

Pointwise Bounds and Blow-up for Systems of Semilinear Elliptic Inequalities at an Isolated Singularity via Nonlinear Potential Estimates

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Abstract

We study the behavior near the origin of C^2 positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{aligned} 0 \leq -\Delta u &\leq f(v) \\ 0 \leq -\Delta v &\leq g(u) \end{aligned} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 2,$$

where $f, g : (0, \infty) \rightarrow (0, \infty)$ are continuous functions. We provide optimal conditions on f and g at ∞ such that solutions of this system satisfy pointwise bounds near the origin. In dimension $n = 2$ we show that this property holds if $\log^+ f$ or $\log^+ g$ grow at most linearly at infinity. In dimension $n \geq 3$ and under the assumption $f(t) = O(t^\lambda)$, $g(t) = O(t^\sigma)$ as $t \rightarrow \infty$, ($\lambda, \sigma \geq 0$), we obtain a new critical curve that optimally describes the existence of such pointwise bounds. Our approach relies in part on sharp estimates of nonlinear potentials which appear naturally in this context.

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1 Introduction

In this paper we study the behaviour near the origin of C^2 positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{aligned} 0 \leq -\Delta u \leq f(v) \\ 0 \leq -\Delta v \leq g(u) \end{aligned} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 2, \quad (1.1)$$

where $f, g : (0, \infty) \rightarrow (0, \infty)$ are continuous functions. More precisely, we consider the following question.

Question 1. For which continuous functions $f, g : (0, \infty) \rightarrow (0, \infty)$ do there exist continuous functions $h_1, h_2 : (0, 1) \rightarrow (0, \infty)$ such that all C^2 positive solutions $u(x)$ and $v(x)$ of the system (1.1) satisfy

$$u(x) = O(h_1(|x|)) \quad \text{as } x \rightarrow 0 \quad (1.2)$$

$$v(x) = O(h_2(|x|)) \quad \text{as } x \rightarrow 0 \quad (1.3)$$

and what are the optimal such h_1 and h_2 when they exist?

We call a function h_1 (resp. h_2) with the above properties a pointwise bound for u (resp. v) as $x \rightarrow 0$.

Question 1 is motivated by the results on the single semilinear inequality

$$0 \leq -\Delta u \leq f(u) \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 2, \quad (1.4)$$

and its higher order version

$$0 \leq -\Delta^m u \leq f(u) \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, m \geq 1, n \geq 2,$$

which are discussed in [16, 17, 18].

Although the literature on semilinear elliptic systems is quite extensive, very little of it deals with semilinear inequalities. We mention the work of Bidaut-Véron and Grillo [2] in which the following coupled inequalities are studied:

$$\begin{aligned} 0 \leq \Delta u \leq |x|^a v^p \\ 0 \leq \Delta v \leq |x|^b u^q \end{aligned} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3, pq < 1, \quad (1.5)$$

and

$$\begin{aligned} \Delta u \geq |x|^a v^p \\ \Delta v \geq |x|^b u^q \end{aligned} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3, pq > 1. \quad (1.6)$$

Another related system of semilinear elliptic inequalities appears in [3] (see also [12]) and contains as a particular case the model

$$\begin{aligned} -\Delta u \geq |x|^a v^p \\ -\Delta v \geq |x|^b u^q \end{aligned} \quad \text{in } \mathbb{R}^n \setminus \{0\}, n \geq 2. \quad (1.7)$$

Our system (1.1) is different in nature from (1.5), (1.6) and (1.7) and its investigation completes the general picture of semilinear elliptic systems of inequalities. In particular (see Theorem 3.7), we will obtain pointwise bounds for positive solutions of the system

$$\begin{aligned} 0 \leq -\Delta u \leq |x|^a v^p \\ 0 \leq -\Delta v \leq |x|^b u^q \end{aligned} \quad \text{in } B_1(0) \setminus \{0\}, n \geq 3$$

which complement the studies of systems (1.5), (1.6) or (1.7).

Remark 1. Let

$$\Gamma(r) = \begin{cases} r^{-(n-2)}, & \text{if } n \geq 3 \\ \log \frac{2}{r}, & \text{if } n = 2. \end{cases} \quad (1.8)$$

Since $\Gamma(|x|)$ is positive and harmonic in $B_1(0) \setminus \{0\}$, the functions $u_0(x) = v_0(x) = \Gamma(|x|)$ are always positive solutions of (1.1). Hence, any pointwise bound for positive solutions of (1.1) must be at least as large as Γ and whenever Γ is such a bound for u (resp. v) it is necessarily optimal. In this case we say that u (resp. v) is *harmonically bounded* at 0.

We shall see that whenever a pointwise bound for positive solutions of (1.1) exists, then u or v (or both) are harmonically bounded at 0.

Our results reveal the fact that the optimal conditions for the existence of pointwise bounds for positive solutions of (1.1) are related to the growth at infinity of the nonlinearities f and g . In dimension $n = 2$ we prove that pointwise bounds exist if $\log^+ f$ or $\log^+ g$ grow at most linearly at infinity (see Theorems 2.1, 2.2 and 2.3). In dimensions $n \geq 3$ we will assume that f and g have a power type growth at infinity, namely

$$\begin{aligned} f(t) &= O(t^\lambda) & \text{as } t \rightarrow \infty \\ g(t) &= O(t^\sigma) & \text{as } t \rightarrow \infty \end{aligned}$$

with $\lambda, \sigma \geq 0$. In this setting, we will find (see Theorem 3.4) that no pointwise bounds exist if the pair (λ, σ) lies above the curve

$$\sigma = \frac{2}{n-2} + \frac{n}{n-2} \frac{1}{\lambda}. \quad (1.9)$$

On the other hand, if (λ, σ) lies below the curve (1.9) then pointwise bounds for positive solutions of (1.1) always exist and their optimal estimates depend on new subregions in the $\lambda\sigma$ plane (see Theorems 3.1, 3.2, 3.3 and 3.4).

We note that the curve (1.9) lies below the Sobolev hyperbola

$$\frac{1}{\sigma+1} + \frac{1}{\lambda+1} = 1 - \frac{2}{n}, \text{ that is, } \sigma = \frac{2\lambda + n + 2}{(n-2)\lambda - 2}$$

which separates the regions of existence and nonexistence for Lane-Emden systems:

$$\begin{aligned} -\Delta u &= v^\lambda \\ -\Delta v &= u^\sigma \end{aligned} \quad \text{in } B_1(0) \subset \mathbb{R}^n, n \geq 3,$$

(see [10, 11, 13, 15]).

As is the case for the single inequality (1.4), our analysis of the system (1.1) in all dimensions $n \geq 2$ relies heavily on the Brezis-Lions representation formula for superharmonic functions (see Appendix A). In two dimensions, an extension of the methods used in [16] to study (1.4) yields an essentially complete answer to Question 1. On the other hand, in three and higher dimensions (this is the most important and interesting possibility for us in this paper) this is not the case. We require additional nontrivial methods and tools to study Question 1 when $n \geq 3$, the most crucial of which are a Moser type iteration (see Lemma 5.5) and certain pointwise estimates (see Corollary 4.1) for the nonlinear potential $N((Ng)^\sigma)$, $\sigma \geq \frac{2}{n-2}$, where N is the Newtonian potential operator over a ball in \mathbb{R}^n , $n \geq 3$, and g is a nonnegative bounded function.

Section 4 in this work is concerned with various pointwise and integral estimates of nonlinear potentials of Havin-Maz'ya type and their connections with Wolff potentials. Since the results in

this section may be of independent interest, we state them in greater generality than is needed for our study of the system (1.1).

In any dimension $n \geq 2$, we prove that our pointwise bounds for positive solutions of (1.1) are optimal. When these bounds are not given by Γ , their optimality follows by constructing (with the help of Lemma 5.1) solutions u and v of (1.1) satisfying suitable coupled conditions on the union of a countable number of balls which cluster at the origin and are harmonic outside these balls. In this case, it is interesting to point out that although our optimal pointwise bounds are radially symmetric functions, these bounds are not achieved by radial solutions of (1.1), because nonnegative radial superharmonic functions in a punctured neighborhood of the origin are harmonically bounded as $x \rightarrow 0$.

We also consider the following analog of Question 1 when the singularity is at ∞ instead of at the origin.

Question 2. For which continuous functions $f, g : (0, \infty) \rightarrow (0, \infty)$ do there exist continuous functions $h_1, h_2 : (1, \infty) \rightarrow (0, \infty)$ such that all C^2 positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{aligned} 0 \leq -\Delta u \leq f(v) \\ 0 \leq -\Delta v \leq g(u) \end{aligned} \quad \text{in } \mathbb{R}^n \setminus B_1(0), \quad n \geq 2, \quad (1.10)$$

satisfy

$$u(x) = O(h_1(|x|)) \quad \text{as } |x| \rightarrow \infty \quad (1.11)$$

$$v(x) = O(h_2(|x|)) \quad \text{as } |x| \rightarrow \infty \quad (1.12)$$

and what are the optimal such h_1 and h_2 when they exist?

This paper is organized as follows. In Sections 2 and 3 we state our main results in dimensions $n = 2$ and $n \geq 3$ respectively. In Section 4 we obtain, using Hedberg inequalities and Wolff potential estimates, some new pointwise and integral bounds for nonlinear potentials of Havin-Maz'ya type. Using these estimates, we collect in Section 5 some preliminary lemmas while Sections 6 and 7 contain the proofs of our main results.

2 Statement of two dimensional results

In this section we state our results for Questions 1 and 2 when $n = 2$.

We say a continuous function $f : (0, \infty) \rightarrow (0, \infty)$ is *exponentially bounded* at ∞ if

$$\log^+ f(t) = O(t) \quad \text{as } t \rightarrow \infty$$

where

$$\log^+ s := \begin{cases} \log s, & \text{if } s > 1 \\ 0, & \text{if } s \leq 1. \end{cases}$$

If $f, g : (0, \infty) \rightarrow (0, \infty)$ are continuous functions then either

- (i) f and g are both exponentially bounded at ∞ ;
- (ii) neither f nor g is exponentially bounded at ∞ ; or
- (iii) one and only one of the functions f and g is exponentially bounded at ∞ .

Our result for Question 1 when $n = 2$ and f and g satisfy (i) (resp. (ii), (iii)) is Theorem 2.1 (resp. 2.2, 2.3) below.

By the following theorem, if the functions f and g are both exponentially bounded at ∞ then all positive solutions u and v of the system (1.1) are harmonically bounded at 0.

Theorem 2.1. *Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system*

$$0 \leq -\Delta u \leq f(v) \quad (2.1)$$

$$0 \leq -\Delta v \leq g(u) \quad (2.2)$$

in a punctured neighborhood of the origin in \mathbb{R}^2 , where $f, g : (0, \infty) \rightarrow (0, \infty)$ are continuous and exponentially bounded at ∞ . Then both u and v are harmonically bounded, that is

$$u(x) = O\left(\log \frac{1}{|x|}\right) \quad \text{as } x \rightarrow 0 \quad (2.3)$$

and

$$v(x) = O\left(\log \frac{1}{|x|}\right) \quad \text{as } x \rightarrow 0. \quad (2.4)$$

By Remark 1, the bounds (2.3) and (2.4) are optimal.

By the following theorem, it is *essentially* the case that if neither of the functions f and g is exponentially bounded at ∞ then neither of the positive solutions u and v of the system (1.1) satisfies an a priori pointwise bound at 0.

Theorem 2.2. *Suppose $f, g : (0, \infty) \rightarrow (0, \infty)$ are continuous functions satisfying*

$$\lim_{t \rightarrow \infty} \frac{\log f(t)}{t} = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\log g(t)}{t} = \infty. \quad (2.5)$$

Let $h : (0, 1) \rightarrow (0, \infty)$ be a continuous function satisfying $\lim_{r \rightarrow 0^+} h(r) = \infty$. Then there exist C^2 positive solutions $u(x)$ and $v(x)$ of the system (2.1, 2.2) in $B_1(0) \setminus \{0\} \subset \mathbb{R}^2$ such that

$$u(x) \neq O(h(|x|)) \quad \text{as } x \rightarrow 0 \quad (2.6)$$

and

$$v(x) \neq O(h(|x|)) \quad \text{as } x \rightarrow 0. \quad (2.7)$$

By the following theorem, if at least one of the functions f and g is exponentially bounded at ∞ then at least one of the positive solutions u and v of the system (1.1) is harmonically bounded at 0.

Theorem 2.3. *Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system*

$$0 \leq -\Delta u$$

$$0 \leq -\Delta v \leq g(u)$$

in a punctured neighborhood of the origin in \mathbb{R}^2 , where $g : (0, \infty) \rightarrow (0, \infty)$ is continuous and exponentially bounded at ∞ . Then v is harmonically bounded, that is

$$v(x) = O\left(\log \frac{1}{|x|}\right) \quad \text{as } x \rightarrow 0. \quad (2.8)$$

If, in addition,

$$-\Delta u \leq f(v)$$

in a punctured neighborhood of the origin, where $f : (0, \infty) \rightarrow (0, \infty)$ is a continuous function satisfying

$$\log^+ f(t) = O(t^\lambda) \quad \text{as } t \rightarrow \infty$$

for some $\lambda > 1$ then

$$u(x) = o\left(\left(\log \frac{1}{|x|}\right)^\lambda\right) \quad \text{as } x \rightarrow 0. \quad (2.9)$$

Note that in Theorems 2.1–2.3 we impose no conditions on the growth of $f(t)$ (or $g(t)$) as $t \rightarrow 0^+$.

By the following theorem, the bounds (2.9) and (2.8) for u and v in Theorem 2.3 are optimal.

Theorem 2.4. *Suppose $\lambda > 1$ is a constant and $\psi : (0, 1) \rightarrow (0, 1)$ is a continuous function satisfying $\lim_{r \rightarrow 0^+} \psi(r) = 0$. Then there exist C^∞ positive solutions $u(x)$ and $v(x)$ of the system*

$$\begin{aligned} 0 \leq -\Delta u &\leq e^{v^\lambda} \\ 0 \leq -\Delta v &\leq e^u \end{aligned} \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^2 \quad (2.10)$$

such that

$$u(x) \neq O\left(\psi(|x|) \left(\log \frac{2}{|x|}\right)^\lambda\right) \quad \text{as } x \rightarrow 0 \quad (2.11)$$

and

$$\frac{v(x)}{\log \frac{1}{|x|}} \rightarrow 1 \quad \text{as } x \rightarrow 0. \quad (2.12)$$

The following theorem generalizes Theorems 2.1 and 2.3 by allowing u and v to be negative and allowing the right sides of (2.1, 2.2) to depend on x .

Theorem 2.5. *Let $U(x)$ and $V(x)$ be C^2 solutions of the system*

$$0 \leq -\Delta U \leq |x|^{-a} e^{|V|^\lambda}, \quad U(x) > -a \log \frac{1}{|x|} \quad (2.13)$$

$$0 \leq -\Delta V \leq |x|^{-a} e^U, \quad V(x) > -a \log \frac{1}{|x|} \quad (2.14)$$

in a punctured neighborhood of the origin in \mathbb{R}^2 where a and λ are positive constants. Then

$$U(x) = O\left(\log \frac{1}{|x|}\right) + o\left(\left(\log \frac{1}{|x|}\right)^\lambda\right) \quad \text{as } x \rightarrow 0 \quad (2.15)$$

$$V(x) = O\left(\log \frac{1}{|x|}\right) \quad \text{as } x \rightarrow 0. \quad (2.16)$$

The analog of Theorem 2.5 when the singularity is at ∞ instead of at the origin is the following result.

