u_0(|x|)(1+|x|^\beta)$$

are C^2 positive solutions of (1.10) and (1.9) respectively.

And second, the upper bound of 1 on β in Theorem 1 is optimal because Korevaar, Mazzeo, Pacard, and Schoen [7] give an explicit formula for a C^2 positive solution $u(x)$ of $\Delta u + u^{n^*} = 0$ in $\mathbf{B}^n \setminus \{0\}$ satisfying (1.8) such that, for each $\beta > 1$ and each C^2 positive solution $u_0(|x|)$ of (1.10), u does not satisfy (1.2).

Theorem 1, because of the weakness of condition (1.9), should be useful to others, who having proved (1.8) by whatever method, will then be able to use Theorem 1 to immediately obtain the more precise asymptotic behavior (1.2). For example, by using Theorem 1, the conclusion (1.4) of Chen and Lin's Theorems A and B below can be immediately sharpened to (1.2) where $\beta \in (0,1)$ is any Hölder exponent for the function K in those theorems. For another example, we use Theorem 1 in our proof of Theorem 2 below.

Before we state Theorem 2, we introduce a notation $\mathcal{C}^\alpha(\Omega)$, $\alpha > 0$, as follows:

Definition 1.1. *1. If α is a positive integer, then $\mathcal{C}^\alpha(\Omega)$ is the usual Hölder space $C^\alpha(\Omega)$.*

2. If $\alpha > [\alpha]$, then $\mathcal{C}^\alpha(\Omega)$ is the set of all functions $f \in C^{[\alpha]}(\Omega)$ such that

$$|\nabla^{[\alpha]} f(x) - \nabla^{[\alpha]} f(y)| \le c(|x-y|)|x-y|^{\alpha-[\alpha]}, \qquad x, y \in \Omega,$$

for some nonnegative continuous function $c(\cdot)$ satisfying $c(0) = 0$.

Note that $\mathcal{C}^\alpha$ is only slightly stronger than C^α and is weaker than $C^{\alpha+\varepsilon}$ for each $\varepsilon > 0$.

Our assumption on K, in Theorem 2 below, is as follows:

Hypothesis H: K is a positive function in $\mathcal{C}^\alpha(\mathbf{B}^n)$ where $\alpha = (n-2)/2$. Also, if $n \ge 6$, then

$$|\nabla^j K(x)| \le c(|x|)|\nabla K(x)|^{\frac{\alpha-j}{\alpha-1}}, \qquad 2 \le j \le [\alpha], \qquad x \in \mathbf{B}^n \tag{1.11}$$

for some nonnegative continuous function $c(\cdot)$ satisfying $c(0) = 0$.

We now state

Theorem 2. *Suppose u is a C^2 positive solution of (1.1) where K satisfies Hypothesis H. Then either u has a removable singularity at the origin or there exists a C^2 positive radial solution $u_0(|x|)$ of (1.3) such that u satisfies (1.2) for each $\beta \in (0,1) \cap (0,\alpha]$.*

Our assumptions on K are very simple for low dimensions. In fact, when $n = 3$ we only assume that $K \in \mathcal{C}^{\frac{1}{2}}$, when $n = 4$ that $K \in C^1$, and when $n = 5$ that $K \in \mathcal{C}^{\frac{3}{2}}$. The assumptions on higher derivatives of K only start when $n \ge 6$.

Also, Theorem 2 is nearly optimal when n is 3 or 4. Indeed, after communicating this theorem to Man Chun Leung, he informed us that if n is 3 or 4, $\varepsilon > 0$ is any small number, and $\varphi\colon (0,1) \to (0,\infty)$ is any large continuous function, then he can construct a positive function $K \in C^{\frac{n-2}{2}-\varepsilon}(\mathbf{B}^n)$ such that (1.1) has a C^2 positive solution $u(x)$ which does not satisfy (1.6).

A new feature of Theorem 2 is that our conditions on K include not only a large set of non-constant K but also the case that K is a positive constant.

Our proof of Theorem 2 for $n \ge 4$ relies heavily on various ingenious methods that Chen and Lin used in their papers [2, 3, 4]. Instead of providing all the details, we shall cite those arguments and indicate what changes we need to make. However, when $n = 3$ our assumption on K is only slightly stronger than $C^{1/2}$ which prevents us from using the Pohozaev identity. Instead we use, as in [21], a delicate version of the method of moving spheres.

The condition (1.11) was first used by Li [9, page 322], who also gave examples of functions satisfying this condition.

If $K \in C^1(\mathbf{B}^n)$, $\nabla K(0) \neq 0$ and u has a non-removable singularity at the origin, then Lin [11] proved that u satisfies (1.4). On the other hand, if $\nabla K(0) = 0$, then in a neighborhood of the origin, by our assumption on K, we have $|\nabla K(x)| \leq C|x|^{\frac{n-2}{2}-1}$, which implies that $|K(x) - K(0)| \leq C|x|^{\frac{n-2}{2}}$. This flatness index $(n-2)/2$ is very important, without which the conclusion of Theorem 2 is not true; counter examples can be found in Chen and Lin's works [2, 11, 4]. It is interesting to note that the flatness index of the asymptotically flat manifold in Schoen and Yau's positive mass theorem [17, 18] is also bounded below by $(n-2)/2$. Perhaps there is a connection between Theorem 2 and the positive mass theorem.

We conclude our introduction by stating Chen and Lin's results mentioned above. They assume

Hypothesis CL. The function $K \in C^0(\mathbf{B}^n) \cap C^1(\mathbf{B}^n \setminus \{0\})$ is positive and in a neighborhood of the origin can be written as

$$K(x) = K(0) + Q(x) + R(x)$$

where $Q \in C^1(\mathbf{R}^n \setminus \{0\})$ is a homogeneous function of degree $\alpha > 0$ satisfying

$$c_1|x|^{\alpha-1} \leq |\nabla Q(x)| \leq c_2|x|^{\alpha-1} \qquad \text{for} \qquad x \in \mathbf{R}^n \setminus \{0\}$$

for some positive constants c_1 and c_2 and where R satisfies

$$\lim_{|x|\to 0^+} |R(x)||x|^{-\alpha} = \lim_{|x|\to 0^+} |\nabla R(x)||x|^{1-\alpha} = 0.$$

They prove the following two theorems, which highlight the importance of the flatness index $(n-2)/2$.

Theorem A [2] [4]. *Suppose u is a C^2 positive solution of (1.1) where K satisfies Hypothesis CL with $\alpha \geq (n-2)/2$. Then either u has a removable singularity at the origin or u satisfies (1.4) for some C^2 positive radial solution $u_0(|x|)$ of (1.3).*

Theorem B [11]. *Suppose u is a C^2 positive solution of (1.1) where K satisfies Hypothesis CL with $0 < \alpha < (n-2)/2$ and in addition Q satisfies either*

$$\int_{\mathbf{R}^n} \nabla Q(x+\xi)(1+|x|^2)^{-n} dx \neq 0 \qquad \textit{for all } \xi \in \mathbf{R}^n$$

or

$$\int_{\mathbf{R}^n} Q(x+\xi)(1+|x|^2)^{-n} dx > 0$$

for all ξ satisfying

$$\int_{\mathbf{R}^n} \nabla Q(x+\xi)(1+|x|^2)^{-n} dx = 0.$$

Then either u has a removable singularity at the origin or u satisfies (1.4) for some C^2 positive radial solution $u_0(|x|)$ of (1.3).

2 Refined asymptotic behavior

In this section, we prove Theorem 1.

Proof of Theorem 1. Define $w\colon (0,\infty) \times S^{n-1} \to \mathbf{R}$ by $w(t,\theta) = e^{-\frac{n-2}{2}t}u(e^{-t}\theta)$. Then w is a C^2 positive solution of

$$w_{tt} - \left(\frac{n-2}{2}\right)^2 w + \Delta_\theta w + \widehat{K}(t,\theta)w^{n^*} = 0 \quad \text{in} \quad (0,\infty) \times S^{n-1} \tag{2.1}$$

satisfying, uniformly in S^{n-1}, the inequalities

$$0 < \liminf_{t\to\infty} w(t,\theta) \leq \limsup_{t\to\infty} w(t,\theta) < \infty \tag{2.2}$$

where $\widehat{K}(t,\theta) := -\Delta u(e^{-t}\theta)/u(e^{-t}\theta)^{n^*}$. Note, for later, that

$$\max_{\theta\in S^{n-1}} |\widehat{K}(t,\theta)-1| = \mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t\to\infty. \tag{2.3}$$

To prove Theorem 1 it suffices to prove that there exists a C^2 positive solution $w_0(t)$ of

$$w_0'' - \left(\frac{n-2}{2}\right)^2 w_0 + w_0^{n^*} = 0 \quad \text{in} \quad \mathbf{R} \tag{2.4}$$

such that

$$\max_{\theta\in S^{n-1}} |w(t,\theta)-w_0(t)| = \mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t\to\infty. \tag{2.5}$$

Our proof of (2.5) makes use of two clever observations of Mazzeo, Pollack, and Uhlenbeck [12] concerning the spectrum of the linearization of (2.1) about certain functions which are independent of θ. These two observations are stated below in the sentence containing equation (2.21) and in the first sentence of the proof of Lemma 2.4. Since each of these observations can be verified by direct calculation using only elementary calculus, our proof of (2.5) can be read independently of [12]. In fact, our proof of Theorem 1 is completely self-contained except for a result of Caffarelli, Gidas, and Spruck [1, Theorem 8.1] which we use below in our proofs of Lemmas 2.1 and 2.2.