Theorem 2.6. *Let $u(y)$ and $v(y)$ be C^2 solutions of the system*

$$0 \leq -\Delta u \leq |y|^a e^{|v|^\lambda}, \quad u(y) > -a \log |y|$$

$$0 \leq \Delta v \leq |y|^a e^u, \quad v(y) > -a \log |y|$$

in the complement of a compact subset of \mathbb{R}^2 where a and λ are positive constants. Then

$$\begin{aligned} u(y) &= O(\log |y|) + o((\log |y|)^\lambda) \\ v(y) &= O(\log |y|) \end{aligned} \quad \text{as } |y| \rightarrow \infty. \quad (2.17)$$

Proof. Apply the Kelvin transform

$$U(x) = u(y), \quad V(x) = v(y), \quad y = \frac{x}{|x|^2}$$

and then use Theorem 2.5. □

3 Statement of three and higher dimensional results

In this section we state our results for Questions 1 and 2 when $n \geq 3$. We will mainly be concerned with the case that the continuous functions $f, g : (0, \infty) \rightarrow (0, \infty)$ in Questions 1 and 2 satisfy

$$f(t) = O(t^\lambda) \quad \text{as } t \rightarrow \infty \quad (3.1)$$

$$g(t) = O(t^\sigma) \quad \text{as } t \rightarrow \infty \quad (3.2)$$

for some nonnegative constants λ and σ . We can assume without loss of generality that $\sigma \leq \lambda$.

If λ and σ are nonnegative constants satisfying $\sigma \leq \lambda$ then (λ, σ) belongs to one of the following four pairwise disjoint subsets of the $\lambda\sigma$ -plane:

$$\begin{aligned} A &:= \left\{ (\lambda, \sigma) : 0 \leq \sigma \leq \lambda \leq \frac{n}{n-2} \right\} \\ B &:= \left\{ (\lambda, \sigma) : \lambda > \frac{n}{n-2} \quad \text{and} \quad 0 \leq \sigma < \frac{2}{n-2} + \frac{n-1}{n-2\lambda} \right\} \\ C &:= \left\{ (\lambda, \sigma) : \lambda > \frac{n}{n-2} \quad \text{and} \quad \frac{2}{n-2} + \frac{n-1}{n-2\lambda} < \sigma \leq \lambda \right\} \\ D &:= \left\{ (\lambda, \sigma) : \lambda > \frac{n}{n-2} \quad \text{and} \quad \sigma = \frac{2}{n-2} + \frac{n-1}{n-2\lambda} \right\}. \end{aligned}$$

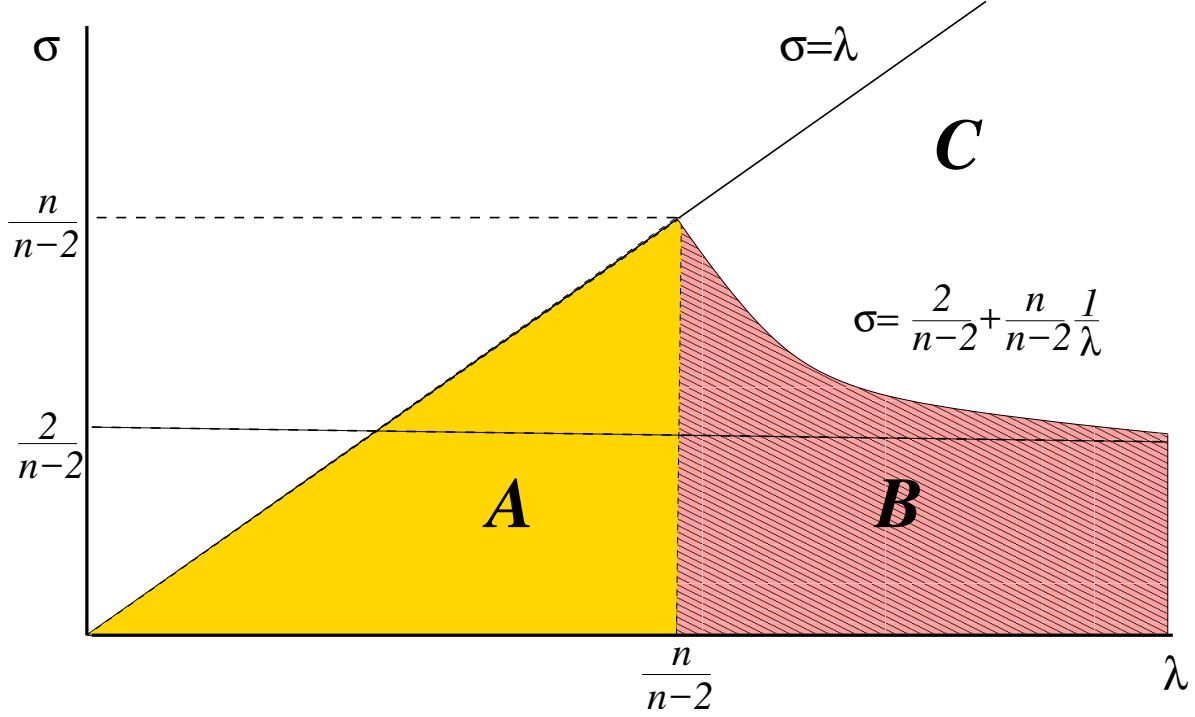


Figure 1: Graph of regions A, B and C.

Note that A , B and C are two dimensional regions in the $\lambda\sigma$ -plane whereas D is the curve separating B and C . (See Figure 1.)

In this section we give a complete answer to Question 1 when $n \geq 3$ and the functions f and g satisfy (3.1, 3.2) where $(\lambda, \sigma) \in A \cup B \cup C$. The following theorem deals with the case that $(\lambda, \sigma) \in A$.

Theorem 3.1. *Let $f, g : (0, \infty) \rightarrow (0, \infty)$ be continuous functions satisfying (3.1, 3.2) where*

$$0 \leq \sigma \leq \lambda \leq \frac{n}{n-2}. \quad (3.3)$$

Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system

$$0 \leq -\Delta u \leq f(v) \quad (3.4)$$

$$0 \leq -\Delta v \leq g(u) \quad (3.5)$$

in a punctured neighborhood of the origin in \mathbb{R}^n , $n \geq 3$. Then both u and v are harmonically bounded, that is

$$u(x) = O(|x|^{-(n-2)}) \quad \text{as } x \rightarrow 0 \quad (3.6)$$

$$v(x) = O(|x|^{-(n-2)}) \quad \text{as } x \rightarrow 0. \quad (3.7)$$

By Remark 1, the bounds (3.6) and (3.7) are optimal.

The following two theorems deal with the case $(\lambda, \sigma) \in B$.

Theorem 3.2. *Let $f, g : (0, \infty) \rightarrow (0, \infty)$ be continuous functions satisfying (3.1, 3.2) where*

$$\lambda > \frac{n}{n-2} \quad \text{and} \quad \sigma < \frac{2}{n-2} + \frac{n}{n-2} \frac{1}{\lambda}. \quad (3.8)$$

Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system (3.4, 3.5) in a punctured neighborhood of the origin in \mathbb{R}^n , $n \geq 3$. Then

$$u(x) = o\left(|x|^{-\frac{(n-2)^2}{n}\lambda}\right) \quad \text{as } x \rightarrow 0 \quad (3.9)$$

and

$$v(x) = O\left(|x|^{-(n-2)}\right) \quad \text{as } x \rightarrow 0. \quad (3.10)$$

By the following theorem the bounds (3.9) and (3.10) for u and v in Theorem 3.2 are optimal.

Theorem 3.3. *Suppose λ and σ satisfy (3.8) and $\psi : (0, 1) \rightarrow (0, 1)$ is a continuous function satisfying $\lim_{r \rightarrow 0^+} \psi(r) = 0$. Then there exist C^∞ positive solutions $u(x)$ and $v(x)$ of the system*

$$\begin{aligned} 0 &\leq -\Delta u \leq v^\lambda \\ 0 &\leq -\Delta v \leq u^\sigma \end{aligned} \quad \text{in } \mathbb{R}^n \setminus \{0\}, n \geq 3 \quad (3.11)$$

such that

$$u(x) \neq O\left(\psi(|x|)|x|^{-\frac{(n-2)^2}{n}\lambda}\right) \quad \text{as } x \rightarrow 0 \quad (3.12)$$

and

$$v(x)|x|^{n-2} \rightarrow 1 \quad \text{as } x \rightarrow 0. \quad (3.13)$$

The following theorem deals with the case that $(\lambda, \sigma) \in C$. In this case there exist pointwise bounds for neither u nor v .

Theorem 3.4. *Suppose λ and σ are positive constants satisfying*

$$\frac{2}{n-2} + \frac{n}{n-2} \frac{1}{\lambda} < \sigma \leq \lambda. \quad (3.14)$$

Let $h : (0, 1) \rightarrow (0, \infty)$ be a continuous function satisfying

$$\lim_{r \rightarrow 0^+} h(r) = \infty.$$

Then there exist C^∞ solutions $u(x)$ and $v(x)$ of the system

$$\left. \begin{aligned} 0 &\leq -\Delta u \leq v^\lambda \\ 0 &\leq -\Delta v \leq u^\sigma \\ u &> 1, v > 1 \end{aligned} \right\} \quad \text{in } \mathbb{R}^n \setminus \{0\}, n \geq 3 \quad (3.15)$$

such that

$$u(x) \neq O(h(|x|)) \quad \text{as } x \rightarrow 0 \quad (3.16)$$

and

$$v(x) \neq O(h(|x|)) \quad \text{as } x \rightarrow 0. \quad (3.17)$$

The following theorem can be viewed as the limiting case of Theorem 3.2 as $\lambda \rightarrow \infty$.

Theorem 3.5. Let $g : (0, \infty) \rightarrow (0, \infty)$ be a continuous function satisfying (3.2) where

$$\sigma < \frac{2}{n-2}.$$

Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system

$$\begin{aligned} 0 &\leq -\Delta u \\ 0 &\leq -\Delta v \leq g(u) \end{aligned}$$

in a punctured neighborhood of the origin in \mathbb{R}^n , $n \geq 3$. Then v is harmonically bounded, that is

$$v(x) = O\left(|x|^{-(n-2)}\right) \quad \text{as } x \rightarrow 0. \quad (3.18)$$

By Remark 1, the bound (3.18) is optimal.

In Theorem 3.7 we will extend some of our results to the more general system

$$\begin{aligned} 0 &\leq -\Delta u \leq |x|^{-\alpha} v^\lambda \\ 0 &\leq -\Delta v \leq |x|^{-\beta} u^\sigma. \end{aligned}$$

Using these extended results and the Kelvin transform, we obtain the following theorem concerning pointwise bounds for positive solutions $U(y)$ and $V(y)$ of the system

$$\begin{aligned} 0 &\leq -\Delta U \leq (V+1)^\lambda \\ 0 &\leq -\Delta V \leq (U+1)^\sigma \end{aligned} \quad (3.19)$$

in the complement of a compact subset of \mathbb{R}^n , $n \geq 3$, where

$$\lambda \geq \sigma \geq 0 \quad \text{and} \quad \sigma < \frac{2}{n-2} + \frac{n-1}{n-2} \frac{1}{\lambda}. \quad (3.20)$$

Note that λ and σ satisfy (3.20) if and only if $(\lambda, \sigma) \in \left(A \setminus \left\{ \left(\frac{n}{n-2}, \frac{n}{n-2} \right) \right\}\right) \cup B$ where A and B are defined at the beginning of this section and graphed in Figure 1.

Theorem 3.6. Let $U(y)$ and $V(y)$ be C^2 nonnegative solutions of the system (3.19) in the complement of a compact subset of \mathbb{R}^n , $n \geq 3$, where λ and σ satisfy (3.20).

Case A. If $\sigma = 0$ then as $|y| \rightarrow \infty$

$$\begin{aligned} U(y) &= o\left(|y|^{\frac{n-2}{n}\left(\frac{2(n-2)}{n}\lambda+2\right)}\right) \\ V(y) &= o\left(|y|^{\frac{2(n-2)}{n}}\right). \end{aligned}$$

Case B. If $0 < \sigma < \frac{2}{n-2}$ then as $|y| \rightarrow \infty$

$$\begin{aligned} U(y) &= o\left(|y|^{\frac{2(n-2)(\lambda+1)}{n}}\right) \\ V(y) &= O(|y|^2). \end{aligned}$$

Case C. If $\sigma = \frac{2}{n-2}$ then as $|y| \rightarrow \infty$

$$\begin{aligned} U(y) &= o\left(|y|^{\frac{2(n-2)(\lambda+1)}{n}} (\log |y|)^{\frac{n-2}{n}\lambda}\right) \\ V(y) &= o(|y|^2 \log |y|). \end{aligned}$$

Case D. Suppose $\sigma > \frac{2}{n-2}$. Let $\varepsilon > 0$ and $D = (n-2)\lambda \left(\frac{2}{n-2} + \frac{n}{n-2}\frac{1}{\lambda} - \sigma\right)$. Then $D > 0$ and as $|y| \rightarrow \infty$

$$\begin{aligned} U(y) &= o\left(|y|^{\frac{2(n-2)(\lambda+1)}{D} + \varepsilon}\right) \\ V(y) &= o\left(|y|^{\frac{2(n-2)(\sigma+1)}{D} + \varepsilon}\right). \end{aligned}$$

Theorem 3.7. Let $u(x)$ and $v(x)$ be C^2 nonnegative solutions of the system

$$0 \leq -\Delta u \leq |x|^{-\alpha} \left(v + |x|^{-(n-2)}\right)^\lambda \quad (3.21)$$

$$0 \leq -\Delta v \leq |x|^{-\beta} \left(u + |x|^{-(n-2)}\right)^\sigma \quad (3.22)$$

in a punctured neighborhood of the origin in \mathbb{R}^n , $n \geq 3$, where $\alpha, \beta \in \mathbb{R}$ and λ and σ satisfy (3.20).

Case A. Suppose $\sigma = 0$.

(A1) If $\beta \leq n$ then as $x \rightarrow 0$

$$\begin{aligned} u(x) &= O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}((n-2)\lambda + \alpha)}\right) \\ v(x) &= O\left(\left(\frac{1}{|x|}\right)^{n-2}\right). \end{aligned}$$

(A2) If $\beta > n$ then as $x \rightarrow 0$

$$\begin{aligned} u(x) &= O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}\left(\frac{n-2}{n}\beta\lambda + \alpha\right)}\right) \\ v(x) &= o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}\beta}\right). \end{aligned}$$

Case B. Suppose $0 < \sigma < \frac{2}{n-2}$. Let

$$\delta = \max\{(n-2)\lambda + \alpha, [(n-2)\sigma - 2 + \beta]\lambda + \alpha\} \quad (3.23)$$

(B1) If $\delta \leq n$ then as $x \rightarrow 0$

$$u(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) \quad (3.24)$$

$$v(x) = \begin{cases} O\left(\left(\frac{1}{|x|}\right)^{n-2}\right), & \text{if } \beta \leq n - (n-2)\sigma \\ o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}[(n-2)\sigma + \beta]}\right), & \text{if } \beta > n - (n-2)\sigma. \end{cases} \quad (3.25)$$

(B2) If $\delta > n$ then as $x \rightarrow 0$

$$u(x) = o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}\delta}\right) \quad (3.26)$$

$$v(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2} + \left(\frac{1}{|x|}\right)^{(n-2)\sigma-2+\beta}\right). \quad (3.27)$$

Case C. Suppose $\sigma = \frac{2}{n-2}$.

(C1) If either

(i) $\beta < n - 2$ and $(n - 2)\lambda + \alpha \leq n$; or

(ii) $\beta \geq n - 2$ and $\beta\lambda + \alpha < n$

then, as $x \rightarrow 0$, u and v satisfy (3.24) and (3.25), that is

$$u(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right)$$

$$v(x) = \begin{cases} O\left(\left(\frac{1}{|x|}\right)^{n-2}\right), & \text{if } \beta \leq n - 2 \\ o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}(\beta+2)}\right), & \text{if } \beta > n - 2. \end{cases}$$

(C2) If neither (i) nor (ii) holds then as $x \rightarrow 0$

$$u(x) = \begin{cases} o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}[(n-2)\lambda+\alpha]}\right), & \text{if } \beta < n - 2 \\ o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}(\beta\lambda+\alpha)} \left(\log \frac{1}{|x|}\right)^{\frac{n-2}{n}\lambda}\right), & \text{if } \beta \geq n - 2 \end{cases} \quad (3.28)$$

$$v(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^\beta \log \frac{1}{|x|}\right). \quad (3.29)$$

Case D. Suppose $\sigma > \frac{2}{n-2}$. Let

$$a := \frac{\lambda}{n}[(n-2)\sigma - 2] \quad \text{and} \quad b := \frac{\alpha}{n}[(n-2)\sigma - 2] + \beta. \quad (3.30)$$

Then $0 < a < 1$.

(D1) If either

(i) $\frac{b}{1-a} < n - 2$ and $(n - 2)\lambda \leq n - \alpha$; or

(ii) $\frac{b}{1-a} \geq n - 2$ and $\frac{b\lambda}{1-a} < n - \alpha$

then, as $x \rightarrow 0$, u and v satisfy (3.24) and (3.25).

(D2) If neither (i) nor (ii) holds then as $x \rightarrow 0$

$$u(x) = \begin{cases} o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}[(n-2)\lambda+\alpha]}\right), & \text{if } \frac{b}{1-a} < n - 2 \\ o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}(\frac{b\lambda}{1-a}+\alpha+\varepsilon)}\right), & \text{if } \frac{b}{1-a} \geq n - 2 \end{cases} \quad (3.31)$$

and

$$v(x) = \begin{cases} O\left(\left(\frac{1}{|x|}\right)^{n-2}\right), & \text{if } \frac{b}{1-a} < n-2 \\ o\left(\left(\frac{1}{|x|}\right)^{\frac{b}{1-a}+\varepsilon}\right), & \text{if } \frac{b}{1-a} \geq n-2 \end{cases} \quad (3.32)$$

for all $\varepsilon > 0$.

4 Nonlinear potentials

In this section we are concerned with pointwise and integral estimates of certain nonlinear potentials using inequalities of Hedberg type and Wolff potential estimates (see [1], [9]). As a consequence, we will prove the following theorem.

Theorem 4.1. *Let $B = B_1(0)$ be the unit ball in \mathbb{R}^n , $n \geq 3$, and let*

$$Nf(x) = \int_B \frac{f(y)}{|x-y|^{n-2}} dy, \quad x \in B. \quad (4.1)$$

Then, for all nonnegative functions $f \in L^\infty(B)$,

$$\|N((Nf)^\sigma)\|_{L^\infty(B)} \leq C \|f\|_{L^s(B)}^{\frac{2s(\sigma+1)}{n}} \|f\|_{L^\infty(B)}^{\frac{(n-2s)\sigma-2s}{n}}, \quad (4.2)$$

if $\sigma > \frac{2}{n-2}$ and $0 < s < \frac{n\sigma}{2(\sigma+1)}$, and

$$\|N((Nf)^\sigma)\|_{L^\infty(B)} \leq C \|f\|_{L^s(B)}^\sigma \log \left(\frac{C \|f\|_{L^\infty(B)}}{\|f\|_{L^s(B)}} \right), \quad (4.3)$$

if $\sigma \geq \frac{2}{n-2}$ and $s = \frac{n\sigma}{2(\sigma+1)}$, where C is a positive constant which does not depend on f .

More precise pointwise estimates of $N((Nf)^\sigma)$ in terms of Wolff potentials

$$\mathbf{W}_\sigma f(x) = \int_0^3 \left(\int_{B_r(x)} f(y) dy \right)^\sigma \frac{dr}{r^{(n-2)\sigma-1}}, \quad x \in B_1(0),$$

along with their analogues for functions f defined on the entire space \mathbb{R}^n , and Riesz or Bessel potentials in place of Nf , will be discussed below (see Theorems 4.2–4.4).

We remark that if $\frac{2}{n-2} \leq \sigma < \frac{n}{n-2}$, then for all nonnegative $f \in L^1(B_1(0))$,

$$C^{-1} \mathbf{W}_\sigma f(x) \leq N((Nf)^\sigma)(x) \leq C \mathbf{W}_\sigma f(x), \quad x \in B_1(0),$$

where C is a positive constant which depends only on σ and n . There are similar pointwise estimates in the range $\frac{n}{n-2} \leq \sigma < \infty$ under the additional assumption that $N((Nf)^\sigma)(x)$ is uniformly bounded, for instance, if $f \in L^\infty(B_1(0))$. These relations between nonlinear potentials $N((Nf)^\sigma)$ and $\mathbf{W}_\sigma f$ are due to Havin and Maz'ya, D. Adams and Meyers (see [1], [9]).

Let μ be a nonnegative Borel measure on \mathbb{R}^n . For $0 < \alpha < n$, the Riesz potential $\mathbf{I}_\alpha \mu$ of order α is defined by

$$\mathbf{I}_\alpha \mu(x) = \int_0^\infty \frac{\mu(B_r(x))}{r^{n-\alpha}} \frac{dr}{r} = \frac{1}{n-\alpha} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n. \quad (4.4)$$

For $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$, the Wolff potential $\mathbf{W}_{\alpha,p}\mu$ is defined by (see [1], [9]):

$$\mathbf{W}_{\alpha,p}\mu(x) = \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \quad (4.5)$$

There is also a nonhomogeneous version applicable for $0 < \alpha \leq \frac{n}{p}$,

$$\mathbf{W}_{\alpha,p}^c\mu(x) = \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r}, \quad x \in \mathbb{R}^n, \quad (4.6)$$

where $c > 0$. Wolff potentials have numerous applications in analysis and PDE (see, for instance, [7], [8], [14], [19]).

We will also use the Havin-Maz'ya potential $\mathbf{U}_{\alpha,p}\mu$, where $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$, defined by:

$$\mathbf{U}_{\alpha,p}\mu(x) = \mathbf{I}_\alpha(\mathbf{I}_\alpha\mu)^{\frac{1}{p-1}}(x), \quad x \in \mathbb{R}^n, \quad (4.7)$$

along with its nonhomogeneous analogue $\mathbf{V}_{\alpha,p}\mu$, where $1 < p < \infty$ and $0 < \alpha \leq \frac{n}{p}$, defined by:

$$\mathbf{V}_{\alpha,p}\mu(x) = \mathbf{J}_\alpha(\mathbf{J}_\alpha\mu)^{\frac{1}{p-1}}(x), \quad x \in \mathbb{R}^n. \quad (4.8)$$

Here Bessel potentials

$$\mathbf{J}_\alpha\mu(x) = \int_{\mathbb{R}^n} G_\alpha(x-t) d\mu(t)$$

with Bessel kernels G_α , $\alpha > 0$, are used in place of Riesz potentials $\mathbf{I}_\alpha\mu$. Clearly, $\mathbf{J}_\alpha\mu(x) \leq c_{\alpha,n}\mathbf{I}_\alpha\mu(x)$, and hence $\mathbf{V}_{\alpha,p}\mu(x) \leq c_{\alpha,n}^{\frac{1}{p-1}}\mathbf{U}_{\alpha,p}\mu(x)$ for all $x \in \mathbb{R}^n$.

Note that the Newtonian potential coincides with

$$\mathbf{I}_2\mu(x) = \mathbf{W}_{1,2}\mu(x) = c_n\mathbf{U}_{1,2}\mu(x), \quad n \geq 3.$$

If $d\mu = f(x)dx$, where $f \geq 0$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, we will denote the corresponding potentials by $\mathbf{I}_\alpha f$, $\mathbf{U}_{\alpha,p}f$, etc.

We will need the Hardy-Littlewood maximal function

$$\mathbf{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy, \quad x \in \mathbb{R}^n.$$

Throughout this section c , c_1 , c_2 , etc., will stand for constants which depend only on α , p , and n . The following pointwise estimates of nonlinear potentials are due to Havin and Maz'ya, and D. R. Adams. (See, for instance, [9, Sec. 10.4.2] and the references therein.)

Theorem 4.2. *Let μ be a locally finite nonnegative Borel measure on \mathbb{R}^n .*

(a) *If $1 < p < \infty$ and $0 < \alpha \leq \frac{n}{p}$, then, for some $c, c_1 > 0$,*

$$\mathbf{V}_{\alpha,p}\mu(x) \geq c_1 \mathbf{W}_{\alpha,p}^c\mu(x) = c_1 \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \quad (4.9)$$

(b) *If $2 - \frac{\alpha}{n} < p < \infty$ and $0 < \alpha \leq \frac{n}{p}$, then, for some $c, c_1 > 0$,*

$$\mathbf{V}_{\alpha,p}\mu(x) \leq c_1 \mathbf{W}_{\alpha,p}^c\mu(x) = c_1 \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \quad (4.10)$$

(c) If $1 < p \leq 2 - \frac{\alpha}{n}$, $0 < \alpha < \frac{n}{p}$, and $\phi(r) = \sup_{x \in \mathbb{R}^n} \mu(B_r(x))$, then

$$\mathbf{V}_{\alpha,p}\mu(x) \leq c_1 \int_0^\infty \left(\frac{\phi(r)}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \quad (4.11)$$

(d) The above estimates with $c = 0$ hold for the potential $\mathbf{U}_{\alpha,p}\mu$ in place of $\mathbf{V}_{\alpha,p}\mu$ if $0 < \alpha < \frac{n}{p}$.

Theorem 4.2 yields that the Wolff potential $\mathbf{W}_{\alpha,p}\mu$ is pointwise equivalent to the Havin-Maz'ya potential $\mathbf{U}_{\alpha,p}\mu$ (and $\mathbf{W}^c_{\alpha,p}\mu$ is equivalent to $\mathbf{V}_{\alpha,p}\mu$ if $c > 0$, up to a choice of c), provided $2 - \frac{\alpha}{n} < p < \infty$.

In the range $1 < p \leq 2 - \frac{\alpha}{n}$ (which excludes the critical case $\alpha = \frac{n}{p}$), the sharp upper estimate (4.10) for $\mathbf{U}_{\alpha,p}\mu$ fails, along with its counterpart for $\mathbf{V}_{\alpha,p}\mu$. However, there are natural substitutes under the additional assumption that the corresponding nonlinear potential is uniformly bounded. The following theorem is due to Adams and Meyer (see [9, Sec. 10.4.2]).

Theorem 4.3. Suppose $\mathbf{V}_{\alpha,p}\mu(x) \leq K$ for all $x \in \mathbb{R}^n$.

(a) If $1 < p < 2 - \frac{\alpha}{n}$ and $0 < \alpha < \frac{n}{p}$, then, for some $c, c_1 > 0$,

$$\mathbf{V}_{\alpha,p}\mu(x) \leq c_1 K^{\frac{(2-p)n-\alpha}{n-\alpha p}} \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{n-1}{n-\alpha p}} e^{-cr} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \quad (4.12)$$

(b) If $p = 2 - \frac{\alpha}{n}$ and $0 < \alpha < \frac{n}{p}$, then, for some $c, c_1, c_2 > 0$,

$$\mathbf{V}_{\alpha,p}\mu(x) \leq c_1 \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \log \left(\frac{c_2 K^{p-1} r^{n-\alpha p}}{\mu(B_r(x))} \right) \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r}, \quad x \in \mathbb{R}^n. \quad (4.13)$$

(c) The above estimates with $c = 0$ hold for the potential $\mathbf{U}_{\alpha,p}\mu$ in place of $\mathbf{V}_{\alpha,p}\mu$.

We now deduce some pointwise bounds for nonlinear potentials.

Theorem 4.4. Let $p > 1$ and $0 < \alpha < n$. Then the following estimates hold.

(a) If $2 - \frac{\alpha}{n} < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $f \in L^s(\mathbb{R}^n)$, where $1 \leq s < \frac{n}{\alpha p}$, then, for all $x \in \mathbb{R}^n$,

$$\mathbf{U}_{\alpha,p}f(x) \leq c (\mathbf{M}f(x))^{\frac{n-\alpha ps}{(p-1)n}} \|f\|_{L^s(\mathbb{R}^n)}^{\frac{\alpha ps}{(p-1)n}}. \quad (4.14)$$

(b) If $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $f \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $0 < s < \frac{n}{\alpha p}$, then, for all $x \in \mathbb{R}^n$,

$$\mathbf{U}_{\alpha,p}f(x) \leq c \|f\|_{L^\infty(\mathbb{R}^n)}^{\frac{n-\alpha ps}{(p-1)n}} \|f\|_{L^s(\mathbb{R}^n)}^{\frac{\alpha ps}{(p-1)n}}. \quad (4.15)$$

(c) If $2 - \frac{\alpha}{n} < p < \infty$, $0 < \alpha \leq \frac{n}{p}$, and $f \in L^s(\mathbb{R}^n)$, where $s = \frac{n}{\alpha p}$, then, for all $x \in \mathbb{R}^n$,

$$\mathbf{V}_{\alpha,p}f(x) \leq c \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \left(\frac{(\mathbf{M}f(x))^{\frac{1}{p-1}}}{(\mathbf{M}f(x))^{\frac{1}{p-1}} + \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}}} + \log^+ \left(\frac{\mathbf{M}f(x)}{\|f\|_{L^s(\mathbb{R}^n)}} \right) \right). \quad (4.16)$$

(d) If $1 < p < \infty$, $0 < \alpha \leq \frac{n}{p}$, and $f \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{n}{\alpha p}$, then, for all $x \in \mathbb{R}^n$,

$$\mathbf{V}_{\alpha,p}f(x) \leq c \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \left(\frac{\|f\|_{L^\infty(\mathbb{R}^n)}^{\frac{1}{p-1}}}{\|f\|_{L^\infty(\mathbb{R}^n)}^{\frac{1}{p-1}} + \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}}} + \log^+ \left(\frac{\|f\|_{L^\infty(\mathbb{R}^n)}}{\|f\|_{L^s(\mathbb{R}^n)}} \right) \right). \quad (4.17)$$

Proof. Suppose first that $2 - \frac{\alpha}{n} < p < \infty$. Fix $R > 0$, and let $d\mu = f(x)dx$, where $f \geq 0$ and $f \in L^s_{\text{loc}}(\mathbb{R}^n)$, $s \geq 1$. Then by (4.10), $\mathbf{V}_{\alpha,p}\mu(x)$ is bounded above by a constant multiple of

$$\begin{aligned} \mathbf{W}^c_{\alpha,p}f(x) &= \int_0^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r} \\ &= \int_0^R \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r} + \int_R^\infty \left(\frac{\mu(B_r(x))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} e^{-cr} \frac{dr}{r} = I_1 + I_2. \end{aligned}$$

Note that

$$\frac{\mu(B_r(x))}{r^{n-\alpha p}} \leq c_n r^{\alpha p} \mathbf{M}f(x).$$

Hence,

$$I_1 \leq C \mathbf{M}f(x)^{\frac{1}{p-1}} \int_0^R r^{\frac{\alpha p}{p-1}} e^{-cr} \frac{dr}{r}.$$

To estimate I_2 , notice that by Hölder's inequality with $s \geq 1$,

$$\mu(B_r(x)) \leq \|f\|_{L^s(\mathbb{R}^n)} c_n r^{n(1-\frac{1}{s})}, \quad x \in \mathbb{R}^n, \quad r > 0.$$

Hence,

$$I_2 \leq C \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \int_R^\infty r^{-\frac{n-\alpha ps}{s(p-1)}} e^{-cr} \frac{dr}{r}.$$

Letting $a = \frac{\mathbf{M}f(x)}{\|f\|_{L^s(\mathbb{R}^n)}}$, and combining the preceding inequalities, we obtain

$$\mathbf{V}_{\alpha,p}f(x) \leq C \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \left(a^{\frac{1}{p-1}} \int_0^R r^{\frac{\alpha p}{p-1}} e^{-cr} \frac{dr}{r} + \int_R^\infty r^{-\frac{n-\alpha ps}{s(p-1)}} e^{-cr} \frac{dr}{r} \right). \quad (4.18)$$

Minimizing the right-hand side over R gives, with $R = a^{-\frac{s}{n}}$,

$$\mathbf{V}_{\alpha,p}f(x) \leq C \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \left(a^{\frac{1}{p-1}} \int_0^{a^{-\frac{s}{n}}} r^{\frac{\alpha p}{p-1}} e^{-cr} \frac{dr}{r} + \int_{a^{-\frac{s}{n}}}^\infty r^{-\frac{n-\alpha ps}{s(p-1)}} e^{-cr} \frac{dr}{r} \right). \quad (4.19)$$

As noted above, the preceding inequality holds for $\mathbf{U}_{\alpha,p}f$ in place of $\mathbf{V}_{\alpha,p}f$ if we set $c = 0$:

$$\begin{aligned} \mathbf{U}_{\alpha,p}f(x) &\leq C \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \left(a^{\frac{1}{p-1}} \int_0^{a^{-\frac{s}{n}}} r^{\frac{\alpha p}{p-1}} \frac{dr}{r} + \int_{a^{-\frac{s}{n}}}^\infty r^{-\frac{n-\alpha ps}{s(p-1)}} \frac{dr}{r} \right) \\ &= C_1 \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} a^{\frac{n-\alpha ps}{(p-1)n}} = C_1 (\mathbf{M}f(x))^{\frac{n-\alpha ps}{(p-1)n}} \|f\|_{L^s(\mathbb{R}^n)}^{\frac{\alpha ps}{(p-1)n}}, \end{aligned}$$

provided $p > 2 - \frac{\alpha}{n}$, $0 < \alpha < \frac{n}{p}$ and $1 \leq s < \frac{n}{\alpha p}$. This proves statement (a) of Theorem 4.4.