Multiplying (2.1) by w_t and then integrating over S^{n-1} we get

$$\frac{d}{dt}Q(t,w) = -\int_{S^{n-1}} [\widehat{K}(t,\theta)-1]w^{n^*}w_t\, dS_\theta \quad \text{for} \quad t>0 \tag{2.6}$$

where

$$Q(t,w) = \frac{1}{2}\int_{S^{n-1}} \left(w_t^2 - \left(\frac{n-2}{2}\right)^2 w^2 + |\nabla_\theta w|^2 + \frac{n-2}{n} w^{\frac{2n}{n-2}}\right) dS_\theta.$$

It follows from (2.1), (2.2), and (2.3) that w and its first order partial derivatives are bounded for t large. Thus, by (2.3) and (2.6), we have $Q(w) := \lim_{t\to\infty} Q(t,w)$ exists and

$$Q(t,w) = Q(w) + \mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t\to\infty. \tag{2.7}$$

Our proof of Theorem 1 will partially consist of some lemmas the first of which is

Lemma 2.1. $Q(w) < 0$.

Proof. Let $w_j(t,\theta) = w(t+t_j,\theta)$ where $t_j\to\infty$. Then, by (2.1), we have

$$w_{jtt} - \left(\frac{n-2}{2}\right)^2 w_j + \Delta_\theta w_j + \widehat{K}(t+t_j,\theta)w_j^{n^*} = 0 \quad \text{in} \quad (-t_j,\infty)\times S^{n-1} \tag{2.8}$$

and it therefore follows from (2.2) and (2.3) that some subsequence of w_j, which we denote again by w_j, converges to w_0 in $C^1_{\text{loc}}(\mathbf{R}\times S^{n-1})$ where w_0 is bounded between positive constants and satisfies

$$w_{0tt} - \left(\frac{n-2}{2}\right)^2 w_0 + \Delta_\theta w_0 + w_0^{n^*} = 0 \quad \text{in} \quad \mathbf{R}\times S^{n-1}. \tag{2.9}$$

Thus, as proved by Caffarelli, Gidas, and Spruck [1, Theorem 8.1], $w_0(t,\theta) = w_0(t)$ is independent of θ, and hence, letting σ_{n-1} be the measure of S^{n-1}, we have

$$\begin{aligned} Q(w) &\underset{j\to\infty}{\longleftarrow} Q(t+t_j,w) = Q(t,w_j) \underset{j\to\infty}{\longrightarrow} Q(t,w_0) \\ &= \frac{\sigma_{n-1}}{2}\left(w_0'(t)^2 - \left(\frac{n-2}{2}\right)^2 w_0(t)^2 + \frac{n-2}{n} w_0(t)^{\frac{2n}{n-2}}\right) \end{aligned}$$

which, as first noted by Fowler [6], is constant and negative because its derivative is the left side of (2.9) multiplied by $\sigma_{n-1}w_0'$ and because w_0 is bounded below on $\mathbf{R}$ by a positive constant. $\square$

Lemma 2.2. *As $t \to \infty$ we have*

$$\max_{\theta \in S^{n-1}} |w(t,\theta) - \overline{w}(t)| \to 0 \tag{2.10}$$

$$\max_{\theta \in S^{n-1}} |w_t(t,\theta) - \overline{w}'(t)| \to 0 \tag{2.11}$$

$$\max_{\theta \in S^{n-1}} |\nabla_\theta w(t,\theta)| \to 0 \tag{2.12}$$

where $\overline{w}(t)$ is the average of $w(t,\theta)$ over S^{n-1}.

Proof. Suppose (2.10) (resp. (2.11), (2.12)) does not hold. Then there exists $\varepsilon > 0$ and $t_j \to \infty$ such that the function $w_j(t,\theta) := w(t + t_j, \theta)$ satisfies

$$\begin{array}{ll} & \max_{\theta \in S^{n-1}} w_j(0,\theta) - \min_{\theta \in S^{n-1}} w_j(0,\theta) \geq \varepsilon \\ \text{(resp.} & \max_{\theta \in S^{n-1}} w_{jt}(0,\theta) - \min_{\theta \in S^{n-1}} w_{jt}(0,\theta) \geq \varepsilon, \\ & \max_{\theta \in S^{n-1}} |\nabla_\theta w_j(0,\theta)| \geq \varepsilon). \end{array} \tag{2.13}$$

Since w_j also satisfies (2.8) we have some subsequence of w_j converges to w_0 in $C^1_{\text{loc}}(\mathbf{R} \times S^{n-1})$ where w_0 is bounded between positive constants and satisfies (2.9). Therefore $w_0(t,\theta) = w_0(t)$ is independent of θ, which contradicts (2.13). $\square$

Lemma 2.3. *As $t \to \infty$ we have*

$$\overline{w}''(t) - \left(\frac{n-2}{2}\right)^2 \overline{w}(t) + \overline{w}(t)^{n^*} = o(1) \tag{2.14}$$

and

$$\frac{\sigma_{n-1}}{2}\left[\overline{w}'(t)^2 - \left(\frac{n-2}{2}\right)^2 \overline{w}(t)^2 + \frac{n-2}{n}\overline{w}(t)^{\frac{2n}{n-2}}\right] = Q(w) + o(1). \tag{2.15}$$

Proof. Using (2.1) we have

$$\begin{aligned} & \overline{w}'' - \left(\frac{n-2}{2}\right)^2 \overline{w} + \overline{w}^{n^*} \\ = \ & \overline{w}'' - \left(\frac{n-2}{2}\right)^2 \overline{w} + \overline{w}^{n^*} - \overline{\left[w_{tt} - \left(\frac{n-2}{2}\right)^2 w + \Delta_\theta w + \widehat{K} w^{n^*}\right]} \\ = \ & \overline{w}^{n^*} - \overline{\widehat{K} w^{n^*}} = o(1) \quad \text{as} \quad t \to \infty \end{aligned}$$

by (2.3) and (2.10). Also

$$\begin{aligned} \text{L.H.S. of (2.15)} &= Q(t,\overline{w}) = Q(t,w) + (Q(t,\overline{w}) - Q(t,w)) \\ &= Q(w) + o(1) \quad \text{as} \quad t \to \infty \end{aligned}$$

by equation (2.7) and Lemma 2.2. $\square$

We now continue with our proof of Theorem 1 by using in this paragraph some methods of Veron [20]. Averaging (2.1) and then subtracting the resulting equation from (2.1) we find that $W := w - \overline{w}$ satisfies

$$W_{tt} - \left(\frac{n-2}{2}\right)^2 W + \Delta_\theta W + \widehat{W} = 0 \quad \text{in} \quad (0,\infty) \times S^{n-1} \tag{2.16}$$

where $\widehat{W} = \widehat{K}w^{n^*} - \overline{\widehat{K}w^{n^*}}$. Multiplying (2.16) by W and integrating over S^{n-1} we get

$$\int_{S^{n-1}} \left(WW_{tt} - \left(\frac{n-2}{2}\right)^2 W^2 + W\Delta_\theta W + W\widehat{W} \right) = 0 \quad \text{in} \quad (0,\infty). \tag{2.17}$$

Let $\psi(t) = \left(\int_{S^{n-1}} W(t,\theta)^2 \right)^{1/2}$. Then ψ is nonnegative and continuous for $t > 0$ and ψ is C^2 on those intervals where ψ is positive. Since

$$|\psi\psi'| = \left| \int_{S^{n-1}} WW_t \right| \le \left(\int_{S^{n-1}} W_t^2 \right)^{1/2} \psi, \quad \text{when} \quad \psi > 0,$$

we have

$$|\psi'| \le \left(\int_{S^{n-1}} W_t^2 \right)^{1/2}, \quad \text{when} \quad \psi > 0,$$

and therefore

$$\psi\psi'' + \psi'^2 = \int_{S^{n-1}} W_t^2 + \int_{S^{n-1}} WW_{tt} \ge \psi'^2 + \int_{S^{n-1}} WW_{tt}, \quad \text{when} \quad \psi > 0. \tag{2.18}$$

Since the smallest nonzero eigenvalue of $-\Delta_\theta$ is $n-1$ we have

$$\int_{S^{n-1}} W\Delta_\theta W \le -(n-1) \int_{S^{n-1}} W^2 \quad \text{in} \quad (0,\infty). \tag{2.19}$$

Also, since $\int_{S^{n-1}} W = 0$, we have

$$\begin{aligned} \int_{S^{n-1}} W\widehat{W} &= \int_{S^{n-1}} W(\widehat{K}w^{n^*} - \overline{\widehat{K}w^{n^*}}) \\ &= \int_{S^{n-1}} W(\widehat{K}w^{n^*} - \overline{w}^{n^*}) = I_1 + I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_{S^{n-1}} Ww^{n^*}(\widehat{K} - 1) \\ &\le \psi(t) \left(\int_{S^{n-1}} (w^{n^*}(\widehat{K} - 1))^2 \right)^{1/2} = \psi(t)\mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t \to \infty \end{aligned}$$

by (2.3), and

$$I_2 := \int_{S^{n-1}} W(w^{n^*} - \overline{w}^{n^*}) = \int_{S^{n-1}} Wn^*\xi^{n^*-1}W$$

where $\xi = \xi(t,\theta)$ is between $w(t,\theta)$ and $\overline{w}(t)$. Since, by (2.10), $\xi(t,\theta) = \overline{w}(t)(1 + o(1))$ as $t \to \infty$, we have $I_2 = n^*\overline{w}^{n^*-1}(1 + o(1))\psi^2$. Thus using (2.18) and (2.19) in (2.17) we obtain for t large and $\psi(t) > 0$ that

$$\psi\psi'' - \left(\frac{n-2}{2}\right)^2 \psi^2 - (n-1)\psi^2 + n^*\overline{w}^{n^*-1}(1 + o(1))\psi^2 + \psi\mathcal{O}(e^{-\beta t}) \ge 0.$$

Hence, for some positive constant C, we have

$$\psi'' - \frac{n^2}{4}\psi + n^*\overline{w}(t)^{n^*-1}(1+o(1))\psi > -Ce^{-\beta t} \tag{2.20}$$

for t large and $\psi(t) > 0$.

Letting $y(t) = \overline{w}(t)^{\frac{-n}{n-2}}\psi(t)$, it follows from (2.2), (2.14), (2.15), and (2.20) that $y(t)$ satisfies

$$Ly := y'' + b(t)y' - a(t)y > -Ce^{-\beta t} \quad \text{for } t \text{ large and } y(t) > 0, \tag{2.21}$$

where C is a positive constant,

$$b(t) = \frac{2n}{n-2}\frac{\overline{w}'(t)}{\overline{w}(t)},$$

and

$$a(t) = \frac{2n}{(n-2)^2}\frac{1}{\overline{w}(t)^2}\left[\frac{-2}{\sigma_{n-1}}Q(w) + o(1)\right] \quad \text{as} \quad t \to \infty.$$

This remarkable change of variables, which is due to Mazzeo, Pollack, and Uhlenbeck [12], has the desirable feature of transforming (2.20) into a differential inequality whose zero order term has a negative coefficient. Hence the differential operator L on the left side of (2.21) satisfies the maximum principle. We are grateful to Rafe Mazzeo for bringing this change of variables to our attention. Note also, by (2.2) and (2.10), that

$$\lim_{t\to\infty} y(t) = 0.$$

We now prove that $y(t)$, and hence $\psi(t)$, tends to zero exponentially as $t \to \infty$. Choose positive constants a_0, b_0, and t_0 such that for $t \geq t_0$ we have $a(t) > a_0$ and $|b(t)| < b_0$. Choose $\varepsilon \in (0,\beta)$ such that $-\varepsilon^2 - \varepsilon b_0 + a_0 > 0$. By increasing t_0, we can assume

$$-\varepsilon^2 - \varepsilon b_0 + a_0 > Ce^{(\varepsilon-\beta)t} \quad \text{for} \quad t \geq t_0.$$

Choose $A > 1$ such that $Ae^{-\varepsilon t_0} > y(t_0)$ and let $\hat{y}(t) = Ae^{-\varepsilon t}$. Then $\hat{y}(t_0) > y(t_0)$ and for $t \geq t_0$ we have

$$\begin{aligned}
-L\hat{y} &= Ae^{-\varepsilon t}(-\varepsilon^2 + \varepsilon b(t) + a(t)) \\
&> Ae^{-\varepsilon t}(-\varepsilon^2 - \varepsilon b_0 + a_0) \\
&> Ae^{-\varepsilon t}Ce^{(\varepsilon-\beta)t} > Ce^{-\beta t}.
\end{aligned}$$

Hence, letting $z(t) = y(t) - \hat{y}(t)$, we have $z(t_0) < 0$, $\lim_{t\to\infty} z(t) = 0$, and $Lz > 0$ for $t \geq t_0$ and $z(t) > 0$. This implies $z(t) \leq 0$ for $t \geq t_0$, for otherwise z would assume a positive maximum at some $t_1 > t_0$ and substituting $t = t_1$ in $Lz > 0$ we would obtain a contradiction. Thus $y(t)$, and hence $\psi(t)$, is $\mathcal{O}(e^{-\varepsilon t})$ as $t \to \infty$, which implies

$$\|W\|_{L^2(\Omega_t^2)} = \mathcal{O}(e^{-\varepsilon t}) \quad \text{as} \quad t \to \infty, \tag{2.22}$$

where

$$\Omega_t^a = (t-a, t+a) \times S^{n-1}.$$

Since $w^{n^*} - \overline{w}^{n^*} = n^*\xi^{n^*-1}(w - \overline{w})$, where $\xi(t,\theta)$, being between $w(t,\theta)$ and $\overline{w}(t)$, is bounded, we have

$$\|w^{n^*} - \overline{w}^{n^*}\|_{L^p(\Omega_t^a)} \leq C\|W\|_{L^p(\Omega_t^a)} \quad \text{for} \quad p > 1,\ t \geq 4,\ 1 \leq a \leq 2$$

where C is a positive constant independent of p, t, and a. Hence

$$\begin{aligned}
\|\widehat{W}\|_{L^p(\Omega_t^a)} &\leq \|\widehat{K}w^{n^*} - \overline{w}^{n^*}\|_{L^p(\Omega_t^a)} + \|\overline{\widehat{K}w^{n^*} - \overline{w}^{n^*}}\|_{L^p(\Omega_t^a)} \\
&\leq 2\|\widehat{K}w^{n^*} - \overline{w}^{n^*}\|_{L^p(\Omega_t^a)} \quad \text{by Jensen's inequality,} \\
&\leq 2\|(\widehat{K}-1)w^{n^*}\|_{L^p(\Omega_t^a)} + 2\|w^{n^*} - \overline{w}^{n^*}\|_{L^p(\Omega_t^a)} \\
&\leq C[e^{-\beta t} + \|W\|_{L^p(\Omega_t^a)}] \quad \text{for} \quad p > 1,\ t \geq 4,\ 1 \leq a \leq 2,
\end{aligned} \tag{2.23}$$

by (2.3), where C is a constant independent of p, t, and a.

Starting with (2.22) and using (2.23) we obtain after a finite number of iterations of standard elliptic theory applied to (2.16) that

$$\|W\|_{C^1(\Omega_t^1)} = \mathcal{O}(e^{-\varepsilon t}) \quad \text{as} \quad t \to \infty. \tag{2.24}$$

Therefore

$$\begin{aligned}
&\overline{w}'^2 - \left(\frac{n-2}{2}\right)^2 \overline{w}^2 + \frac{n-2}{n}\overline{w}^{\frac{2n}{n-2}} \\
&= \frac{2}{\sigma_{n-1}} Q(t, \overline{w}) \\
&= \frac{2}{\sigma_{n-1}} [(Q(t,\overline{w}) - Q(t,w)) + Q(t,w)] \\
&= \frac{2}{\sigma_{n-1}} Q(w) + \mathcal{O}(e^{-\varepsilon t}) \quad \text{as} \quad t \to \infty
\end{aligned}$$

by (2.7). Hence, as noted in [1, p. 291],

$$\overline{w}(t) = w_0(t) + \mathcal{O}(e^{-\varepsilon t}) \quad \text{as} \quad t \to \infty \tag{2.25}$$

where $w_0(t)$ is some periodic solution of (2.4) satisfying

$$w_0'^2 - \left(\frac{n-2}{2}\right)^2 w_0^2 + \frac{n-2}{n} w_0^{\frac{2n}{n-2}} = \frac{2}{\sigma_{n-1}} Q(w) \quad \text{in} \quad \mathbf{R}. \tag{2.26}$$

Thus, by (2.24),

$$\max_{\theta \in S^{n-1}} |w(t,\theta) - w_0(t)| = \mathcal{O}(e^{-\varepsilon t}) \quad \text{as} \quad t \to \infty. \tag{2.27}$$

Since $\psi(t)$ satisfies (2.20), it follows from (2.25) that $\psi(t)$ also satisfies

$$L_0 \psi > -Ce^{-\beta t}, \text{ for } t \text{ large and } \psi(t) > 0, \tag{2.28}$$

where $L_0 = \frac{d^2}{dt^2} + \left(n^* w_0^{n^*-1} - \frac{n^2}{4} + o(1)\right)$.

Lemma 2.4. *For each continuous function h satisfying*

$$h(t) = \mathcal{O}(e^{-\beta t}) \quad \textit{as} \quad t \to \infty,$$

the problem

$$\begin{aligned}
&L_0 \zeta = h \\
&\zeta(t) = \mathcal{O}(e^{-\beta t}) \quad \textit{as} \quad t \to \infty
\end{aligned}$$

has a C^2 solution $\zeta(t)$, which, when $h \equiv 0$, is nontrivial.

Proof. Using the fact that $w_0(t)$ satisfies (2.4) one verifies by direct calculation that a fundamental set of solutions of

$$\zeta'' + \left(n^* w_0^{n^*-1} - \frac{n^2}{4}\right)\zeta = 0$$

is

$$\begin{aligned}
\zeta_1(t) &= e^{-t}\left(-w_0'(t) + \frac{n-2}{2} w_0(t)\right) \\
\zeta_2(t) &= e^{t}\left(w_0'(t) + \frac{n-2}{2} w_0(t)\right).
\end{aligned}$$

These solutions appeared in [12] and [7].

Under the change of variables $x(t) = \binom{x_1(t)}{x_2(t)} = \binom{\zeta(t)}{\zeta'(t)}$, the equation $L_0\zeta = h$ becomes

$$x' = A(t)x + \hat{f}(t,x) + \hat{g}(t),$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -n^* w_0^{n^*-1} + \frac{n^2}{4} & 0 \end{pmatrix}$$
$$\hat{f}(t,x) = \begin{pmatrix} 0 \\ x_1 o(1) \end{pmatrix} \quad \text{and} \quad \hat{g}(t) = \begin{pmatrix} 0 \\ h(t) \end{pmatrix},$$

and a fundamental matrix for $x' = A(t)x$ is

$$X(t) = \begin{pmatrix} \zeta_1(t) & \zeta_2(t) \\ \zeta_1'(t) & \zeta_2'(t) \end{pmatrix} = P(t)e^{Bt},$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P(t) = \begin{pmatrix} e^t\zeta_1(t) & e^{-t}\zeta_2(t) \\ e^t\zeta_1'(t) & e^{-t}\zeta_2'(t) \end{pmatrix}.$$

Note that $P(t)$ is periodic and hence bounded. Define $z(t)$ by $x(t) = P(t)z(t)$. Then

$$z' = Bz + f(t,z) + g(t) \tag{2.29}$$

where

$$f(t,z) = P(t)^{-1}\hat{f}(t, P(t)z) \quad \text{and} \quad g(t) = P(t)^{-1}\hat{g}(t) = \mathcal{O}(e^{-\beta t}).$$