If $f \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then noticing that $\mathbf{M}f(x) \leq \|f\|_{L^\infty(\mathbb{R}^n)}$ for every $x \in \mathbb{R}^n$, we deduce from this a cruder estimate:

$$\mathbf{U}_{\alpha,p}f(x) \leq C_1 \|f\|_{L^\infty(\mathbb{R}^n)}^{\frac{n-\alpha ps}{(p-1)n}} \|f\|_{L^s(\mathbb{R}^n)}^{\frac{\alpha ps}{(p-1)n}}. \quad (4.20)$$

In the case $1 < p \leq 2 - \frac{\alpha}{n}$, we have $\alpha \leq (2-p)n < \frac{n}{p}$. Using (4.11) instead of (4.10), we see that (4.20) still holds. Therefore (4.20) holds for all $1 < p < \frac{n}{\alpha}$ and $1 \leq s < \frac{n}{\alpha p}$.

If $0 < s < 1$, then obviously $\|f\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)}^{1-s} \|f\|_{L^s(\mathbb{R}^n)}^s$. Consequently, (4.20) for $0 < s < 1$ follows from the corresponding estimate with $s = 1$, which proves statement (b) of Theorem 4.4.

To prove statement (c), notice that in the case $s = \frac{n}{\alpha p} \geq 1$ and $p > 2 - \frac{\alpha}{n}$, we clearly have, by looking at the cases $0 < a < 1$ and $a \geq 1$,

$$a^{\frac{1}{p-1}} \int_0^{a^{-\frac{s}{n}}} r^{\frac{\alpha p}{p-1}} e^{-cr} \frac{dr}{r} + \int_{a^{-\frac{s}{n}}}^\infty e^{-cr} \frac{dr}{r} \leq C \left(\left(\frac{a}{a+1} \right)^{\frac{1}{p-1}} + \log^+ a \right).$$

Hence (4.19) yields

$$\mathbf{V}_{\alpha,p} f(x) \leq c \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}} \left(\frac{(\mathbf{M}f(x))^{\frac{1}{p-1}}}{(\mathbf{M}f(x))^{\frac{1}{p-1}} + \|f\|_{L^s(\mathbb{R}^n)}^{\frac{1}{p-1}}} + \log^+ \left(\frac{\mathbf{M}f(x)}{\|f\|_{L^s(\mathbb{R}^n)}} \right) \right), \quad (4.21)$$

for $s = \frac{n}{\alpha p} \geq 1$ and $p > 2 - \frac{\alpha}{n}$, which proves statement (c).

In the case $s = \frac{n}{\alpha p} > 1$ and $1 < p \leq 2 - \frac{\alpha}{n}$, as in the estimates of $\mathbf{U}_{\alpha,p} f$ above, we again use (4.11) in place of (4.10), together with $\mathbf{M}f(x) \leq \|f\|_{L^\infty(\mathbb{R}^n)}$, to complete the proof of statement (d). \square

Remark 1. Inequality (4.15) can be deduced directly applying Hedberg's inequality (see [1], Proposition 3.1.2(a)) to $\mathbf{I}_\alpha g$ where $g = (\mathbf{I}_\alpha f)^{\frac{1}{p-1}}$, followed by Sobolev's inequality.

Remark 2. Theorem 4.1 is immediate from statements (b) and (d) of Theorem 4.4 with $\alpha = 2$ and $p = 1 + \frac{1}{\sigma}$.

We will need the following corollary of Theorem 4.1 in the next section.

Corollary 4.1. *Let $g \in L^\infty(B)$ be a nonnegative function where $B = B_R(x_0)$ is a ball in \mathbb{R}^n , $n \geq 3$, and let*

$$Ng(x) = \int_B \frac{g(y)}{|x-y|^{n-2}} dy, \quad x \in B.$$

Then for $\sigma \geq \frac{2}{n-2}$ we have

$$\|N((Ng)^\sigma)\|_{L^\infty(B)} \leq \begin{cases} C \|g\|_{L^1(B)}^{\frac{2\sigma+2}{n}} \|g\|_{L^\infty(B)}^{\frac{(n-2)\sigma-2}{n}}, & \text{if } \sigma > \frac{2}{n-2} \\ C \|g\|_{L^1(B)}^\sigma \log \left(\frac{C|B| \|g\|_{L^\infty(B)}}{\|g\|_{L^1(B)}} \right), & \text{if } \sigma = \frac{2}{n-2} \end{cases}$$

where $C = C(n, \sigma)$ is a positive constant.

Proof. Apply Theorem 4.1 with $s = 1$ to the function $f(x) = g(x_0 + Rx)$. \square

5 Preliminary lemmas

In this section we provide some lemmas needed for the proofs of our results in Sections 2 and 3.

Lemma 5.1. *Let $\varphi : (0, 1) \rightarrow (0, 1)$ be a continuous function such that $\lim_{r \rightarrow 0^+} \varphi(r) = 0$. Let $\{x_j\}_{j=1}^\infty$ be a sequence in \mathbb{R}^n , where $n \geq 3$ (resp. $n = 2$), such that*

$$0 < 4|x_{j+1}| < |x_j| < \frac{1}{2} \quad (5.1)$$

and

$$\sum_{j=1}^{\infty} \varphi(|x_j|) < \infty. \quad (5.2)$$

Let $\{r_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence satisfying

$$0 < r_j \leq |x_j|/2. \quad (5.3)$$

Then there exist a positive constant $A = A(n)$ and a positive function $u \in C^\infty(\Omega \setminus \{0\})$ where $\Omega = \mathbb{R}^n$ (resp. $\Omega = B_2(0) \subset \mathbb{R}^2$) such that

$$0 \leq -\Delta u \leq \frac{\varphi(|x_j|)}{r_j^n} \quad \text{in } B_{r_j}(x_j) \quad (5.4)$$

$$-\Delta u = 0 \quad \text{in } \Omega \setminus \left(\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \right) \quad (5.5)$$

$$u \geq \frac{A\varphi(|x_j|)}{r_j^{n-2}} \quad \left(\text{resp. } u \geq A\varphi(|x_j|) \log \frac{1}{r_j} \right) \quad \text{in } B_{r_j}(x_j) \quad (5.6)$$

$$u \geq 1 \quad \text{in } \Omega \setminus \{0\}. \quad (5.7)$$

Proof. Let $\psi : \mathbb{R}^n \rightarrow [0, 1]$ be a C^∞ function whose support is $\overline{B_1(0)}$. Define $\psi_j : \mathbb{R}^n \rightarrow [0, 1]$ by $\psi_j(y) = \psi(\eta)$ where $y = x_j + r_j\eta$. Then

$$\int_{\mathbb{R}^n} \psi_j(y) dy = \int_{\mathbb{R}^n} \psi(\eta) r_j^n d\eta = r_j^n I \quad (5.8)$$

where $I = \int_{\mathbb{R}^n} \psi(\eta) d\eta > 0$. Let $\varepsilon_j := \varphi(|x_j|)$ and

$$f := \sum_{j=1}^{\infty} M_j \psi_j \quad \text{where } M_j = \frac{\varepsilon_j}{r_j^n}. \quad (5.9)$$

Since the functions ψ_j have disjoint supports, $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and by (5.9), (5.8), and (5.2) we have

$$\begin{aligned} \int_{\mathbb{R}^n} f(y) dy &= \sum_{j=1}^{\infty} M_j \int_{\mathbb{R}^n} \psi_j(y) dy = I \sum_{j=1}^{\infty} M_j r_j^n \\ &= I \sum_{j=1}^{\infty} \varepsilon_j < \infty. \end{aligned} \quad (5.10)$$

Case I. Suppose $n \geq 3$. Then for $x = x_j + r_j\xi$ and $|\xi| < 1$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy &\geq \int_{|y-x_j| < r_j} \frac{M_j \psi_j(y)}{|x-y|^{n-2}} dy = \int_{|\eta| < 1} \frac{M_j \psi(\eta) r_j^n}{r_j^{n-2} |\xi - \eta|^{n-2}} d\eta \\ &= \frac{\varepsilon_j}{r_j^{n-2}} \int_{|\eta| < 1} \frac{\psi(\eta)}{|\xi - \eta|^{n-2}} d\eta \\ &\geq \frac{J\varepsilon_j}{r_j^{n-2}} \quad \text{where } J = \min_{|\xi| \leq 1} \int_{|\eta| < 1} \frac{\psi(\eta) d\eta}{|\xi - \eta|^{n-2}} > 0. \end{aligned}$$

Thus letting

$$u(x) := \int_{\mathbb{R}^n} \frac{B}{|x-y|^{n-2}} f(y) dy + 1 \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}$$

where $\frac{B}{|x|^{n-2}}$ is a fundamental solution of $-\Delta$ we have u satisfies (5.6) with $A = BJ$ and

$$-\Delta u(x) = f(x) = M_j \psi_j(x) \leq \frac{\varepsilon_j}{r_j^n} \quad \text{for } x \in B_{r_j}(x_j).$$

Also $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and u clearly satisfies (5.5) and (5.7).

Case II. Suppose $n = 2$. Then for $x = x_j + r_j \xi$ and $|\xi| < 1$ we have

$$\begin{aligned} \int_{|y|<2} \left(\log \frac{4}{|x-y|} \right) f(y) dy &\geq \int_{|y-x_j|<r_j} \left(\log \frac{4}{|x-y|} \right) M_j \psi_j(y) dy \\ &= \int_{|\eta|<1} \left(\log \frac{1}{r_j} + \log \frac{4}{|\xi-\eta|} \right) M_j \psi(\eta) r_j^2 d\eta \\ &\geq \varepsilon_j \left[\log \frac{1}{r_j} \int_{|\eta|<1} \psi(\eta) d\eta \right] = I \varepsilon_j \log \frac{1}{r_j}. \end{aligned}$$

Thus letting

$$u(x) := \int_{|y|<2} \frac{1}{2\pi} \left(\log \frac{4}{|x-y|} \right) f(y) dy + 1 \quad \text{for } x \in B_2(0) \setminus \{0\}$$

we have u satisfies (5.6) with $A = \frac{I}{2\pi}$ and

$$-\Delta u(x) = f(x) = M_j \psi_j(x) \leq \frac{\varepsilon_j}{r_j^2} \quad \text{for } x \in B_{r_j}(x_j).$$

Also $u \in C^\infty(B_2(0) \setminus \{0\})$ and u clearly satisfies (5.5) and (5.7). □

Lemma 5.2. *Let u be a C^2 nonnegative superharmonic function in $B_{2\varepsilon}(0) \setminus \{0\} \subset \mathbb{R}^2$ satisfying*

$$\log^+(-\Delta u(x)) = O\left(H\left(\log \frac{1}{|x|}\right)\right) \quad \text{as } x \rightarrow 0 \quad (5.11)$$

where $\varepsilon \in (0, 1/2)$ and $H : (0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function satisfying $\lim_{t \rightarrow \infty} H(t) = \infty$. Then

$$u(x) = O\left(\log \frac{1}{|x|}\right) + o\left(H\left(\log \frac{2}{|x|}\right)\right) \quad \text{as } x \rightarrow 0. \quad (5.12)$$

Proof. Let $x_j \in B_{\frac{\varepsilon}{2}}(0) \setminus \{0\}$ be a sequence which converges to the origin. It suffices to prove (5.12) with x replaced with x_j .

By (5.11) there exists $A > 0$ such that

$$\log^+(-\Delta u(y)) \leq AH\left(\log \frac{2}{|x_j|}\right) \quad \text{for } |y - x_j| \leq \frac{|x_j|}{2}.$$

Thus

$$0 \leq -\Delta u(y) \leq \exp\left(AH\left(\log\frac{2}{|x_j|}\right)\right) \quad \text{for } |y - x_j| \leq \frac{|x_j|}{2}. \quad (5.13)$$

Define $r_j \geq 0$ by

$$\int_{|y-x_j|<r_j} e^{AH\left(\log\frac{2}{|x_j|}\right)} dy = \int_{|y-x_j|<\frac{|x_j|}{2}} -\Delta u(y) dy \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

by Lemma A.1. Thus

$$r_j = o\left(\exp\left(-\frac{A}{2}H\left(\log\frac{2}{|x_j|}\right)\right)\right) \quad \text{as } j \rightarrow \infty \quad (5.14)$$

and by (5.13)

$$\int_{|y-x_j|<\frac{|x_j|}{2}} \left(\log\frac{1}{|y-x_j|}\right) (-\Delta u(y)) dy \leq \int_{|y-x_j|<r_j} \left(\log\frac{1}{|y-x_j|}\right) \exp\left(AH\left(\log\frac{2}{|x_j|}\right)\right) dy.$$

Hence by Lemma A.1 we get

$$\begin{aligned} u(x_j) &\leq C \left[\log\frac{1}{|x_j|} + \int_{|y-x_j|>\frac{|x_j|}{2}, |y|<\varepsilon} \left(\log\frac{1}{|y-x_j|}\right) (-\Delta u(y)) dy \right] \\ &\quad + C \int_{|y-x_j|<r_j} \left(\log\frac{1}{|y-x_j|}\right) \exp\left(AH\left(\log\frac{2}{|x_j|}\right)\right) dy \\ &\leq C \left[\log\frac{1}{|x_j|} + r_j^2 \left(\log\frac{1}{r_j}\right) \exp\left(AH\left(\log\frac{2}{|x_j|}\right)\right) \right] \\ &\leq C \log\frac{1}{|x_j|} + o\left(H\left(\log\frac{2}{|x_j|}\right)\right) \quad \text{as } j \rightarrow \infty \end{aligned}$$

by (5.14). □

Lemma 5.3. *Let u be a C^2 nonnegative function in $B_{3\varepsilon}(0) \setminus \{0\} \subset \mathbb{R}^n$, $n \geq 3$, satisfying*

$$0 \leq -\Delta u(x) = O\left(\left(\frac{1}{|x|}\right)^\gamma \left(\log\frac{1}{|x|}\right)^q\right) \quad \text{as } x \rightarrow 0 \quad (5.15)$$

where $\varepsilon \in (0, 1/8)$, $\gamma \in \mathbb{R}$, and $q \geq 0$ are constants.

(i) *If $q = 0$ then*

$$u(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\gamma\frac{n-2}{n}}\right) \quad \text{as } x \rightarrow 0. \quad (5.16)$$

(ii) *If $q > 0$ and $\gamma \geq n$ then*

$$u(x) = o\left(\left(\frac{1}{|x|}\right)^{\gamma\frac{n-2}{n}} \left(\log\frac{1}{|x|}\right)^{q\frac{n-2}{n}}\right) \quad \text{as } x \rightarrow 0. \quad (5.17)$$

(iii) If $q = 0$, $\gamma > n$, and $v(x)$ is a C^2 nonnegative solution of

$$0 \leq -\Delta v \leq |x|^{-\beta} \left(u + |x|^{-(n-2)} \right)^\sigma \quad \text{in } B_{3\varepsilon}(0) \setminus \{0\} \quad (5.18)$$

where $\beta \in \mathbb{R}$ and $\sigma \geq 2/(n-2)$ then as $x \rightarrow 0$ we have

$$v(x) = \begin{cases} O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\frac{2}{n}[(n-2)\sigma-2]+\beta}\right) & \text{if } \sigma > \frac{2}{n-2} \\ O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^\beta \log \frac{1}{|x|}\right) & \text{if } \sigma = \frac{2}{n-2}. \end{cases} \quad (5.19)$$

Proof. Let $\{x_j\} \subset B_\varepsilon(0) \setminus \{0\}$ be a sequence which converges to the origin. It suffices to prove (5.16), (5.17), (5.19), and (5.20) with x replaced with x_j .