Since $f(t,z) = M(t)z$ for some 2×2 matrix $M(t)$ which tends to 0 as $t \to \infty$, it follows from standard ODE methods (see [5, Chapter 13, Theorem 4.1 and its proof]) that (2.29) has a nontrivial solution $z(t)$ satisfying $|z(t)| = \mathcal{O}(e^{-\beta t})$ as $t \to \infty$. Transforming this solution $z(t)$ back to $\zeta(t)$, we obtain Lemma 2.4. $\square$

Letting $y_0(t) = w_0(t)^{-\frac{n}{n-2}}\psi(t)$, it follows from (2.4), (2.26), and (2.28) that $y_0(t)$ satisfies

$$L_1 y_0 > -Ce^{-\beta t} \qquad \text{for } t \text{ large and } y_0(t) > 0,$$

where C is a positive constant and

$$L_1 = \frac{d^2}{dt^2} + \frac{2n}{n-2}\frac{w_0'(t)}{w_0(t)}\frac{d}{dt} + \frac{2n}{(n-2)^2}\frac{1}{w_0(t)^2}\left[\frac{2}{\sigma_{n-1}}Q(w) + o(1)\right].$$

Furthermore, by (2.10),

$$\lim_{t\to\infty} y_0(t) = 0.$$

It follows from Lemma 2.4 that there exists a nontrivial C^2 solution $y_1(t)$ of $L_1 y_1 = 0$ satisfying

$$y_1(t) = \mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t \to \infty,$$

and that there exists a C^2 solution $y_2(t)$ of $L_1 y_2 = -Ce^{-\beta t}$ satisfying

$$y_2(t) = \mathcal{O}(e^{-\beta t}) \quad \text{as} \quad t \to \infty.$$

Choose $t_0 > 0$ such that $y_1(t_0) \neq 0$ and the zero order term of L_1 is negative for $t \geq t_0$. Choose a constant A such that

$$y_2(t_0) + Ay_1(t_0) > y_0(t_0)$$

and let $y_3(t) = y_2(t) + Ay_1(t)$. Then letting $z(t) = y_0(t) - y_3(t)$ we have $z(t_0) < 0$, $\lim_{t\to\infty} z(t) = 0$, and $L_1 z > 0$ for $t \geq t_0$ and $y_0(t) > 0$. This implies $z(t) \leq 0$ for $t \geq t_0$, for otherwise z would assume a positive maximum at some $t_1 > t_0$ and substituting $t = t_1$ in $L_1 z > 0$ we would obtain a contradiction. Thus $y_0(t)$, and hence $\psi(t)$, is $\mathcal{O}(e^{-\beta t})$ as $t \to \infty$. Hence (2.22) holds with ε replaced with β and the discussion after (2.22) shows that (2.27) holds with ε replaced with β. This completes the proof of Theorem 1. $\square$

3 Proof of Theorem 2 for $n = 3$

In this section, we first assume $u(x) \le C|x|^{-\frac{1}{2}}$ for some $C > 0$ and in step 1 we show that

$$u(x) = \bar{u}(|x|)(1 + \circ(1)).$$

Then in step 2 we show that if u has a non-removable singularity at 0, then u is comparable to $|x|^{-\frac{1}{2}}$. In step 3 we show the proof of $u(x) \le C|x|^{-\frac{n-2}{2}}$ for $n = 3$. The proof in Step 3 is a small modification of the proof in [2].

Step 1: Proof of asymptotic symmetry for $n = 3$

In this step, we show that

$$u = \bar{u}(1 + \circ(1)). \tag{3.1}$$

under the assumption $u(x) \le C|x|^{-\frac{1}{2}}$. Note that (3.1) is much weaker than $u = u_0(1 + \circ(1))$.

Here we quote a well known fact: $u(x) \le C|x|^{-\frac{n-2}{2}}$ implies the following inequalities by a simple rescaling of u and the Harnack inequality:

$$\max_{|x|=r} u(x) \le c_1 \min_{|x|=r} u(x), \qquad |\nabla u(x)| \le c_1 |x|^{-\frac{n}{2}}, \quad 0 < r < \frac{1}{2}$$

for some $c_1 > 0$ independent of r (See [3]).

Suppose (3.1) is not satisfied for $n = 3$. Then there exists a sequence $r_i \to 0$ as $i \to \infty$ and $\epsilon_0 > 0$ such that

$$\max_{|x|=r_i} u(x) \ge (1 + \epsilon_0) \min_{|x|=r_i} u(x). \tag{3.2}$$

For those r_i in (3.2) we have

Lemma 3.1. *There exists $C_0 > 0$ independent of i such that*

$$\bar{u}(r_i) r_i^{\frac{1}{2}} \ge C_0.$$

Proof of Lemma 3.1: If $\bar{u}(r_i) r_i^{\frac{1}{2}} \to 0$, let $\tilde{v}_i(y) = r_i^{\frac{1}{2}} u(r_i y)$ for $0 < |y| < 1/r_i$. By the spherical Harnack inequality we have $\tilde{v}_i(e) \to 0$ where $e = (1, 0, 0)$. Let $\hat{v}_i(y) = \tilde{v}_i(y)/\tilde{v}_i(e)$, then the equation that $\hat{v}_i$ satisfies is

$$\Delta \hat{v}_i(y) + K(r_i y) \tilde{v}_i(e)^4 \hat{v}_i(y)^5 = 0 \qquad 0 < |y| < 1/r_i.$$

By the Harnack inequality, $\hat{v}_i$ is uniformly bounded over any compact subset of $\mathbf{R}^n \setminus \{0\}$. Therefore there exists $\hat{v}$ such that a subsequence of $\hat{v}_i$ converges uniformly to some $\hat{v}$ over any compact subset of $\mathbf{R}^n \setminus \{0\}$. Since $\tilde{v}_i(e) \to 0$, we have

$$\Delta \hat{v}(y) = 0 \qquad \text{in} \quad \mathbf{R}^n \setminus \{0\}.$$

Since $\hat{v}$ is non-negative, $\hat{v}$ is of the form

$$\hat{v}(y) = a|y|^{-1} + b, \qquad a, b \ge 0.$$

However, since $\hat{v}$ is radially symmetric and $\hat{v}_i$ converges to $\hat{v}$ over $|y| = 1$, it is impossible to have (3.2). Lemma 3.1 is established. □

In the following argument we shall rescale u at r_i, then the key point of our argument is that we can find standard bubbles which are important for the method of moving spheres. For $n = 3$, the method of moving spheres does not require too much on the smoothness of K, so we have the freedom to shift the center of the standard bubble along any direction we want. In the argument that follows, we will choose one direction to shift the center of the standard bubble for a large distance and then we shall apply the method of moving spheres to get a contradiction.

Let

$$v_i(y) = r_i^{\frac{1}{2}} u(r_i y), \qquad 0 < |y| < r_i^{-1}.$$

Since $v_i(y)|y|^{\frac{1}{2}} \leq C$ for y in any compact subset of $\mathbf{R}^3$. We know that $\{v_i\}$ converge uniformly to some function v in $\mathbf{R}^3 \setminus \{0\}$. This v satisfies

$$\Delta v(y) + K_0 v(y)^5 = 0 \qquad \text{in} \quad \mathbf{R}^3 \setminus \{0\}.$$

With no loss of generality we assume $K_0 = n(n-2) = 3$. By a well known result of Caffarelli, Gidas and Spruck [1] this v can not have a singularity at the origin, because otherwise v will be radially symmetric, which will lead to a violation of (3.2). So v is defined globally. By the famous classification theorem of Caffarelli, Gidas and Spruck [1], v has to be a standard bubble. i.e.

$$v(y) = \Big(\frac{\mu}{1+\mu^2|y-R_0 e|^2}\Big)^{\frac{1}{2}}$$

where μ is a positive constant and we choose the coordinate system so that $R_0 > 0$ and $e = (1,0,0)$. Note that R_0 can not be zero since v is not radially symmetric.

Let

$$g_i(y) = v_i(y + (R+R_0)e)$$

where R will be determined to be a large positive number later. Since v_i is not defined at the origin, g_i is not defined at $-(R+R_0)e$. Let

$$v_1(y) = v(y+(R+R_0)e) = \Big(\frac{\mu}{1+\mu^2|y+Re|^2}\Big)^{\frac{1}{2}}.$$

Then $\{g_i\}$ converge uniformly to v_1 over all compact subsets of $\mathbf{R}^3 - Z$ where $Z = -(R+R_0)e$.

Let

$$v_1^\lambda(y) = \Big(\frac{\lambda}{|y|}\Big)^{n-2} v_1\Big(\frac{\lambda^2 y}{|y|^2}\Big).$$

This v_1^λ is the Kelvin transformation of v_1. From now on we shall use y^λ to denote $\frac{\lambda^2 y}{|y|^2}$.

Remark 3.1. The reason we have to consider g_i, which is a shift v_i by a very large distance, is because we want to stay away from the singularity when we apply the method of moving spheres later. Note that R_0 could be any positive number, but no matter how small R_0 is, as long as R is large enough, we can always start our method of moving spheres from a radius (λ_0) only slightly less than R to a radius (λ_1) slightly bigger than R. When R is sufficiently large, we can make λ_1 as close to R as we want, therefore, the singular point does not cause us trouble any more.

The following lemma is on the difference between v_1 and v_1^λ.