For the proof of part (i) we can assume $\gamma \geq n$ because increasing γ to n weakens condition (5.15) and does not change the estimate (5.16).

By (5.15) there exists $A > 0$ such that

$$-\Delta u(y) < \frac{A}{(2|y|)^\gamma} \left(\log \frac{1}{2|y|} \right)^q \quad \text{for } 0 < |y| < 2\varepsilon.$$

Thus

$$-\Delta u(y) \leq \frac{A}{|x_j|^\gamma} \left(\log \frac{1}{|x_j|} \right)^q \quad \text{for } |y - x_j| < \frac{|x_j|}{2}. \quad (5.21)$$

Define $r_j \geq 0$ by

$$\int_{|y-x_j| < r_j} \frac{A}{|x_j|^\gamma} \left(\log \frac{1}{|x_j|} \right)^q dy = \int_{|y-x_j| < \frac{|x_j|}{2}} -\Delta u(y) dy \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad (5.22)$$

by Lemma A.1. Then

$$r_j = o\left(|x_j|^{\frac{\gamma}{n}} \left(\log \frac{1}{|x_j|} \right)^{-\frac{q}{n}}\right) \ll |x_j| \quad \text{as } j \rightarrow \infty \quad (5.23)$$

because $\gamma \geq n$ and $q \geq 0$. Hence by Lemma A.1 and (5.21) we have

$$\begin{aligned} u(x_j) &\leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j| > \frac{|x_j|}{2}, |y| < 2\varepsilon} \frac{-\Delta u(y) dy}{|y-x_j|^{n-2}} + \int_{|y-x_j| < \frac{|x_j|}{2}} \frac{-\Delta u(y) dy}{|y-x_j|^{n-2}} \right] \\ &\leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j| < \frac{|x_j|}{2}} \frac{-\Delta u(y) dy}{|y-x_j|^{n-2}} \right] \\ &\leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j| < r_j} \frac{A \left(\frac{1}{|x_j|}\right)^\gamma \left(\log \frac{1}{|x_j|}\right)^q}{|y-x_j|^{n-2}} dy \right] \\ &\leq C \left[\left(\frac{1}{|x_j|}\right)^{n-2} + \left(\frac{1}{|x_j|}\right)^\gamma \left(\log \frac{1}{|x_j|}\right)^q r_j^2 \right] \\ &\leq C \left[\left(\frac{1}{|x_j|}\right)^{n-2} + o\left(\left(\frac{1}{|x_j|}\right)^\gamma \left(\log \frac{1}{|x_j|}\right)^q\right)^{\frac{n-2}{n}} \right] \quad \text{as } j \rightarrow \infty \end{aligned}$$

by (5.23), which proves parts (i) and (ii).

We now prove part (iii). For $|x - x_j| < \frac{|x_j|}{4}$ we have by (5.18) and Lemma A.1 that

$$\begin{aligned} \frac{-\Delta v(x)}{|x|^{-\beta}} &\leq \left(u(x) + |x|^{-(n-2)} \right)^\sigma \\ &\leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j| < \frac{|x_j|}{2}} \frac{-\Delta u(y) dy}{|y-x|^{n-2}} + \int_{|y-x_j| > \frac{|x_j|}{2}, |y| < 2\varepsilon} \frac{-\Delta u(y) dy}{|y-x|^{n-2}} \right]^\sigma \\ &\leq C \left[\frac{1}{|x_j|^{\sigma(n-2)}} + \left(\left(N_{B_{\frac{|x_j|}{2}}}(x_j)(-\Delta u) \right)(x) \right)^\sigma \right] \end{aligned}$$

where $(N_\Omega f)(x) := \int_\Omega \frac{f(y)}{|y-x|^{n-2}} dy$.

Thus by Lemma A.1

$$\begin{aligned} v(x_j) &\leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j| < \frac{|x_j|}{4}} \frac{\Delta v(y)}{|y-x_j|^{n-2}} dy + \int_{|y-x_j| > \frac{|x_j|}{4}, |y| < 2\varepsilon} \frac{-\Delta v(y)}{|y-x_j|^{n-2}} dy \right] \\ &\leq C \left[\frac{1}{|x_j|^{n-2}} + \frac{|x_j|^{-\beta}}{|x_j|^{\sigma(n-2)-2}} + |x_j|^{-\beta} (H(-\Delta u))(x_j) \right] \end{aligned} \quad (5.24)$$

where $Hf = N_{B_{\frac{|x_j|}{4}}}(x_j) \left(N_{B_{\frac{|x_j|}{2}}}(x_j) f \right)^\sigma$.

Case I. Suppose $(n-2)\sigma > 2$. Then using (5.21) and (5.22) with $q = 0$ in Corollary 4.1 we get

$$(H(-\Delta u))(x_j) = o\left(\frac{1}{|x_j|^{\frac{2}{n}(\sigma(n-2)-2)}} \right) \quad \text{as } j \rightarrow \infty.$$

Thus (5.19) follows from (5.24).

Case II. Suppose $(n-2)\sigma = 2$. Then using (5.21), (5.22), and (5.23) with $q = 0$ in Corollary 4.1 we get

$$\begin{aligned} (H(-\Delta u))(x_j) &\leq C \left(r_j^n \frac{A}{|x_j|^\gamma} \right)^\sigma \log \left(\frac{C|x_j|^n (A/|x_j|^\gamma)}{r_j^n A/|x_j|^\gamma} \right) \\ &= C|x_j|^{n\sigma-\gamma\sigma} \left(\frac{r_j}{|x_j|} \right)^{n\sigma} \log \left(C \left(\frac{|x_j|}{r_j} \right)^n \right) \\ &= o\left(|x_j|^{n\sigma-\gamma\sigma} |x_j|^{(\gamma-n)\sigma} \log \frac{1}{|x_j|^{\gamma-n}} \right) \\ &= o\left(\log \frac{1}{|x_j|} \right) \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Thus (5.20) follows from (5.24). □

Lemma 5.4. Suppose $u(x)$ and $v(x)$ are C^2 nonnegative solutions of the system

$$0 \leq -\Delta u \quad (5.25)$$

$$0 \leq -\Delta v \leq |x|^{-\beta} \left(u + |x|^{-(n-2)} \right)^\sigma \quad (5.26)$$

in a punctured neighborhood of the origin in \mathbb{R}^n , $n \geq 3$, where $\beta \in \mathbb{R}$.

(i) If $0 \leq \sigma < \frac{2}{n-2}$ then

$$v(x) = O\left(|x|^{-(n-2)} + |x|^{2-(n-2)\sigma-\beta}\right) \quad \text{as } x \rightarrow 0. \quad (5.27)$$

(ii) If σ and λ satisfy (3.20) and

$$-\Delta u \leq |x|^{-\alpha} \left(v + |x|^{-(n-2)}\right)^\lambda \quad (5.28)$$

in a punctured neighborhood of the origin, where $\alpha \in \mathbb{R}$, then for some $\gamma > n$ we have

$$-\Delta u(x) = O(|x|^{-\gamma}) \quad \text{as } x \rightarrow 0. \quad (5.29)$$

Proof. Choose $\varepsilon \in (0, 1)$ such that $u(x)$ and $v(x)$ are C^2 nonnegative solutions of the system (5.25, 5.26) in $B_{2\varepsilon}(0) \setminus \{0\}$. Let $\{x_j\}_{j=1}^\infty$ be a sequence in \mathbb{R}^n such that

$$0 < 4|x_{j+1}| < |x_j| < \varepsilon/2.$$

It suffices to prove (5.27) and (5.29) with x replaced with x_j .

By Lemma A.1 we have

$$\int_{|x|<\varepsilon} -\Delta u(x) dx < \infty \quad \text{and} \quad \int_{|x|<\varepsilon} -\Delta v(x) dx < \infty \quad (5.30)$$

and, for $|x - x_j| < \frac{|x_j|}{4}$,

$$u(x) \leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j|<\frac{|x_j|}{2}} \frac{1}{|x-y|^{n-2}} (-\Delta u(y)) dy \right] \quad (5.31)$$

$$v(x) \leq C \left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j|<\frac{|x_j|}{2}} \frac{1}{|x-y|^{n-2}} (-\Delta v(y)) dy \right] \quad (5.32)$$

where $C > 0$ does not depend on j or x .

By (5.30), we have as $j \rightarrow \infty$ that

$$\int_{|y-x_j|<\frac{|x_j|}{2}} -\Delta u(y) dy \rightarrow 0 \quad \text{and} \quad \int_{|y-x_j|<\frac{|x_j|}{2}} -\Delta v(y) dy \rightarrow 0. \quad (5.33)$$

Define $f_j, g_j : \overline{B_2(0)} \rightarrow [0, \infty)$ by

$$f_j(\xi) = -r_j^n \Delta u(x_j + r_j \xi) \quad \text{and} \quad g_j(\xi) = -r_j^n \Delta v(x_j + r_j \xi)$$

where $r_j = |x_j|/4$. Making the change of variables $y = x_j + r_j \zeta$ in (5.33), (5.32), and (5.31) we get

$$\int_{|\zeta|<2} f_j(\zeta) d\zeta \rightarrow 0 \quad \text{and} \quad \int_{|\zeta|<2} g_j(\zeta) d\zeta \rightarrow 0, \quad (5.34)$$

and

$$v(x_j + r_j \xi) \leq \frac{C}{r_j^{n-2}} [1 + (N_2 g_j)(\xi)] \quad \text{for } |\xi| < 1 \quad (5.35)$$

$$u(x_j + r_j \xi) \leq \frac{C}{r_j^{n-2}} [1 + (N_2 f_j)(\xi)] \quad \text{for } |\xi| < 1. \quad (5.36)$$

where $(N_R f)(\xi) := \int_{|\zeta| < R} |\xi - \zeta|^{-(n-2)} f(\zeta) d\zeta$.

We now prove part (i). If $\sigma = 0$ then part (i) follows from Lemma 5.3(i). Hence we can assume $0 < \sigma < 2/(n-2)$. Define $\varepsilon \in (0, 1)$ and $\gamma > 0$ by

$$\sigma = \frac{2}{n-2}(1-\varepsilon)^2 \quad \text{and} \quad \gamma = \frac{n}{n-2}(1-\varepsilon).$$

It follows from (5.34) and Riesz potential estimates (see [6, Lemma 7.12]) that $N_2 f_j \rightarrow 0$ in $L^\gamma(B_2(0))$ and hence

$$(N_2 f_j)^\sigma \rightarrow 0 \quad \text{in } L^{\frac{n}{2(1-\varepsilon)}}(B_2(0)).$$

Thus by Hölder's inequality

$$\int_{B_1(0)} \Gamma (N_2 f_j)^\sigma d\xi \leq \|\Gamma\|_{\frac{n}{n+2\varepsilon-2}} \|(N_2 f_j)^\sigma\|_{\frac{n}{2(1-\varepsilon)}} \rightarrow 0 \quad (5.37)$$

where Γ is given by (1.8). By (5.35) and (5.34) we have

$$\begin{aligned} v(x_j) &\leq \frac{C}{|x_j|^{n-2}} \left(1 + \int_{B_2(0)} \Gamma g_j d\xi \right) \\ &\leq \frac{C}{|x_j|^{n-2}} \left(1 + \int_{B_1(0)} \Gamma g_j d\xi \right) \end{aligned} \quad (5.38)$$

and for $|\xi| < 1$ it follows from (5.26) and (5.36) that

$$\begin{aligned} g_j(\xi) &= r_j^n (-\Delta v(x_j + r_j \xi)) \\ &\leq C r_j^n |x_j|^{-\beta} \left(u(x_j + r_j \xi) + |x_j|^{-(n-2)} \right)^\sigma \\ &\leq C |x_j|^{n-\beta-(n-2)\sigma} (1 + ((N_2 f_j)(\xi))^\sigma). \end{aligned} \quad (5.39)$$

Substituting (5.39) in (5.38) and using (5.37), we get

$$v(x_j) \leq C \left(|x_j|^{-(n-2)} + |x_j|^{2-(n-2)\sigma-\beta} \right)$$

which completes the proof of part (i).

Next we prove part (ii). Since increasing λ and/or σ weakens the conditions (5.26, 5.28) on u and v we can assume instead of (3.20) that

$$\lambda \geq \sigma \geq \frac{2}{n-2} \quad \text{and} \quad \sigma < \frac{2}{n-2} + \frac{n}{n-2} \frac{1}{\lambda}. \quad (5.40)$$

Since, for $R \in (0, \frac{1}{2}]$,

$$\int_{2R < |\zeta| < 2} \frac{g_j(\zeta) d\zeta}{|\xi - \zeta|^{n-2}} \leq \frac{1}{R^{n-2}} \int_{|\zeta| < 2} g_j(\zeta) d\zeta \quad \text{for } |\xi| < R$$

and

$$\int_{4R < |\eta| < 2} \frac{f_j(\eta) d\eta}{|\zeta - \eta|^{n-2}} \leq \frac{1}{(2R)^{n-2}} \int_{|\eta| < 2} f_j(\eta) d\eta \quad \text{for } |\zeta| < 2R,$$

it follows from (5.34), (5.35) and (5.36) that for $R \in (0, \frac{1}{2}]$ we have

$$v(x_j + r_j \xi) \leq Cr_j^{2-n} \left[\frac{1}{R^{n-2}} + N_{2R} g_j(\xi) \right] \quad \text{for } |\xi| < R$$

and

$$u(x_j + r_j \zeta) \leq Cr_j^{2-n} \left[\frac{1}{R^{n-2}} + N_{4R} f_j(\zeta) \right] \quad \text{for } |\zeta| < 2R$$

where C is independent of ξ , ζ , j , and R . It therefore follows from (5.26, 5.28) that for $R \in (0, \frac{1}{2}]$ we have

$$\begin{aligned} r_j^{-n} f_j(\xi) &= -\Delta u(x_j + r_j \xi) \\ &\leq Cr_j^{-\alpha} \left(r_j^{2-n} \left[\frac{1}{R^{n-2}} + (N_{2R} g_j)(\xi) \right] \right)^\lambda \\ &\leq Cr_j^{-\alpha - (n-2)\lambda} \left[\frac{1}{R^{(n-2)\lambda}} + ((N_{2R} g_j)(\xi))^\lambda \right] \quad \text{for } |\xi| < R, \end{aligned} \quad (5.41)$$

and

$$\begin{aligned} r_j^{-n} g_j(\zeta) &= -\Delta v(x_j + r_j \zeta) \\ &\leq Cr_j^{-\beta} \left(r_j^{2-n} \left[\frac{1}{R^{n-2}} + (N_{4R} f_j)(\zeta) \right] \right)^\sigma \\ &\leq Cr_j^{-b} \left[\frac{1}{R^{(n-2)\sigma}} + ((N_{4R} f_j)(\zeta))^\sigma \right] \quad \text{for } |\zeta| < 2R, \end{aligned}$$

where $b = \beta + (n-2)\sigma$. Thus for $\xi \in \mathbb{R}^n$ we have

$$\begin{aligned} ((N_{2R} g_j)(\xi))^\lambda &\leq \left(Cr_j^{n-b} N_{2R} \left[\frac{1}{R^{(n-2)\sigma}} + (N_{4R} f_j)^\sigma \right] (\xi) \right)^\lambda \\ &\leq Cr_j^{(n-b)\lambda} \left[R^{(2-(n-2)\sigma)\lambda} + ((M_{4R} f_j)(\xi))^\lambda \right] \end{aligned}$$

where $M_R f_j := N_R ((N_R f_j)^\sigma)$. Hence by (5.41) there exists a positive constant a which depends only on n , α , β , λ , and σ such that

$$f_j(\xi) \leq C \frac{1}{(Rr_j)^a} \left(1 + ((M_{4R} f_j)(\xi))^\lambda \right) \quad \text{for } |\xi| < R \leq \frac{1}{2}. \quad (5.42)$$

By (5.40) there exists $\varepsilon = \varepsilon(n, \lambda, \sigma) \in (0, 1)$ such that

$$\sigma < \frac{n}{n-2+\varepsilon} \quad \text{and} \quad \sigma < \frac{2-\varepsilon}{n-2+\varepsilon} + \frac{n}{n-2+\varepsilon} \frac{1}{\lambda}. \quad (5.43)$$

To prove for some $\gamma > n$ that (5.29) holds with $x = x_j$, it suffices by the definition of r_j and f_j to show for some $\gamma > 0$ that the sequence

$$\{r_j^\gamma f_j(0)\} \quad \text{is bounded.} \quad (5.44)$$

To prove (5.44) and thereby complete the proof of Lemma 5.4(ii), we need the following result.

Lemma 5.5. *Suppose the sequence*

$$\{r_j^\alpha f_j\} \quad \text{is bounded in } L^p(B_{4R}(0)) \quad (5.45)$$

for some constants $\alpha \geq 0$, $p \in [1, \infty)$, and $R \in (0, \frac{1}{2}]$. Let $\beta = \alpha\lambda\sigma + a$ where a is as in (5.42). Then there exists a constant $C_0 = C_0(n, \lambda, \sigma) > 0$ such that the sequence

$$\{r_j^\beta f_j\} \quad \text{is bounded in } L^q(B_R(0)) \quad (5.46)$$

provided $q \in [1, \infty]$ and

$$\frac{1}{p} - \frac{1}{q} \leq C_0. \quad (5.47)$$

Proof. It follows from (5.42) that

$$r_j^\beta f_j(\xi) \leq \frac{C}{R^a} \left(1 + ((M_{4R}(r_j^\alpha f_j))(\xi))^\lambda\right) \quad \text{for } |\xi| < R. \quad (5.48)$$

We can assume

$$p \leq n/2 \quad (5.49)$$

for otherwise it follows from Riesz potential estimates (see [6, Lemma 7.12]) and (5.45) that the sequence $\{N_{4R}(r_j^\alpha f_j)\}$ is bounded in $L^\infty(B_{4R}(0))$ and hence by (5.48) we see that (5.46) holds for all $q \in [1, \infty]$.

Define p_2 by

$$\frac{1}{p} - \frac{1}{p_2} = \frac{2 - \varepsilon}{n}. \quad (5.50)$$

where $\varepsilon = \varepsilon(n, \lambda, \sigma)$ is as in (5.43). By (5.49), $p_2 \in (p, \infty)$ and by Riesz potential estimates we have

$$\|(N_{4R}f_j)^\sigma\|_{p_2/\sigma} = \|N_{4R}f_j\|_{p_2}^\sigma \leq C\|f_j\|_p^\sigma \quad (5.51)$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(B_{4R}(0))}$. Since, by (5.43),

$$\frac{1}{p_2} = \frac{1}{p} - \frac{2 - \varepsilon}{n} \leq 1 - \frac{2 - \varepsilon}{n} = \frac{n - 2 + \varepsilon}{n} < \frac{1}{\sigma}$$

we have

$$p_2/\sigma > 1. \quad (5.52)$$

We can assume

$$p_2/\sigma \leq n/2 \quad (5.53)$$

for otherwise by Riesz potential estimates and (5.51) we have

$$\|M_{4R}(r_j^\alpha f_j)\|_\infty \leq C\|(N_{4R}(r_j^\alpha f_j))^\sigma\|_{p_2/\sigma} \leq C\|r_j^\alpha f_j\|_p^\sigma$$

which is bounded by (5.45). Hence (5.48) implies (5.46) holds for all $q \in [1, \infty]$.

Define p_3 and q by

$$\frac{\sigma}{p_2} - \frac{1}{p_3} = \frac{2 - \varepsilon}{n} \quad \text{and} \quad q = \frac{p_3}{\lambda}. \quad (5.54)$$

By (5.52) and (5.53), $p_3 \in (1, \infty)$ and by Riesz potential estimates

$$\begin{aligned} \|(M_{4R}f_j)^\lambda\|_q &= \|M_{4R}f_j\|_{p_3}^\lambda \\ &\leq C\|(N_{4R}f_j)^\sigma\|_{p_2/\sigma}^\lambda \leq C\|f_j\|_p^{\lambda\sigma} \end{aligned}$$

by (5.51). It follows therefore from (5.48) that

$$\|r_j^\beta f_j\|_{L^q(B_R(0))} \leq \frac{C}{R^a} \left(1 + \|r_j^\alpha f_j\|_p^{\lambda\sigma}\right)$$

which is a bounded sequence by (5.45). To complete the proof of Lemma 5.5, it suffices to show

$$\frac{1}{p} - \frac{1}{q} \geq C_0 \quad (5.55)$$

for some $C_0 = C_0(n, \lambda, \sigma) > 0$ because if (5.46) holds for some $q \geq 1$ satisfying (5.55) then it clearly holds for all $q \geq 1$ satisfying (5.47).

By (5.50) and (5.54) we have

$$\begin{aligned} \frac{1}{p} - \frac{1}{q} &= \frac{1}{p} - \frac{\lambda}{p_3} = \frac{1}{p} + \frac{(2-\varepsilon)\lambda}{n} - \frac{\lambda\sigma}{p_2} \\ &= \frac{1}{p} + \frac{(2-\varepsilon)\lambda}{n} + \frac{(2-\varepsilon)\lambda\sigma}{n} - \frac{\lambda\sigma}{p} \\ &= -\frac{\lambda\sigma - 1}{p} + \frac{(2-\varepsilon)\lambda\sigma + (2-\varepsilon)\lambda}{n}. \end{aligned} \quad (5.56)$$

Case I. Suppose $\lambda\sigma \leq 1$. Then by (5.56) and (5.40)

$$\frac{1}{p} - \frac{1}{q} \geq \frac{(2-\varepsilon)\lambda\sigma + (2-\varepsilon)\lambda}{n} \geq C_1(n) > 0.$$

Case II. Suppose $\lambda\sigma > 1$. Then, by (5.56),

$$\begin{aligned} \frac{1}{p} - \frac{1}{q} &\geq 1 - \sigma\lambda + \frac{(2-\varepsilon)\lambda\sigma + (2-\varepsilon)\lambda}{n} \\ &= \frac{1}{n} [n + (2-\varepsilon)\lambda - \lambda\sigma(n - (2-\varepsilon))] \\ &= \frac{(n - (2-\varepsilon))\lambda}{n} \left[\frac{2-\varepsilon}{n - (2-\varepsilon)} + \frac{n}{n - (2-\varepsilon)} \frac{1}{\lambda} - \sigma \right] \\ &= C_2(n, \lambda, \sigma) > 0 \end{aligned}$$

by (5.43).

Thus (5.55) holds with $C_0 = \min(C_1, C_2)$. This completes the proof of Lemma 5.5. \square

We return now to the proof of Lemma 5.4(ii). By (5.34), the sequence $\{f_j\}$ is bounded in $L^1(B_2(0))$. Starting with this fact and iterating Lemma 5.5 a finite number of times (m times is enough if $m > 1/C_0$) we see that there exists $R_0 \in (0, \frac{1}{2})$ and $\gamma > n$ such that sequence $\{r_j^\gamma f_j\}$ is bounded in $L^\infty(B_{R_0}(0))$. In particular (5.44) holds. This completes the proof of Lemma 5.4(ii). \square

6 Proofs of two dimensional results

In this section we prove Theorems 2.1–2.5. The following theorem with $h(t) = t^\lambda$ immediately implies Theorems 2.1 and 2.3. We stated Theorems 2.1 and 2.3 separately in order to clearly highlight the differences between possibilities (i) and (iii) which are stated at the beginning of Section 2.

Theorem 6.1. *Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system*

$$0 \leq -\Delta u \tag{6.1}$$

$$0 \leq -\Delta v \leq g(u) \tag{6.2}$$

in a punctured neighborhood of the origin in \mathbb{R}^2 , where $g : (0, \infty) \rightarrow (0, \infty)$ is a continuous function satisfying

$$\log^+ g(t) = O(t) \quad \text{as } t \rightarrow \infty. \tag{6.3}$$

Then v is harmonically bounded, that is

$$\limsup_{x \rightarrow 0} \frac{v(x)}{\log \frac{1}{|x|}} < A \tag{6.4}$$

for some constant $A > 0$.

If, in addition,

$$-\Delta u \leq f(v) \tag{6.5}$$

in a punctured neighborhood of the origin, where $f : (0, \infty) \rightarrow (0, \infty)$ is a continuous function satisfying

$$\log^+ f(t) = O(h(t)) \quad \text{as } t \rightarrow \infty \tag{6.6}$$

for some continuous nondecreasing function $h : (0, \infty) \rightarrow (0, \infty)$ satisfying $\lim_{t \rightarrow \infty} h(t) = \infty$ then

$$u(x) = O\left(\log \frac{1}{|x|}\right) + o\left(h\left(A \log \frac{2}{|x|}\right)\right) \quad \text{as } x \rightarrow 0. \tag{6.7}$$

For simplicity and to motivate Theorem 2.5, we stated Theorem 2.3 for the special case $h(t) = t^\lambda$ rather than for more general h as in Theorem 6.1. Also, the bound (2.9) in Theorem 2.3 is optimal by Theorem 2.4, whereas in general we can only show the bound (6.7) in Theorem 6.1 is essentially optimal (see Theorem 6.2).

Proof of Theorem 6.1. Since u is positive and superharmonic in a punctured neighborhood of the origin, there exists a constant $\varepsilon \in (0, 1/4)$ such that $u > \varepsilon$ in $B_{2\varepsilon}(0) \setminus \{0\}$. Choose a positive constant K such that $g(t) \leq e^{Kt}$ for $t > \varepsilon$. Then v is a C^2 positive solution of

$$0 \leq -\Delta v \leq e^{Ku} \quad \text{in } B_{2\varepsilon}(0) \setminus \{0\}. \tag{6.8}$$

Since u and v are positive and superharmonic in $B_{2\varepsilon}(0) \setminus \{0\}$, we have by Lemma A.1 that

$$-\Delta u, -\Delta v \in L^1(B_\varepsilon(0)) \tag{6.9}$$

and

$$\begin{aligned} u(x) &= m_1 \log \frac{1}{|x|} + \frac{1}{2\pi} \int_{|y| < \varepsilon} \left(\log \frac{1}{|x-y|} \right) (-\Delta u(y)) dy + h_1(x) \\ v(x) &= m_2 \log \frac{1}{|x|} + \frac{1}{2\pi} \int_{|y| < \varepsilon} \left(\log \frac{1}{|x-y|} \right) (-\Delta v(y)) dy + h_2(x) \end{aligned} \quad \text{for } 0 < |x| < \varepsilon \tag{6.10}$$

where $m_1, m_2 \geq 0$ are constants and $h_1, h_2 : B_\varepsilon(0) \rightarrow \mathbb{R}$ are harmonic functions.

Suppose for contradiction there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset B_{\frac{\varepsilon}{2}}(0) \setminus \{0\}$ such that $x_j \rightarrow 0$ as $j \rightarrow \infty$ and

$$\frac{v(x_j)}{\log \frac{1}{|x_j|}} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \quad (6.11)$$

Since, for $|x - x_j| < \frac{|x_j|}{4}$,

$$\int_{|y-x_j| > \frac{|x_j|}{2}, |y| < \varepsilon} \left(\log \frac{1}{|x-y|} \right) (-\Delta u(y)) dy \leq \left(\log \frac{4}{|x_j|} \right) \int_{|y| < \varepsilon} -\Delta u(y) dy,$$

and similarly for v , it follows from (6.9) and (6.10) that

$$\begin{aligned} u(x) &\leq C \log \frac{1}{|x_j|} + \frac{1}{2\pi} \int_{|y-x_j| < \frac{|x_j|}{2}} \left(\log \frac{1}{|x-y|} \right) (-\Delta u(y)) dy \\ v(x) &\leq C \log \frac{1}{|x_j|} + \frac{1}{2\pi} \int_{|y-x_j| < \frac{|x_j|}{2}} \left(\log \frac{1}{|x-y|} \right) (-\Delta v(y)) dy \end{aligned} \quad \text{for } |x - x_j| < \frac{|x_j|}{4} \quad (6.12)$$

where C does not depend on j or x .

Substituting $x = x_j$ in (6.12) and using (6.11) we get

$$\frac{1}{\log \frac{1}{|x_j|}} \int_{|y-x_j| < \frac{|x_j|}{2}} \left(\log \frac{1}{|x_j-y|} \right) (-\Delta v(y)) dy \rightarrow \infty \quad \text{as } j \rightarrow \infty. \quad (6.13)$$

Also, (6.9) implies

$$\int_{|y-x_j| < \frac{|x_j|}{2}} -\Delta u(y) dy \rightarrow 0 \quad \text{and} \quad \int_{|y-x_j| < \frac{|x_j|}{2}} -\Delta v(y) dy \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (6.14)$$

Let $r_j = \frac{|x_j|}{4}$ and define $f_j, g_j : \overline{B_2(0)} \rightarrow [0, \infty)$ by

$$f_j(\zeta) = -r_j^2 \Delta u(x_j + r_j \zeta) \quad \text{and} \quad g_j(\zeta) = -r_j^2 \Delta v(x_j + r_j \zeta).$$

Making the change of variables $y = x_j + r_j \zeta$ in (6.14), (6.13), and (6.12) and using (6.8) we get

$$\int_{|\zeta| < 2} f_j(\zeta) d\zeta \rightarrow 0 \quad \text{and} \quad \int_{|\zeta| < 2} g_j(\zeta) d\zeta \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (6.15)$$

$$\frac{1}{M_j} \int_{|\zeta| < 2} \left(\log \frac{4}{|\zeta|} \right) g_j(\zeta) d\zeta \rightarrow \infty \quad \text{as } j \rightarrow \infty, \quad (6.16)$$

and

$$\begin{aligned} g_j(\xi) &\leq -\Delta v(x_j + r_j \xi) \leq e^{Ku(x_j + r_j \xi)} \\ &\leq e^{M_j + u_j(\xi)} \quad \text{for } |\xi| < 1 \end{aligned} \quad (6.17)$$

where

$$u_j(\xi) := \frac{K}{2\pi} \int_{|\zeta| < 2} \left(\log \frac{4}{|\xi - \zeta|} \right) f_j(\zeta) d\zeta$$

and $M_j = C \log \frac{1}{|x_j|}$ for some constant C independent of j and ξ .