Lemma 3.2. *For $R >> 1$, there exists $c_1(\mu, R) > 0$ such that for $\lambda_0 = R-2$,*

$$\begin{aligned} v_1(y) - v_1^{\lambda_0}(y) &\geq c_1(|y|^2 - \lambda_0^2)|y|^{-3} \quad \text{for} \ \ |y| \geq \lambda_0 \\ \frac{\partial(v_1(y) - v_1^{\lambda_0}(y))}{\partial \nu} &> c_1 > 0 \qquad \text{on} \ \ \partial B_{\lambda_0} \end{aligned}$$

where ν stands for the unit outer normal vector of ∂B_{λ_0}. For $\lambda_1 = R + R_0/5$,

$$v_1(x) - v_1^\lambda(x) < 0 \quad \text{for} \quad |x| > \lambda_1.$$

Proof: Using the formula

$$\left(\frac{|y|}{\lambda}|y^\lambda - \eta|\right)^2 - |y-\eta|^2 = \frac{(|y|^2-\lambda^2)(|\eta|^2-\lambda^2)}{\lambda^2} \qquad y, \eta \in \mathbf{R}^n, \quad y \neq 0, \tag{3.3}$$

which follows from the law of cosines, we find using the mean value theorem that

$$\begin{aligned} v_1(y)-v_1^\lambda(y) &= (\frac{\mu}{1+\mu^2|y+Re|^2})^{1/2}-(\frac{\mu}{(\frac{|y|^2}{\lambda^2})(1+\mu^2|y^\lambda+Re|^2})^{1/2} \\ &= \frac{1}{2}\mu^{-1/2}\cdot\frac{(|y|^2-\lambda^2)(\mu^{-2}+R^2-\lambda^2)}{\lambda^2\xi^{3/2}} \end{aligned} \tag{3.4}$$

Where ξ is some number between $\mu^{-2}+|y+Re|^2$ and $(\frac{|y|}{\lambda})^2(\mu^{-2}+|y^\lambda+Re|)$ so it is not hard to see that $|\xi|\le c_3|y|^2$ when $\lambda=\lambda_0$ for some $c_3(\mu,R)>0$. Consequently there exists $c_4(R,\mu)>0$ such that

$$v_1(y)-v_1^{\lambda_0}(y)\ge c_4(|y|^2-\lambda_0^2)|y|^{-3}.$$

If R is sufficiently big, for $\lambda=\lambda_1$, we have $v_1<v_1^{\lambda_1}$ for $|y|>\lambda_1$. The last statement is proved. The boundary derivative estimate is a result of the Hopf Lemma since both v_1 and $v_1^{\lambda_0}$ satisfy the same equation over $\mathbf{R}^3\setminus\bar{B}_{\lambda_0}$ and $v_1=v_1^{\lambda_0}$ on ∂B_{λ_0}. Lemma 3.2 is established. $\square$

Now we determine $R>>1$ to be the one in Lemma 3.2. From now on in this step we always assume λ to stay in $[\lambda_0,\lambda_1]$. Also in this step we define

$$\Sigma_\lambda=B(0,\delta r_i^{-1})\setminus\bar{B}_\lambda$$

with $\delta>0$ to be determined later.

Note that the singularity of g_i is at $Z=-(R+R_0)e$ and the peak of v_1 is at $-Re$. g_i converge to v_1 over all the compact subsets of $\mathbf{R}^3\setminus\{Z\}$. The equation that g_i satisfies is

$$\Delta g_i(y)+K(r_iy+r_i(R+R_0)e)g_i(y)^5=0,\qquad |y|<r_i^{-1}/2,\quad y\neq Z.$$

Let g_i^λ be defined as the Kelvin transformation of g_i with respect to $B(0,\lambda)$. Then the equation for g_i^λ is

$$\Delta g_i^\lambda(y)+K(r_iy^\lambda+r_i(R+R_0)e)g_i^\lambda(y)^5=0,\qquad |y|>\lambda.$$

We are going to compare g_i and g_i^λ for $\lambda\in[\lambda_0,\lambda_1]$. Note that g_i^λ is well defined on $|y|>\lambda$ for $\lambda\in[\lambda_0,\lambda_1]$. Now we have the following lemma on the difference between g_i and g_i^λ.

Lemma 3.3. *There exist $\epsilon_0(R,R_0,\mu)>0$ and $i_0(R,\delta)>1$ such that for $i\ge i_0$*

$$g_i(y)-g_i^{\lambda_0}(y)\ge\epsilon_0(|y|-\lambda_0)|y|^{-2}+\epsilon_0r_i^{1/2}(\lambda_0^{-1}-|y|^{-1}),\quad \lambda_0<|y|<r_i^{-1}/2,y\neq-(R+R_0)e.$$

Moreover, there exists $z_1\in\Sigma_{\lambda_1}\setminus\{-(R+R_0)e\}$ independent of i such that

$$g_i(z_1)<g_i^{\lambda_1}(z_1)-\epsilon_0.$$

Proof of Lemma 3.3: Since $\{g_i\}$ converge uniformly to v_1 over any compact subset of $\mathbf{R}^3\setminus\{-(R+R_0)e\}$, we see from Lemma 3.2 that for any fixed large $R_1>>R$, we have

$$g_i(y)-g_i^{\lambda_0}(y)>2\epsilon_2(|y|-\lambda_0)|y|^{-2},\quad |y+(R+R_0)e|>R_1^{-1},\quad |y|<R_1$$

for sufficiently large i. Note that ϵ_2 is determined only by v_1 and $v_1^{\lambda_0}$ and at this moment we stay carefully away from the singularity. Now for $|y+(R+R_0)e|<R_1^{-1}$, since $g_i^{\lambda_0}$ is well defined there, we have

$$|g_i^{\lambda_0}(y)-g_i^{\lambda_0}(y_i^*)|<\epsilon_2/10$$

where $y_i^*\in\partial B(-(R+R_0)e,R_1^{-1})$ so that $g_i(y_i^*)=\min_{\partial B(-(R+R_0)e,R_1^{-1})}g_i$. This is true because R_1 is large. Then by the super harmonicity of g_i, $g_i(y)\ge g_i(y_i^*)$ for any $y\in B(-(R+R_0)e,R_1^{-1})$, $y\neq-(R+R_0)e$. Therefore for $|y|<R_1$ and $y\neq-(R+R_0)e$ we have

$$g_i(y)-g_i^{\lambda_0}(y)>\frac{3}{2}\epsilon_2(|y|-\lambda_0)|y|^{-2}.$$

The conclusion holds on this region. For $R_1 \le |y| \le \delta r_i^{-1}$ we take advantage of the super harmonicity of g_i. First we observe that

$$g_i(y) \ge m r_i^{\frac{1}{2}} \qquad \text{on} \quad |y| = \delta r_i^{-1}$$

where m is the minumum of u over $B_1 \setminus \{0\}$ and δ is any fixed constant. So $g_i(y) >> |y|^{-1}$ on $|y| = \delta r_i^{-1}$ for all large i.

Secondly by the definition of g_i^λ, we always have

$$g_i^{\lambda_0}(y) \le (\mu^{-\frac{1}{2}} - 2\epsilon)|y|^{-1} \quad \text{for} \quad |y| > R_1$$

for R_1 sufficiently large and some $\epsilon > 0$. On the other hand,

$$g_i(y) \ge (\mu^{-\frac{1}{2}} - \epsilon)|y|^{-1} \quad |y| = R_1.$$

So by the super harmonicity of g_i we have

$$g_i(y) \ge (\mu^{-\frac{1}{2}} - \frac{3}{2}\epsilon)|y|^{-1} + \epsilon r_i^{\frac{1}{2}}(\lambda_0^{-1} - |y|^{-1}) \quad R_1 \le |y| \le \delta r_i^{-1}, y \ne -(R+R_0)e.$$

Lemma 3.3 is implied by the above inequality by choosing ϵ_0 small enough. $\Box$

Let

$$w_\lambda = g_i(y) - g_i^\lambda(y).$$

Observe that for $\lambda \in [\lambda_0, \lambda_1]$,$g_i^\lambda$ is smooth in $B(0, r_i^{-1}/2) \setminus \overline{B_\lambda}$.

Now consider the equation that w_λ satisfies,

$$\Delta w_\lambda + b_\lambda w_\lambda = Q_\lambda$$

where

$$b_\lambda = K(r_i y + r_i(R+R_0)e)\frac{g_i(y)^5 - g_i^\lambda(y)^5}{g_i(y) - g_i^\lambda(y)}$$

and

$$Q_\lambda(y) = (K(r_i y^\lambda + r_i(R+R_0)e) - K(r_i y + r_i(R+R_0)e))g_i^\lambda(y)^5.$$

Remark 3.2. Later we shall apply the method of moving spheres to function $w_\lambda + h_\lambda$, where h_λ will be a very important test function to construct. The reason we need h_λ is that w_λ does not satisfy the maximum principle for all $\lambda \in [\lambda_0, \lambda_1]$. h_λ is a "perturbation" of w_λ that makes the method of moving spheres work for $w_\lambda + h_\lambda$. The main feature of h_λ is to satisfy the following inequality:

$$\Delta h_\lambda + b_\lambda h_\lambda + Q_\lambda \le 0 \qquad \text{in} \quad \Sigma_\lambda.$$

This is always satisfied by making

$$\Delta h_\lambda + Q_\lambda \le 0 \quad \text{and} \quad h_\lambda \le 0 \quad \text{in} \quad \Sigma_\lambda.$$

The construction of h_λ involves the Green's function $G^\lambda(y,\eta)$ for the Laplacian in $\mathbf{R}^n \setminus \bar{B}_\lambda$ which is given by

$$G^\lambda(y,\eta) = \frac{1}{n(n-2)\omega_n}(|y-\eta|^{2-n} - (\frac{\lambda}{|y|})^{n-2}|\frac{\lambda^2 y}{|y|^2} - \eta|^{2-n}) \tag{3.5}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$.

In the future the estimate of h_λ will rely heavily on the estimate of the Green's function. So we give an estimate of the Green's function first. The information at ∂B_λ is traced carefully.

Lemma 3.4. *There exists $c_1(n) > 0$ such that for $|\eta - y| < \frac{1}{3}(|y| - \lambda)$, we have*

$$G^\lambda(y,\eta) \ge c_1 |y-\eta|^{2-n}. \tag{3.6}$$

For $|\eta - y| \ge \frac{1}{3}(|y| - \lambda)$ and $|y| \le 10\lambda$, there exist constants $c_2(n), c_3(n) > 0$ such that

$$c_2 \frac{(|y| - \lambda)(|\eta|^2 - \lambda^2)}{\lambda |y-\eta|^n} \le G^\lambda(y,\eta) \le c_3 \frac{(|y| - \lambda)(|\eta|^2 - \lambda^2)}{\lambda |y-\eta|^n}. \tag{3.7}$$

Moreover, if we only assume $\lambda < |y| < 10\lambda$ and $|\eta| > \lambda$, we have the second inequality of (3.7).

Remark 3.3. By the self-adjointness of $G^\lambda(y,\eta)$ we can reverse the roles of y and η in Lemma 3.4 and the conclusions still hold.

Proof of Lemma 3.4: If $|\eta-y|<\frac{1}{3}(|y|-\lambda)$ then $|\eta-y^\lambda|>2|y-\eta|$. Therefore

$$G^\lambda(y,\eta)\ge \frac{1}{n(n-2)\omega_n}(|y-\eta|^{2-n}-2^{2-n}|y-\eta|^{2-n})\ge c_1|y-\eta|^{2-n}$$

which proves (3.6). When $|\eta-y|\ge\frac{1}{3}(|y|-\lambda)$ and $\lambda\le|y|<10\lambda$ we have

$$\frac{|y|}{\lambda}|y^\lambda-\eta|<10(|\eta-y|+|y-y^\lambda|)<70|\eta-y|.$$

On the other hand, we always have

$$\frac{|y|}{\lambda}|y^\lambda-\eta|>|\eta-y|\qquad\text{for}\qquad y,\eta\in\mathbf{R}^n\setminus\bar{B}_\lambda. \tag{3.8}$$

because the Green's function is positive or by (3.3). Consequently

$$\begin{aligned}
G^\lambda(y,\eta) &= \frac{1}{n(n-2)\omega_n}(|y-\eta|^{2-n}-(\frac{|y|}{\lambda}|y^\lambda-\eta|)^{2-n})\\
&\ge \frac{1}{n\omega_n}\frac{\frac{|y|}{\lambda}|y^\lambda-\eta|-|y-\eta|}{(\frac{|y|}{\lambda}|y^\lambda-\eta|)^{n-1}}\\
&\ge \frac{1}{n70^{n-1}\omega_n}\frac{(\frac{|y|}{\lambda}|y^\lambda-\eta|)^2-|y-\eta|^2}{|y-\eta|^{n-1}(\frac{|y|}{\lambda}|y^\lambda-\eta|+|y-\eta|)}\\
&\ge \frac{1}{n71^n\omega_n}\frac{(\frac{|y|}{\lambda}|y^\lambda-\eta|)^2-|y-\eta|^2}{|y-\eta|^n}\\
&= \frac{1}{n71^n\omega_n}\frac{(|y|^2-\lambda^2)(|\eta|^2-\lambda^2)}{\lambda^2|y-\eta|^n}\ge c_2\frac{(|y|-\lambda)(|\eta|^2-\lambda^2)}{\lambda|y-\eta|^n}.
\end{aligned}$$

where we have used (3.3). So we have proved the first inequality of (3.7). To prove the second, we only need to reverse all the inequalities in the above in view of (3.8). Lemma 3.4 is established. □

We compare g_i and g_i^λ on the boundary $|y|=\delta r_i^{-1}$. For g_i we have $g_i(y)\ge mr_i^{1/2}$. For g_i^λ we have

$$g_i^\lambda(y)\le C\delta^{-1}r_i,\qquad |y|=\delta r_i^{-1}.$$

No matter how small δ is, we always have $g_i(y)>>g_i^\lambda(y)$ for $|y|=\delta r_i^{-1}$ as long as i is sufficiently large.

To construct h_λ we need the following estimate: By the assumption on K,

$$|K(r_iy^\lambda+r_i(R+R_0)e)-K(r_iy+r_i(R+R_0)e)|\le c(\delta)r_i^{1/2}|y|^{1/2},\qquad |y|<\delta r_i^{-1}.$$

Then

$$|Q_\lambda(y)|\le c(\delta)r_i^{1/2}|y|^{-4.5},\quad y\in\Sigma_\lambda\setminus\{-(R+R_0)e\}. \tag{3.9}$$

Here we abuse the notation by using $c(\cdot)$ to indicate a non negative continuous function that vanishes at 0.

Define

$$h_\lambda(y)=\epsilon_1r_i^{1/2}(|y|^{-1}-\lambda^{-1})+\int_{\Sigma_\lambda}G^\lambda(y,\eta)Q_\lambda(\eta)d\eta,$$

We want ϵ_1 and δ to be chosen so that $\epsilon_0>>\epsilon_1>>c(\delta)$ where ϵ_0 appeared in Lemma 3.3.

From the definition we see that

$$\Delta h_\lambda+Q_\lambda=0\quad\text{in}\quad\Sigma_\lambda.$$

We need to show that $h_\lambda\le 0$ in Σ_λ.

It follows immediately from (3.9) that

$$\int_{\Sigma_\lambda} G^\lambda(y,\eta)Q_\lambda(\eta)d\eta \le c(\delta)M_i^{-1}\int_{\Sigma_\lambda} G^\lambda(y,\eta)|\eta|^{-4.5}d\eta.$$

To estimate the term on the right we discuss two situations:

Situation 1: $\lambda < |y| < 4\lambda$.

Let

$$\begin{array}{rcl} \Omega_a &=& \{\eta\in\Sigma_\lambda;\quad |\eta-y| < (|y|-\lambda)/3 \quad \}. \\ \Omega_b &=& \{\eta\in\Sigma_\lambda;\quad |\eta-y| > (|y|-\lambda)/3 \quad \text{and} \quad |\eta| < 8\lambda\}. \\ \Omega_c &=& \{\eta\in\Sigma_\lambda;\quad |\eta| > 8\lambda\}. \end{array}$$

Then we use

$$G^\lambda(y,\eta) \le \frac{1}{3\omega_3}|y-\eta|^{-1}, \qquad \eta\in\Omega_a$$

to obtain

$$\int_{\Omega_a} G^\lambda(y,\eta)|\eta|^{-4.5}d\eta \le c_6(|y|-\lambda).$$

where $c_6 = c_6(n)$. Then for $\eta\in\Omega_b$ we use (3.7) to get

$$\int_{\Omega_b} G^\lambda(y,\eta)|\eta|^{-4.5}d\eta \le c_7(|y|-\lambda),$$

where $c_7 = c_7(n)$. Note that in the estimate above, we used the estimates for $G^\lambda(y,\eta)$ in Lemma 3.4 and the fact that

$$|\eta|-\lambda| \le 4|\eta-y|, \qquad \text{for} \quad \eta\in\Omega_b.$$

Next for $\eta\in\Omega_c$ we use (3.7) and $|y-\eta| > |\eta|/2$ to get

$$\int_{\Omega_c} G^\lambda(y,\eta)|\eta|^{-4.5}d\eta \le c_8(|y|-\lambda).$$

where $c_8 = c_8(n)$.

Situation 2: $|y| > 4\lambda$**.**

Let

$$\begin{array}{rcl} E_1 &=& \{\quad \eta\in\Sigma_\lambda;\quad |\eta| < |y|/2, \quad \}. \\ E_2 &=& \{\quad \eta\in\Sigma_\lambda;\quad |\eta-y| < |y|/2, \quad \}. \\ E_3 &=& \{\quad \eta\in\Sigma_\lambda;\quad |\eta-y| > |y|/2, \quad |y|/2 < |\eta| < 2|y|.\}. \\ E_4 &=& \{\quad \eta\in\Sigma_\lambda;\quad |\eta| > 2|y|. \quad \}. \end{array}$$

In this situation we always use

$$G^\lambda(y,\eta) \le \frac{1}{3\omega_3}|y-\eta|^{-1}.$$

Then by elementary estimates we have

$$\int_{E_j} G^\lambda(y,\eta)|\eta|^{-4.5}d\eta \le c_{10}|y|^{-1}, \quad j=1,2,3,4,$$

where $c_{10} = c_{10}(n)$.

Therefore there exists $c_{11} = c_{11}(n)$ such that

$$h_\lambda(y) \le \begin{cases} \epsilon_1 r_i^{\frac{1}{2}}(|y|^{-1}-\lambda^{-1}) + c_{11}c(\delta)r_i^{\frac{1}{2}}(|y|-\lambda), & \lambda < |y| < 4\lambda, \\ \epsilon_1 r_i^{\frac{1}{2}}(|y|^{-1}-\lambda^{-1}) + c_{11}c(\delta)r_i^{\frac{1}{2}}, & |y| > 4\lambda. \end{cases}$$

So for any fixed ϵ_1 we can determine δ to be small enough so that $h_\lambda < 0$ in Σ_λ.