Let $\Omega_j = \{\xi \in B_1(0) : u_j(\xi) > M_j\}$. Then letting $p_j = \pi/(K \int_{|\zeta|<2} f_j(\zeta) d\zeta)$ and using Jensen's inequality, it follows from (6.17) that

$$\begin{aligned} \int_{\Omega_j} g_j(\xi)^{p_j} d\xi &\leq \int_{|\xi|<2} e^{2p_j u_j(\xi)} d\xi \\ &\leq \int_{|\xi|<2} \left(\int_{|\zeta|<2} \frac{4}{|\xi - \zeta|} \frac{f_j(\zeta)}{\int_{|\zeta|<2} f_j} d\zeta \right) d\xi \\ &= \int_{|\zeta|<2} \left(\int_{|\xi|<2} \frac{4 d\xi}{|\xi - \zeta|} \right) \frac{f_j(\zeta)}{\int_{|\zeta|<2} f_j} d\zeta \\ &\leq \int_{|\xi|<4} \frac{4 d\xi}{|\xi|} < \infty. \end{aligned}$$

(The idea of using Jensen's inequality as above is due to Brezis and Merle [5].) Thus by (6.15) and Hölder's inequality

$$\limsup_{j \rightarrow \infty} \int_{\Omega_j} \left(\log \frac{4}{|\zeta|} \right) g_j(\zeta) d\zeta < \infty.$$

Hence, defining $\hat{g}_j : B_1(0) \rightarrow [0, \infty)$ by

$$\hat{g}_j(\xi) = \begin{cases} g_j(\xi), & \text{for } \xi \in B_1(0) \setminus \Omega_j \\ 0, & \text{for } \xi \in \Omega_j \end{cases}$$

it follows from (6.15) and (6.16) that

$$\frac{1}{M_j} \int_{|\zeta|<1} \left(\log \frac{4}{|\zeta|} \right) \hat{g}_j(\zeta) d\zeta \rightarrow \infty \quad \text{as } j \rightarrow \infty. \quad (6.18)$$

By (6.15) and (6.17) we have

$$\int_{|\zeta|<1} \hat{g}_j(\zeta) d\zeta \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad (6.19)$$

and

$$\hat{g}_j(\xi) \leq e^{2M_j} \quad \text{in } B_1(0). \quad (6.20)$$

Define $\rho_j \geq 0$ by

$$\int_{|\zeta|<\rho_j} e^{2M_j} d\zeta = \int_{|\zeta|<1} \hat{g}_j(\zeta) d\zeta. \quad (6.21)$$

Then by (6.20) and (6.18) we find for large j that $\rho_j \in (0, 1]$ and

$$\begin{aligned} \int_{|\zeta|<\rho_j} (e^{2M_j} - \hat{g}_j(\zeta)) \log \frac{4}{|\zeta|} d\zeta &\geq \left(\log \frac{4}{\rho_j} \right) \int_{|\zeta|<\rho_j} (e^{2M_j} - \hat{g}_j(\zeta)) d\zeta \\ &= \left(\log \frac{4}{\rho_j} \right) \int_{\rho_j < |\zeta| < 1} \hat{g}_j(\zeta) d\zeta \\ &\geq \int_{\rho_j < |\zeta| < 1} \left(\log \frac{4}{|\zeta|} \right) \hat{g}_j(\zeta) d\zeta. \end{aligned}$$

Thus

$$\int_{|\zeta|<1} \left(\log \frac{4}{|\zeta|} \right) \hat{g}_j(\zeta) d\zeta \leq \int_{|\zeta|<\rho_j} e^{2M_j} \log \frac{4}{|\zeta|} d\zeta.$$

It follows therefore from (6.18) that

$$\frac{1}{M_j} e^{2M_j} \rho_j^2 \log \frac{4}{\rho_j} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \quad (6.22)$$

Also, by (6.21) and (6.19), $\rho_j e^{M_j} \rightarrow 0$ as $j \rightarrow \infty$. Hence, for large j , we see that

$$\frac{1}{M_j} e^{2M_j} \rho_j^2 \log \frac{4}{\rho_j} \leq \frac{1}{M_j} e^{2M_j} e^{-2M_j} \log 4 e^{M_j} = O(1) \quad \text{as } j \rightarrow \infty,$$

which contradicts (6.22) and thereby proves (6.4).

Since $v(x)$ is positive and superharmonic, v is bounded below in some punctured neighborhood of the origin by some constant $\delta \in (0, 1)$. Hence by (6.4) we have

$$\delta \leq v(x) \leq A \log \frac{1}{|x|} \quad \text{for } |x| \text{ small and positive.}$$

Also by (6.6) there exists a positive constant C such that

$$\log^+ f(t) \leq Ch(t) \quad \text{for } t \geq \delta.$$

Hence for $|x|$ small and positive we have by (6.5) that

$$\begin{aligned} \log^+ (-\Delta u(x)) &\leq \log^+ f(v(x)) \leq Ch(v(x)) \\ &\leq Ch \left(A \log \frac{1}{|x|} \right) = CH \left(\log \frac{1}{|x|} \right) \end{aligned}$$

where $H(t) = h(At)$. Thus (6.7) follows from Lemma 5.2. □

Proof of Theorem 2.2. Define $F, M : (0, \infty) \rightarrow (0, \infty)$ by

$$F(t) = \min\{f(t), g(t)\} \quad \text{and} \quad M(t) = \min_{\tau \geq t} \frac{\log F(\tau)}{\tau}.$$

Then M is nondecreasing. By (2.5), $M(t) \rightarrow \infty$ as $t \rightarrow \infty$ and there exists $K > 0$ such that $F(t) > 1$ for $t \geq K$. Thus

$$tM(t) \leq \min_{\tau \geq t} \log F(\tau) \quad \text{for } t \geq K. \quad (6.23)$$

Define $\varphi : (0, 1) \rightarrow (0, 1)$ by $\varphi(r) = r$ and let $\{x_j\}_{j=1}^\infty$, $\{r_j\}_{j=1}^\infty$, and A be as in Lemma 5.1. By holding x_j fixed and decreasing r_j we can assume

$$A\varphi(|x_j|) \log \frac{1}{r_j} \geq K, \quad (6.24)$$

$$A\varphi(|x_j|) M \left(A\varphi(|x_j|) \log \frac{1}{r_j} \right) > 2 \quad (6.25)$$

and

$$(h(|x_j|))^2 < A\varphi(|x_j|) \log \frac{1}{r_j}. \quad (6.26)$$

Let $\Omega = B_2(0)$. By Lemma 5.1 there exists a positive function $u \in C^\infty(\Omega \setminus \{0\})$ which satisfies (5.4)–(5.7). By (5.6) and (6.26) we have

$$u(x_j) \neq O(h(|x_j|)) \quad \text{as } j \rightarrow \infty$$

which implies (2.6). Also for $x \in B_{r_j}(x_j)$ and $-\Delta u(x) > 0$ it follows from (5.6), (6.24), (6.23), (6.25) and (5.4) that

$$\begin{aligned} \log F(u(x)) &\geq \left(A\varphi(|x_j|) \log \frac{1}{r_j} \right) M \left(A\varphi(|x_j|) \log \frac{1}{r_j} \right) \\ &> 2 \log \frac{1}{r_j} \\ &\geq \log(-\Delta u(x)). \end{aligned}$$

Thus u satisfies

$$0 \leq -\Delta u \leq F(u) \tag{6.27}$$

in $B_{r_j}(x_j)$. By (5.5), u satisfies (6.27) in $\Omega \setminus (\{0\} \cup \bigcup_{j=1}^\infty B_{r_j}(x_j))$. Thus u satisfies (6.27) in $\Omega \setminus \{0\}$. Taking $v = u$ completes to proof of Theorem 2.2. \square

The following theorem with $h(t) = t^\lambda$, $\lambda > 1$, immediately implies Theorem 2.4.

Theorem 6.2. *Suppose $h : (0, \infty) \rightarrow (0, \infty)$ and $\psi : (0, 1) \rightarrow (0, 1)$ are continuous nondecreasing functions satisfying*

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \infty \quad \text{and} \quad \lim_{r \rightarrow 0^+} \psi(r) = 0. \tag{6.28}$$

Then there exist C^∞ positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{aligned} 0 &\leq -\Delta u \leq e^{h(v)} \\ 0 &\leq -\Delta v \leq e^u \end{aligned} \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^2 \tag{6.29}$$

such that

$$u(x) \neq O\left(\psi(|x|)h\left(\log \frac{2}{|x|}\right)\right) \quad \text{as } x \rightarrow 0 \tag{6.30}$$

and

$$\frac{v(x)}{\log \frac{1}{|x|}} \rightarrow 1 \quad \text{as } x \rightarrow 0. \tag{6.31}$$

Proof. Let $v(x) = \log \frac{4}{|x|}$. Then v satisfies (6.29)₂ and (6.31). Define $\varphi : (0, 1) \rightarrow (0, 1)$ by $\varphi = \sqrt{\psi}$.

Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^2$ be as in Lemma 5.1 and $r_j = e^{-\frac{1}{2}h\left(\log \frac{2}{|x_j|}\right)}$. By taking a subsequence if necessary, it follows from (6.28)₁ that r_j satisfies (5.3).

Therefore, by Lemma 5.1, there exists a positive function $u \in C^\infty(B_2(0) \setminus \{0\})$ and a positive constant A such that u satisfies (5.4)–(5.7). Thus u satisfies (6.29)₁ in $B_2(0) \setminus (\{0\} \cup \bigcup_{j=1}^\infty B_{r_j}(x_j))$. Also for $x \in B_{r_j}(x_j)$ we have

$$0 \leq -\Delta u(x) \leq \frac{\varphi(|x_j|)}{r_j^2} \leq \frac{1}{r_j^2} = e^{h\left(\log \frac{4}{2|x_j|}\right)} < e^{h\left(\log \frac{4}{|x|}\right)} = e^{h(v(x))}.$$

Hence u satisfies (6.29)₁ in $B_2(0) \setminus \{0\}$.

Finally

$$\begin{aligned} \frac{u(x_j)}{\psi(|x_j|)h\left(\log\frac{2}{|x_j|}\right)} &\geq \frac{A\varphi(|x_j|)\log\frac{1}{r_j}}{\psi(|x_j|)h\left(\log\frac{2}{|x_j|}\right)} \\ &= \frac{A/2}{\sqrt{\psi(|x_j|)}} \rightarrow \infty \quad \text{as } j \rightarrow \infty \end{aligned}$$

which proves (6.30). □

Proof of Theorem 2.5. Define functions u and v by

$$u(x) = U(x) + a \log \frac{1}{|x|}, \quad v(x) = V(x) + a \log \frac{1}{|x|}.$$

Then u and v are C^2 positive solutions of

$$\begin{aligned} 0 &\leq -\Delta u \\ 0 &\leq -\Delta v \leq e^u \end{aligned}$$

in a punctured neighborhood of the origin. Thus (2.16) follows from Theorem 2.3. Hence by (2.13)

$$\begin{aligned} \log^+(-\Delta u(x)) &= \log^+(-\Delta U(x)) \leq \log^+(|x|^{-a}e^{|V|^\lambda}) \\ &= a \log \frac{1}{|x|} + |V|^\lambda \leq a \log \frac{1}{|x|} + C \left(\log \frac{1}{|x|} \right)^\lambda. \end{aligned}$$

Thus (2.15) follows from Lemma 5.2. □

7 Proofs of three and higher dimensional results

In this section we prove Theorems 3.1–3.7.

Proof of Theorem 3.1. Since increasing σ and/or λ weakens the conditions (3.1, 3.2), we can assume $\sigma = \lambda = \frac{n}{n-2}$.

As in the first paragraph of the proof of Theorem 6.1, there exist positive constants K and ε such that u and v are positive solutions of the system

$$\begin{aligned} 0 &\leq -\Delta u \leq K v^{\frac{n}{n-2}} \\ 0 &\leq -\Delta v \leq K u^{\frac{n}{n-2}} \end{aligned} \quad \text{in } B_\varepsilon(0) \setminus \{0\}.$$

Let $w = u + v$. Then in $B_\varepsilon(0) \setminus \{0\}$ we have

$$\begin{aligned} 0 &\leq -\Delta w = -\Delta u - \Delta v \leq K \left(u^{\frac{n}{n-2}} + v^{\frac{n}{n-2}} \right) \\ &\leq K w^{\frac{n}{n-2}}. \end{aligned}$$

Thus by [16, Theorem 2.1]

$$u(x) + v(x) = w(x) = O\left(|x|^{-(n-2)}\right) \quad \text{as } x \rightarrow 0$$

which proves (3.6) and (3.7). □

Proof of Theorem 3.5. As in the first paragraph of the proof of Theorem 6.1, we can assume the function g is given by $g(t) = t^\sigma$ and then Theorem 3.5 follows immediately from Lemma 5.4(i) with $\beta = 0$. \square

Proof of Theorem 3.7. We prove Theorem 3.7 one case at a time.

Case A. Suppose $\sigma = 0$. Then by (3.22) and Lemma 5.3(i) applied to v we have

$$v(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}\beta}\right). \quad (7.1)$$

It follows therefore from (3.21) that

$$-\Delta u(x) = O\left(\left(\frac{1}{|x|}\right)^{(n-2)\lambda+\alpha} + \left(\frac{1}{|x|}\right)^{\frac{n-2}{n}\beta\lambda+\alpha}\right)$$

and hence by Lemma 5.3(i)

$$u(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}((n-2)\lambda+\alpha)} + \left(\frac{1}{|x|}\right)^{\frac{n-2}{n}(\frac{n-2}{n}\beta\lambda+\alpha)}\right). \quad (7.2)$$

Case A of Theorem 3.7 follows immediately from (7.1) and (7.2).

The reasoning used to prove Cases B, C, and D of Theorem 3.7 is as follows. Either u satisfies

$$-\Delta u(x) = O\left(\left(\frac{1}{|x|}\right)^n\right) \quad \text{as } x \rightarrow 0 \quad (7.3)$$

or it doesn't.

Step I. If u satisfies (7.3) then we prove below that u and v satisfy (3.24) and (3.25).

Step II. If u does not satisfy (7.3) then, for example, to prove Theorem 3.7 in Case B, we prove below that the condition $\delta \leq n$ in (B1) does not hold and u and v satisfy (3.26) and (3.27).

These two steps complete the proof of Case B as follows: If the condition $\delta \leq n$ in (B1) holds then by Step II, u satisfies (7.3) and hence by Step I, u and v satisfy (3.24, 3.25). On the other hand, if the condition $\delta > n$ in (B2) holds then by Steps I and II, u and v satisfy either (3.24, 3.25) or (3.26, 3.27). But since (3.24, 3.25) implies (3.26, 3.27), we have u and v satisfy (3.26, 3.27).

Similar reasoning will be used in Cases C and D.

Step I. Suppose u satisfies (7.3). Then by Lemma 5.3(i) with $\gamma = n$ we see that u satisfies (3.24) as $x \rightarrow 0$. Hence by (3.22),

$$0 \leq -\Delta v = O\left(\left(\frac{1}{|x|}\right)^{(n-2)\sigma+\beta}\right) \quad \text{as } x \rightarrow 0.$$

Thus by Lemma 5.3(i) applied to v we have as $x \rightarrow 0$ that

$$v(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{\frac{n-2}{n}((n-2)\sigma+\beta)}\right)$$

which implies v satisfies (3.25) as $x \rightarrow 0$. This completes the proof of Step I.

Step II. Suppose

$$-\Delta u(x) \neq O\left(\left(\frac{1}{|x|}\right)^n\right) \quad \text{as } x \rightarrow 0. \quad (7.4)$$

By Lemma 5.4(ii)

$$-\Delta u(x) = O\left(\left(\frac{1}{|x|}\right)^{\gamma_1}\right) \quad \text{as } x \rightarrow 0 \quad (7.5)$$

for some $\gamma_1 > n$.

We now complete the proof Theorem 3.7 by completing the proof of Step II one case at a time.

Case B. Suppose $0 < \sigma < \frac{2}{n-2}$. Then by Lemma 5.4(i) we have $v(x)$ satisfies (3.27). Hence by (3.21)

$$-\Delta u(x) = O\left(\left(\frac{1}{|x|}\right)^{(n-2)\lambda+\alpha} + \left(\frac{1}{|x|}\right)^{[(n-2)\sigma-2+\beta]\lambda+\alpha}\right). \quad (7.6)$$

By (7.4) the maximum δ of the two exponents on $\frac{1}{|x|}$ in (7.6) is greater than n . Thus by Lemma 5.3(i), u satisfies (3.26). This completes the proof of Step II in Case B.

Case C. Suppose $\sigma = \frac{2}{n-2}$. Then by (7.5) and Lemma 5.3(iii) we have $v(x)$ satisfies (3.29). Hence by (3.21)

$$-\Delta u(x) = \begin{cases} O\left(\left(\frac{1}{|x|}\right)^{(n-2)\lambda+\alpha > n}\right), & \text{if } \beta < n-2 \\ o\left(\left(\frac{1}{|x|}\right)^{\beta\lambda+\alpha} \left(\log \frac{1}{|x|}\right)^\lambda\right), & \text{if } \beta \geq n-2. \end{cases}$$

Thus by (7.4) neither (i) nor (ii) in the statement of Case C holds. Hence by Lemma 5.3(i),(ii), u satisfies (3.28). This completes the proof of Step II in Case C.