Then the standard method of moving sphere method can be applied here to finish the proof of the asymptotic symmetry for $n = 3$. First for $\lambda = \lambda_0$, by the estimate of h_λ and Lemma 3.3 we see that $w_{\lambda_0} + h_{\lambda_0} \geq 0$ in Σ_{λ_0}. Then we consider the equation for $w_\lambda + h_\lambda$:

$$\Delta(w_\lambda + h_\lambda) + b_\lambda(w_\lambda + h_\lambda) = Q_\lambda + \Delta h_\lambda + b_\lambda h_\lambda \quad \text{in} \quad \Sigma_\lambda \setminus Z_i.$$

where Z_i stands for the singular point $-(R+R_0)e$. Since $h_\lambda \leq 0$ and $\Delta h_\lambda + Q_\lambda \leq 0$ in Σ_λ, $w_\lambda + h_\lambda$ satisfies the maximum principle for all $\lambda \in [\lambda_0, \lambda_1]$. Also we observe from the estimate of h_λ that $w_\lambda + h_\lambda$ is always positive on $|y| = \delta r_i^{-1}$. Let

$$\bar{\lambda}_i = \sup\{\lambda_1 \geq \lambda \geq \lambda_0; \quad w_\mu + h_\mu \geq 0 \quad \text{in} \quad \Sigma_\mu \setminus Z_i, \forall \mu \in [\lambda_0, \lambda) \quad \}.$$

On one hand, by the fact that $w_\lambda + h_\lambda$ always satisfies the maximum principle and the fact that it is always positive on $|y| = \delta r_i^{-1}$ we know that $\bar{\lambda}_i = \lambda_1$. On the other hand, by Lemma 3.3 we know $\bar{\lambda}_i < \lambda_1$ since $w_{\lambda_1} + h_{\lambda_1} < -\frac{\epsilon_0}{2}$ for all large i. This contradiction finishes the proof for the asymptotic symmetry for $n = 3$.

Step 2: If u has a non-removable singularity, then u is comparable to $|x|^{-\frac{1}{2}}$

First we cite a few known facts that are true for all dimension $n \geq 3$ if $u(x) \leq C|x|^{-\frac{n-2}{2}}$ is assumed.

Let $w(t) = \bar{u}(r)r^{\frac{n-2}{2}}$ where $e^t = r$ and $\bar{u}$ is the spherical average of u. Then standard computation leads to (see Chen-Lin [3], [4]):

$$(\frac{n-2}{2})^2 w - C_1 w^{n^*} \leq w_{tt} \leq (\frac{n-2}{2})^2 w - C_2 w^{n^*}. \tag{3.10}$$

Here we cite two lemmas in [3], [4]:

Lemma 3.5. *There exists $\epsilon_1 > 0$ such that $w_{tt} > 0$ whenever $w(t) \leq \epsilon_1$.*

Proof of Lemma 3.5: The proof follows immediately from (3.10). $\square$.

Lemma 3.6. *By making ϵ_1 smaller if necessary, we have the following:*

1. *Suppose $w(t)$ is non-increasing in (t_0, t_1) with $w(t_0) \leq \epsilon_1$. Then the inequality*

$$t_1 - t_0 \leq \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)} + C\epsilon_1^{\frac{4}{n-2}} \tag{3.11}$$

holds where $C(n)$ is a constant independent of ϵ_1. Further more, if t_1 is a local minimum of $w(t)$, we have

$$t_1 - t_0 \geq \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)}.$$

2. *Suppose $w(t)$ is nondecreasing in (t_1, t_2) with $w(t_2) \leq \epsilon_1$. Then the inequality*

$$t_2 - t_1 \leq \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)} + C\epsilon_1^{\frac{4}{n-2}} \tag{3.12}$$

holds where $C(n)$ is a constant independent of ϵ_1. Further more, if t_1 is a local minimum of $w(t)$, we have

$$t_2 - t_1 \geq \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)}.$$

Remark 3.4. Lemma 3.6 stated here is slightly more precise than Lemma 5.1 in [3] and (2.5)(2.6) in [4]. The reason is we specify that the last term in (3.11) and (3.12) is small if ϵ_1 is small. This observation is crutial for $n = 3$.

Lemma 3.6 is important for our argument. First we can see from its statement that it describes the fluctuation of $w(t)$. Secondly, it describes the relationship between $w(t_0)$ and $w(t_1)$ and indicates that $t_1 - t_0$ tends to infinity if $w(t_1) \to 0$ and $w(t_0) = \epsilon$. This is a picture of pathology since this is not the case a standard bubble (with no singularity) or u_0 (a global solution with a singularity) should satisfy. Moreover, we have seen that at a local minimum, u is very close to its spherical average. These facts will be taken advantage in our argument in an essential way. Thirdly, this lemma gives a rather precise description of u between $\log t_0$ and $\log t_1$. This has been used by Chen-Lin in deriving a contradiction for high dimensions. They used this description to get a contradiction from the Pohozaev Identity. However, for $n = 3$, our assumption on K does not allow us to use the Pohozaev Identity. We shall use a different argument.

Now we finish our proof of Theorem 2 for $n = 3$. We have known that $w(t) \leq C$ for all $t < 0$. If $\liminf_{t \to -\infty} w(t) > 0$, then we can apply Theorem 1 directly to obtain the convergent rate. So here we assume $\liminf_{t \to -\infty} w(t) = 0$ and we want to show that $\limsup_{t \to -\infty} w(t) = 0$. Once we have this, a standard argument shows that u has a removable singularity at the origin. See page 238 of [4].

Suppose for contradiction $\limsup_{t \to -\infty} w(t) > 0$. Then by Lemmas 3.5 and 3.6, we can find a sequence $r_i \to 0$ such that $w(\log r_i) \to 0$ and $w(\log r_i)$ are local minimums of w. Also, let r_i^* be the point on the left of r_i such that $w(\log r_i^*) = \epsilon_1$ and w is monotone decreasing on $(\log r_i^*, \log r_i)$. Let $\bar{r}_i$ be the point that $w(\log \bar{r}_i) = \epsilon_1$ and $w(\cdot)$ is monotone increasing over $(\log r_i, \log \bar{r}_i)$. For the convenience of notation, we define $t_i^* = \log r_i^*$, $t_i = \log r_i$, $\bar{t}_i = \log \bar{r}_i$.

The idea of this proof is the following. u decreases like a harmonic function on $r_i^* \leq r \leq r_i$. However, $\bar{u}(r_i)$ and $\bar{u}(\bar{r}_i)$ are pretty close. By Lemma 3.6 both $t_i - t_i^*$ and $\bar{t}_i - t_i$ are tending to infinity, which is where the problem is. So we construct a harmonic function slightly smaller than u on r_i^* and $\bar{r}_i$. This function will be greater than $u(r_i)$ at $r = r_i$, which violates the maximum principle. The asymptotic symmetry $u = \bar{u}(1 + O(r))$ and the smallness of ϵ_1 are important here.

By Lemma 3.6 we have

$$2 \log \frac{w(t_i^*)}{w(t_i)} \leq t_i - t_i^* \leq 2 \log \frac{w(t_i^*)}{w(t_i)} + C\epsilon_1, \tag{3.13}$$

$$2 \log \frac{w(\bar{t}_i)}{w(t_i)} \leq \bar{t}_i - t_i \leq 2 \log \frac{w(\bar{t}_i)}{w(t_i)} + C\epsilon_1, \tag{3.14}$$

Recall that C is independent of ϵ_1 and i. We derive from (3.13) and (3.14) that

$$\bar{u}(r_i^*) r_i^* r_i^{-1} \leq \bar{u}(r_i) \leq e^{C\epsilon_1} \bar{u}(r_i^*) r_i^* r_i^{-1}, \tag{3.15}$$

$$\bar{u}(\bar{r}_i) \leq \bar{u}(r_i) \leq e^{C\epsilon_1} \bar{u}(\bar{r}_i). \tag{3.16}$$

Now we define a harmonic function H over $r_i^* \leq |x| \leq \bar{r}_i$.

$$H(x) = a_i |x|^{-1} + b_i$$

where

$$a_i = (1 - \epsilon_1) \frac{\bar{u}(r_i^*) - \bar{u}(\bar{r}_i)}{(r_i^*)^{-1} - (\bar{r}_i)^{-1}}$$

$$b_i = (1 - \epsilon_1) \frac{\bar{u}(\bar{r}_i)(r_i^*)^{-1} - \bar{u}(r_i^*)(\bar{r}_i)^{-1}}{(r_i^*)^{-1} - (\bar{r}_i)^{-1}}$$

By direct computation

$$H(r_i^*) = (1 - \epsilon_1)\bar{u}(r_i^*), \qquad H(\bar{r}_i) = (1 - \epsilon_1)\bar{u}(\bar{r}_i).$$

Since $u = \bar{u}(1 + \circ(1))$, we know for sufficiently large i,

$$u(x) > H(x), \qquad \text{for} \quad |x| = r_i^* \quad \text{and} \quad |x| = \bar{r}_i.$$

Since u is super-harmonic, we have

$$u(x) > H(x), \qquad |x| = r_i.$$

Consequently,

$$(1 + \epsilon_1)\bar{u}(r_i) > H(r_i).$$

By the definition of H we have

$$\frac{1+\epsilon_1}{1-\epsilon_1}\bar{u}(r_i) > \frac{\bar{u}(r_i^*)r_i^{-1} - \bar{u}(\bar{r}_i)r_i^{-1} + \bar{u}(\bar{r}_i)(r_i^*)^{-1} - \bar{u}(r_i^*)(\bar{r}_i)^{-1}}{(r_i^*)^{-1} - (\bar{r}_i)^{-1}}$$

which is equivalent to

$$\frac{1+\epsilon_1}{1-\epsilon_1}\frac{\bar{u}(r_i)}{\bar{u}(r_i^*)}\frac{r_i}{r_i^*} + \frac{\bar{u}(\bar{r}_i)}{\bar{u}(r_i^*)} + \frac{r_i}{\bar{r}_i} > \frac{1+\epsilon_1}{1-\epsilon_1}\frac{\bar{u}(r_i)}{\bar{u}(r_i^*)}\frac{r_i}{\bar{r}_i} + 1 + \frac{\bar{u}(\bar{r}_i)}{\bar{u}(r_i^*)}\frac{r_i}{r_i^*}.$$

It follows from Lemma 3.6 that the second term, the third term on the left and the first term on the right tend to 0 as i tends to ∞.This, combined with (3.15) and (3.16), gives

$$\frac{1+\epsilon_1}{1-\epsilon_1}e^{C\epsilon_1} + \circ(1) + \circ(1) > \circ(1) + 1 + e^{-C\epsilon_1}.$$

Clearly this is impossible for ϵ_1 sufficiently small. So we have proved $\limsup_{t\to-\infty} w(t) = 0$. So we have shown that if the singularity is not removable, then $u(x)$ is comparable to $|x|^{-\frac{n-2}{2}}$ when $n=3$.

Step 3: $u(x) \le C|x|^{-\frac{1}{2}}$

The proof in [2] can be modified under our assumption on K. So here we only state the outline and pinpoint the changes and important estimates. Also we write down the structure for the proof of

$$u(x) \le C|x|^{-\frac{n-2}{2}} \tag{3.17}$$

for all dimension n.

Suppose for contradiction that (3.17) does not hold, i.e. there is a sequence $\bar{x}_i \to 0$ such that

$$u(\bar{x}_i)|\bar{x}_i|^{\frac{n-2}{2}} \to \infty.$$

Then by a standard selection process and the classification theorem of Caffarelli-Gidas-Spruck, there exists a sequence $x_i \to 0$ which are local maximums of u such that $u(x_i)|x_i|^{\frac{n-2}{2}} \to \infty$ and $u(x_i)^{-1}u(u(x_i)^{-\frac{2}{n-2}}\cdot + x_i)$ converge uniformly on compact subsets of $\mathbf{R}^n$ to a function U satisfying

$$\Delta U + K(0)U^{n^*} = 0 \quad \text{in} \quad \mathbf{R}^n, \qquad U(0) = 1 = \max_{\mathbf{R}^n} U.$$

With no loss of generality we assume $K(0) = n(n-2)$. By the classification Theorem of Caffarelli, Gidas and Spruck, the expression for U is

$$U(y) = (1+|y|^2)^{-\frac{n-2}{2}}.$$

From now on we let $M_i = u(x_i)$.

Let $v_i(y) = M_i^{-1}u_i(M_i^{-\frac{2}{n-2}}y + x_i)$ and let v_i^λ be its Kelvin transformation. Let $w_\lambda = v_i - v_i^\lambda$ then w_λ satisfies

$$\Delta w_\lambda + b_\lambda w_\lambda = Q_\lambda$$

where

$$Q_\lambda = (K(x_i + M_i^{-\frac{2}{n-2}}y^\lambda) - K(x_i + M_i^{-\frac{2}{n-2}}y))(v_i^\lambda)^{n^*}$$

and

$$b_\lambda = \begin{cases} K(x_i + M_i^{-\frac{2}{n-2}}y)\frac{v_i^{n^*} - (v_i^\lambda)^{n^*}}{v_i - v_i^\lambda} & v_i(y) \neq v_i^\lambda(y) \\ \\ \frac{n+2}{n-2}K(x_i + M_i^{-\frac{2}{n-2}}y)v_i(y)^{\frac{4}{n-2}} & v_i(y) = v_i^\lambda(y). \end{cases}$$

Let $\lambda_0 = \frac{1}{2}$ and $\lambda_1 = 2$, we have, like Lemma 3.6

$$v_i(y) - v_i^{\lambda_0}(y) \ge \epsilon_0(|y| - \lambda_0)|y|^{1-n} + \epsilon_0 M_i^{-1}(\lambda_0^{2-n} - |y|^{2-n}) \quad y \in \Sigma_{\lambda_0} \setminus Z_i.$$

Now for $n=3$, by our assumption on K we have

$$Q_\lambda(y) \le c(\delta) M_i^{-1} |y|^{-4.5} \qquad \text{in} \qquad \Sigma_\lambda \tag{3.18}$$

Base on this, we construct h_λ as

$$h_\lambda(y) = \epsilon_1 M_i^{-1} (\frac{1}{|y|} - \frac{1}{\lambda}) + \int_{\Sigma_\lambda} G^\lambda(y,\eta) Q_\lambda(\eta) d\eta.$$

where ϵ_1 is a small positive number.

The method of moving spheres can be applied to $w_\lambda + h_\lambda$ for $\lambda \in [\lambda_0, \lambda_1]$. In [2] Chen-Lin assumed $K \in C^\beta$ for any $\beta > \frac{1}{2}$. It can be weakened to the current version since we can choose δ small enough so that the first term of h_λ dominates. By elementary estimate we can show that h_λ is non-positive in Σ_λ. Therefore, by the definition of h_λ we can apply the method of moving spheres to $w_\lambda + h_\lambda$ from λ_0 to λ_1, which leads to a contradiction, since v_i and v_i^λ converge to U and U^λ in C^2 norm over finite domain. $U < U^\lambda$ outside B_λ if $\lambda > 1$. h_λ tends to 0 uniformly over any finite domain.

4 Proof of Theorem 2 for $n \ge 4$

Step One: Spherical Harnack Inequality:

In this step, we indicate the proof of

$$u(x) \le C|x|^{-\frac{n-2}{2}} \tag{4.1}$$

for $n \ge 4$. We still use the notations v_i, v_i^λ, w_λ, Q_λ, b_λ, etc. The proof is similar to the one in [2]. We only pinpoint the changes and important estimates.

Case 1: $n = 4$

For $n \ge 4$, K is differentiable, it is proved in [2] that $\nabla K(0) = 0$ under the assumption for contradiction. Once we have this, by the assumption on K we have, for $\lambda \in [\frac{1}{2}, 2]$, that

$$|K(x_i + M_i^{-\frac{2}{n-2}} y) - K(x_i + M_i^{-\frac{2}{n-2}} y^\lambda)| \le c(\delta) M_i^{-1} (|y| - \lambda).$$

Consequently,

$$|Q_\lambda(y)| \le c(\delta) M_i^{-1} |y|^{-1-n}.$$

In [2], $K \in C^2$ for $n = 4, 5$ but this is not necessary, the above estimate is enough for $n = 4$. We construct h_λ as follows:

$$h_\lambda(y) = \epsilon_1 M_i^{-1} (\frac{1}{|y|^2} - \frac{1}{\lambda^2}) + \int_{\Sigma_\lambda} G^\lambda(y,\eta) Q_\lambda(\eta) d\eta.$$

where ϵ_1 is, as in the case $n = 3$, a small positive number. Note that when δ is small, $c(\delta)$ is small correspondingly. The test function h_λ can still be non-positive. Then the argument of the method of moving spheres can be applied to $w_\lambda + h_\lambda$ ($\lambda \in [\frac{1}{2}, 2]$) to get a contradiction.

Case 2: $n \ge 5$

In this case, let $D_i = |\nabla K(x_i)|$. The argument in [2] can be modified slightly to prove that

$$D_i^{\frac{1}{\alpha-1}} M_i^{\frac{2}{n-2}} \to 0 \tag{4.2}$$

where $\alpha = (n-2)/2$. Note that even though $K \in C^2$ for $n = 5$ in [2], but this is not needed, the argument they used to prove $\nabla K(0) = 0$ can be applied to this case under the assumption of K for $n = 5$. Once we have (4.2), then

$$|K(x_i + M_i^{-\frac{2}{n-2}} y) - K(x_i + M_i^{-\frac{2}{n-2}} y^\lambda)| \le 3c(\delta) M_i^{-1} |y|^{1/2} (|y| - \lambda), \quad \text{for} \quad n = 5$$

and

$$|K(x_i + M_i^{-\frac{2}{n-2}} y) - K(x_i + M_i^{-\frac{2}{n-2}} y^\lambda)| \le 3c(\delta) M_i^{-1} |y|^{\alpha-1} (|y| - \lambda) \quad \text{for} \quad n \ge 6.$$

Let

$$h_\lambda(y) = \frac{\epsilon_2}{M_i}(|y|^{2-n} - \lambda^{2-n}) + \int_{\Sigma_\lambda} G^\lambda(y,\eta)Q_\lambda(\eta)d\eta.$$

We require δ to be so small that $\epsilon_0 >> \epsilon_2 >> \epsilon_1$.

Then the moving sphere method works for $w_\lambda + h_\lambda$ for $\lambda \in [\frac{1}{2}, 2]$. The spherical Harnack inequality is proved. $\square$

Remark 4.1. In [2], the proof of $\nabla K(0) = 0$ or (4.2) under the assumption for contradiction is involved with a combination of two Kelvin transformations and a small translation in the method of moving planes. Their approach can be re-interpreted by the method of moving spheres and the amount of computation can be greatly reduced.

Step Two

In this step, we show that for $n \geq 4$, if the singularity at the origin is not removable, then $u(x)$ is comparable to $|x|^{-\frac{n-2}{2}}$. More specifically, we have known that $w(t) \leq C$. If $\liminf_{t\to-\infty} w(t) = 0$, we want to show that $\limsup_{t\to-\infty} w(t) = 0$. Then u has a removable singularity at 0.

This is essentially proved by the arguments in Lin [11] and Chen-Lin [3]. In [11] Lin proved that if $\nabla K(0) \neq 0$, then either u has a removable singularity or $u(x)$ is comparable to $|x|^{-\frac{n-2}{2}}$. So we can assume $\nabla K(0) = 0$. By the assumption of K we have

$$|x \cdot \nabla K(x)| \leq C|x|^\alpha, \qquad \alpha = (n-2)/2.$$

This fact is exactly what Chen-Lin used in [4] (page 237 and page 238) to get a contradiction from the Pohozaev Identity. So we don't repeat the argument here. Therefore we have shown that for $n \geq 5$,

$$\liminf_{t\to-\infty} w(t) = 0 \quad \text{implies} \quad \limsup_{t\to-\infty} w(t) = 0.$$

Consequently, u has a removable singularity at the origin. Here we would like to point out that for $n = 4$, there is no need to consider the case $\nabla K(0) \neq 0$ first. The estimate in Lemma 3.6 is enough to derive a contradiction from the Pohozaev Identity directly.

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