Case D. Suppose $\sigma > \frac{2}{n-2}$ and a and b are defined by (3.30). Then by (3.20)

$$1 > 1 - a = \frac{n-2}{n} \lambda \left[\frac{n}{n-2} \frac{1}{\lambda} + \frac{2}{n-2} - \sigma \right] > 0. \quad (7.7)$$

By (7.5) and Lemma 5.3(iii) we have

$$v(x) = O\left(\left(\frac{1}{|x|}\right)^{p_0}\right) \quad \text{as } x \rightarrow 0$$

for some $p_0 > \max\{n-2, \frac{b}{1-a}\}$. Hence by (3.21)

$$-\Delta u(x) = O\left(\left(\frac{1}{|x|}\right)^{\gamma_0 := \alpha + p_0 \lambda}\right) \quad \text{as } x \rightarrow 0$$

and $\gamma_0 > n$ by (7.4). Thus by Lemma 5.3(iii) v satisfies (5.19) with $\gamma = \gamma_0 = \alpha + p_0 \lambda > n$, that is

$$v(x) = O\left(\left(\frac{1}{|x|}\right)^{n-2}\right) + o\left(\left(\frac{1}{|x|}\right)^{p_1}\right) \quad \text{as } x \rightarrow 0 \quad (7.8)$$

where

$$\begin{aligned} p_1 &:= \frac{\gamma_0}{n} [(n-2)\sigma - 2] + \beta = \frac{\alpha + p_0 \lambda}{n} [(n-2)\sigma - 2] + \beta \\ &= p_0 a + b. \end{aligned}$$

By (7.7) the sequence defined by $p_{j+1} = ap_j + b$ decreases to $\frac{b}{1-a}$. Thus after iterating a finite number of times the process of obtaining p_1 from p_0 and using (7.8) we obtain as $x \rightarrow 0$ that v satisfies (3.32) for all $\varepsilon > 0$. Hence by (3.21)

$$-\Delta u(x) = \begin{cases} O\left(\left(\frac{1}{|x|}\right)^{(n-2)\lambda+\alpha}\right), & \text{if } \frac{b}{1-a} < n-2 \\ O\left(\left(\frac{1}{|x|}\right)^{\frac{b\lambda}{1-a}+\alpha+\varepsilon}\right), & \text{if } \frac{b}{1-a} \geq n-2 \end{cases} \quad (7.9)$$

for all $\varepsilon > 0$. By (7.4) the exponents on $\frac{1}{|x|}$ in (6.9) are greater than n . (That is neither (i) nor (ii) in the statement of Case D hold.) Thus, by Lemma 5.3(i), u satisfies (3.31) for all $\varepsilon > 0$. This completes the proof of Step II in Case D. \square

Proof of Theorem 3.2. Since increasing σ weakens the condition (3.2) on g and since the bounds (3.9), (3.10) do not depend on σ , we can assume without loss of generality that

$$\lambda > \frac{n}{n-2} \quad \text{and} \quad \frac{2}{n-2} < \sigma < \frac{2}{n-2} + \frac{n}{n-2} \frac{1}{\lambda}.$$

As in the first paragraph of the proof of Theorem 6.1, there exists a constant $K > 0$ such that u and v are C^2 positive solutions of

$$\begin{aligned} 0 &\leq -\Delta u \leq K v^\lambda \\ 0 &\leq -\Delta v \leq K u^\sigma \end{aligned}$$

in a punctured neighborhood of the origin in \mathbb{R}^n . By scaling we can assume $K = 1$.

We now apply Theorem 3.7, Case D with $\alpha = \beta = 0$. Let a and b be defined by (3.30). Then $b = 0$, $0 = \frac{b}{1-a} < n-2$ and $(n-2)\lambda > n = n - \alpha$. Thus neither (i) nor (ii) in Theorem 3.7, Case D, hold. Hence Theorem 3.2 follows from (3.31) and (3.32). \square

Proof of Theorem 3.3. Let $v(x) = |x|^{-(n-2)}$ then v satisfies (3.11)₂ and (3.13). Define $\varphi : (0, 1) \rightarrow (0, 1)$ by $\varphi = \sqrt{\psi}$. Let $\{x_j\}$ be as in Lemma 5.1 and $r_j = (2|x_j|)^{\frac{n-2}{n}\lambda}$. By taking a subsequence if necessary, r_j satisfies (5.3). Therefore by Lemma 5.1 there exists a positive function $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and a positive constant $A = A(n)$ such that u satisfies (5.4)–(5.7). Thus u satisfies (3.11)₁ in $\mathbb{R}^n \setminus (\{0\} \cup \cup_{j=1}^\infty B_{r_j}(x_j))$. Also for $x \in B_{r_j}(x_j)$ we have

$$0 \leq -\Delta u(x) \leq \frac{\varphi(|x_j|)}{r_j^n} < \left(\frac{1}{2|x_j|}\right)^{(n-2)\lambda} < v(x)^\lambda.$$

Hence u satisfies (3.11)₁ in $\mathbb{R}^n \setminus \{0\}$.

Finally,

$$\begin{aligned} \frac{u(x_j)}{\psi(|x_j|)|x_j|^{-\frac{(n-2)^2}{n}\lambda}} &\geq \frac{A\varphi(|x_j|)}{r_j^{n-2}\psi(|x_j|)|x_j|^{-\frac{(n-2)^2}{n}\lambda}} \\ &= \frac{A}{2^{\frac{(n-2)^2}{n}\lambda}\sqrt{\psi(|x_j|)}} \rightarrow \infty \quad \text{as } j \rightarrow \infty \end{aligned}$$

which proves (3.12). \square

Proof of Theorem 3.4. It follows from (3.14) that $\lambda > \frac{n}{n-2}$. Denote the problem (3.15) by $P(\lambda, \sigma)$. If $\hat{\lambda} \geq \lambda$ and $\hat{\sigma} \geq \sigma$ are constants and (u, v) solves $P(\lambda, \sigma)$ then clearly (u, v) solves $P(\hat{\lambda}, \hat{\sigma})$. We can therefore assume

$$\sigma < \frac{n}{n-2}. \quad (7.10)$$

Since the first inequality in (3.14) holds

$$\begin{aligned} \text{if and only if} \quad & (n-2)\sigma > 2 + \frac{n}{\lambda} = n - (n-2) + \frac{n}{\lambda} = n - \frac{(n-2)\lambda - n}{\lambda} \\ \text{if and only if} \quad & n - (n-2)\sigma < \frac{(n-2)\lambda - n}{\lambda} \end{aligned}$$

we see by (7.10) that

$$\frac{1}{n - (n-2)\sigma} > \frac{\lambda}{(n-2)\lambda - n},$$

or, in other words,

$$\beta > \alpha\lambda > 0 \quad \text{where} \quad \beta := \frac{1}{n - (n-2)\sigma} \quad \text{and} \quad \alpha := \frac{1}{(n-2)\lambda - n}. \quad (7.11)$$

Let $\varphi(r) = r$ and let $\{x_j\}_{j=1}^\infty$, $\{r_j\}_{j=1}^\infty$, and $A = A(n)$ be as in Lemma 5.1. Define $\psi_j > 0$ as a function of r_j by

$$r_j = \left(\frac{(A\psi_j)^\lambda}{\varphi(|x_j|)} \right)^\alpha. \quad (7.12)$$

Then

$$\frac{A\psi_j}{r_j^{n-2}} = \frac{(A\psi_j)\varphi(|x_j|)^{\alpha(n-2)}}{(A\psi_j)^{\lambda\alpha(n-2)}} = \left(\frac{\varphi(|x_j|)^{n-2}}{(A\psi_j)^{\lambda(n-2) - \frac{1}{\alpha}n}} \right)^\alpha. \quad (7.13)$$

By decreasing r_j (and thereby decreasing ψ_j) we can assume

$$\frac{A\varphi(|x_j|)}{r_j^{n-2}} > h(|x_j|)^2, \quad \sum_{j=1}^\infty \psi_j < \infty, \quad (7.14)$$

$$\psi_j^{\alpha\lambda - \beta} \geq \frac{\varphi(|x_j|)^{\alpha - \sigma\beta}}{A^{\sigma\beta + \alpha\lambda}} \quad \text{and} \quad \frac{A\psi_j}{r_j^{n-2}} > h(|x_j|)^2 \quad (7.15)$$

by (7.13). It follows from (7.12) and (7.15) that

$$\left(\frac{(A\psi_j)^\lambda}{\varphi(|x_j|)} \right)^\alpha = r_j \geq \left(\frac{\psi_j}{(A\varphi(|x_j|))^\sigma} \right)^\beta$$

which implies

$$0 < \frac{\varphi(|x_j|)}{r_j^n} = \left(\frac{A\psi_j}{r_j^{n-2}} \right)^\lambda \quad (7.16)$$

and

$$0 < \frac{\psi_j}{r_j^n} \leq \left(\frac{A\varphi(|x_j|)}{r_j^{n-2}} \right)^\sigma. \quad (7.17)$$

Let $\psi : (0, 1) \rightarrow (0, 1)$ be a continuous function such that $\psi(|x_j|) = \psi_j$. By Lemma 5.1 there exist positive functions $u, v \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that u satisfies (5.4)–(5.7) and v satisfies

$$0 \leq -\Delta v \leq \frac{\psi(|x_j|)}{r_j^n} \quad \text{in } B_{r_j}(x_j) \quad (7.18)$$

$$-\Delta v = 0 \quad \text{in } \mathbb{R}^n \setminus \left(\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \right) \quad (7.19)$$

$$v \geq \frac{A\psi(|x_j|)}{r_j^{n-2}} \quad \text{in } B_{r_j}(x_j) \quad (7.20)$$

and

$$v \geq 1 \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (7.21)$$

Theorem 3.4 follows from (5.4)–(5.7), (7.18)–(7.21), (7.16), (7.17), (7.14)₁, and (7.15)₂. \square

Proof of Theorem 3.6. Let $u(x)$ and $v(x)$ be the Kelvin transforms of $U(y)$ and $V(y)$ respectively. Then

$$U(y) = |x|^{n-2}u(x), \quad V(y) = |x|^{n-2}v(x), \quad x = \frac{y}{|y|^2} \quad (7.22)$$

$$\begin{aligned} \Delta U &= |x|^{n+2}\Delta u, & \Delta V &= |x|^{n+2}\Delta v \\ U + 1 &= |x|^{n-2}(u + |x|^{-(n-2)}), & V + 1 &= |x|^{n-2}(v + |x|^{-(n-2)}) \end{aligned}$$

and thus $u(x)$ and $v(x)$ are C^2 nonnegative solutions of the system (3.21, 3.22) in a punctured neighborhood of the origin where

$$\alpha = n + 2 - (n - 2)\lambda \quad \text{and} \quad \beta = n + 2 - (n - 2)\sigma. \quad (7.23)$$

Using Theorem 3.7 we get the following results.

Case A. Suppose $\sigma = 0$. Then $\beta = n + 2$,

$$\frac{n-2}{n}\beta = \frac{(n-2)(n+2)}{n},$$

and

$$\begin{aligned} \frac{n-2}{n} \left(\frac{n-2}{n}\beta\lambda + \alpha \right) &= \frac{n-2}{n} \left(\frac{(n-2)(n+2)}{n}\lambda + n + 2 - (n-2)\lambda \right) \\ &= \frac{n-2}{n} \left[\left(\frac{n+2}{n} - 1 \right) (n-2)\lambda + n + 2 \right] \\ &= \frac{n-2}{n} \left[\frac{2(n-2)}{n}\lambda + n + 2 \right] \geq (n-2)\frac{n+2}{n} > n - 2. \end{aligned}$$

Thus by Theorem 3.7(A2) we have

$$\begin{aligned} u(x) &= o \left(\left(\frac{1}{|x|} \right)^{\frac{n-2}{n} \left[\frac{2(n-2)}{n}\lambda + n + 2 \right]} \right) \quad \text{as } x \rightarrow 0 \\ v(x) &= o \left(\left(\frac{1}{|x|} \right)^{(n-2)\left(1 + \frac{2}{n}\right)} \right) \quad \text{as } x \rightarrow 0. \end{aligned}$$

Hence Case A of Theorem 3.6 follows from (7.22).

Case B. Suppose $0 < \sigma < \frac{2}{n-2}$. Then

$$\begin{aligned}(n-2)\lambda + \alpha &= n+2, & (n-2)\sigma + \beta &= n+2 \\ \lambda[(n-2)\sigma - 2 + \beta] + \alpha &= \lambda n + n + 2 - (n-2)\lambda = n+2 + 2\lambda\end{aligned}$$

and

$$\delta = \max\{n+2, n+2+2\lambda\} = n+2+2\lambda > n.$$

Thus by Theorem 3.7(B2) we have

$$\begin{aligned}u(x) &= o\left(\left(\frac{1}{|x|}\right)^{(n-2)\frac{n+2+2\lambda}{n}}\right) \quad \text{as } x \rightarrow 0 \\ v(x) &= O\left(\left(\frac{1}{|x|}\right)^n\right) \quad \text{as } x \rightarrow 0.\end{aligned}$$

Hence Case B of Theorem 3.6 follows from (7.22).

Case C. Suppose $\sigma = \frac{2}{n-2}$. Then

$$\begin{aligned}\beta &= n+2 - (n-2)\sigma = n > n-2 \\ \alpha &= n+2 - (n-2)\lambda\end{aligned}$$

and

$$\beta\lambda + \alpha = n\lambda + n + 2 - (n-2)\lambda = n + 2(\lambda + 1) > n + 2.$$

Thus by Theorem 3.7(C2) we have

$$\begin{aligned}u(x) &= o\left(\left(\frac{1}{|x|}\right)^{(n-2)\left(1+\frac{2(\lambda+1)}{n}\right)} \left(\log \frac{1}{|x|}\right)^{\frac{n-2}{n}\lambda}\right) \quad \text{as } x \rightarrow 0 \\ v(x) &= o\left(\left(\frac{1}{|x|}\right)^n \log \frac{1}{|x|}\right) \quad \text{as } x \rightarrow 0.\end{aligned}$$

Hence Case C of Theorem 3.6 follows from (7.22).

Case D. Suppose $\sigma > \frac{2}{n-2}$. Let a and b be defined by (3.30). Then by (7.23) and direct calculation (Maple is helpful), we find

$$\frac{b}{1-a} = (n-2) \left[1 + \frac{2\sigma + 2}{D}\right] > n-2$$

and

$$\frac{b\lambda}{1-a} - (n-\alpha) = \frac{2n(\lambda+1)}{D} > 0 \tag{7.24}$$

by (3.20). Thus neither (i) nor (ii) in Theorem 3.7(D1) hold. Also (7.24) implies

$$\frac{b\lambda}{1-a} + \alpha = n \left[1 + \frac{2(\lambda+1)}{D}\right].$$

Hence by Theorem 3.7(D2) we have

$$\begin{aligned} u(x) &= o\left(\left(\frac{1}{|x|}\right)^{(n-2)\left(1+\frac{2(\lambda+1)}{D}+\varepsilon\right)}\right) \quad \text{as } x \rightarrow 0 \\ v(x) &= o\left(\left(\frac{1}{|x|}\right)^{(n-2)\left(1+\frac{2(\sigma+1)}{D}+\varepsilon\right)}\right) \quad \text{as } x \rightarrow 0. \end{aligned}$$

Thus Case D of Theorem 3.6 follows from (7.22). □

A Brezis-Lions result

We use repeatedly the following special case of a result of Brezis and Lions [4].

Lemma A.1. *Suppose u is a C^2 nonnegative superharmonic function in $B_{2\varepsilon}(0) \setminus \{0\} \subset \mathbb{R}^n$, $n \geq 2$, for some $\varepsilon > 0$. Then*

$$\int_{|x|<\varepsilon} -\Delta u(x) dx < \infty$$

and for $0 < |x| < \varepsilon$ we have

$$u(x) = m\Gamma(|x|) + \int_{|y|<\varepsilon} \omega\Gamma(|x-y|)(-\Delta u(y)) dy + h(x)$$

where Γ is given by (1.8), $\omega = \omega(n) > 0$ and $m \geq 0$ are constants, $\omega(2) = \frac{1}{2\pi}$, and $h : B_\varepsilon(0) \rightarrow \mathbb{R}$ is harmonic.

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