Pointwise Bounds and Blow-up for Nonlinear Fractional Parabolic Inequalities

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Abstract

We investigate pointwise upper bounds for nonnegative solutions $u(x,t)$ of the nonlinear initial value problem

$$0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R},\ n \geq 1, \tag{0.1}$$

$$u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0) \tag{0.2}$$

where λ and α are positive constants. To do this we first give a definition—tailored for our study of (0.1), (0.2)—of fractional powers of the heat operator $(\partial_t - \Delta)^\alpha : Y \to X$ where X and Y are linear spaces whose elements are real valued functions on $\mathbb{R}^n \times \mathbb{R}$ and $0 < \alpha < \alpha_0$ for some α_0 which depends on n, X and Y.

We then obtain, when they exist, optimal pointwise upper bounds on $\mathbb{R}^n \times (0,\infty)$ for nonnegative solutions $u \in Y$ of the initial value problem (0.1), (0.2) with particular emphasis on those bounds as $t \to 0^+$ and as $t \to \infty$.

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1 Introduction

In this paper we study pointwise upper bounds for nonnegative solutions $u(x,t)$ of the nonlinear inequalities

$$0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R},\ n \geq 1, \tag{1.1}$$

satisfying the initial condition

$$u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0) \tag{1.2}$$

where λ and α are positive constants.

To do this, we first give in Section 2 a definition—appropriate for our analysis of the initial value problem (1.1), (1.2)—of fractional powers of the heat operator

$$(\partial_t - \Delta)^\alpha : Y \to X \tag{1.3}$$

where Δ is the Laplacian with respect to $x \in \mathbb{R}^n$, X and Y are linear spaces whose elements are real valued functions on $\mathbb{R}^n \times \mathbb{R}$, and $0 < \alpha < \alpha_0$ for some $\alpha_0 > 0$ which depends on n, X and Y.

With the definition of (1.3) in hand, we obtain, when they exist, optimal pointwise upper bounds on $\mathbb{R}^n \times (0,\infty)$ for nonnegative solutions $u \in Y$ of the initial value problem (1.1), (1.2) with particular emphasis on these bounds as $t \to 0^+$ and as $t \to \infty$. These results are stated in Section 3 and proved in Section 8.

Since the operator (1.3) is nonlocal, we must require the initial condition (1.2) to hold in $\mathbb{R}^n \times (-\infty, 0)$ (not just in $\mathbb{R}^n \times \{0\}$) and nonnegative solutions of (1.1), (1.2) may not tend pointwise to zero as $t \to 0^+$ (see Theorem 3.5) even though they satisfy the initial condition (1.2).

Of course any estimates we obtain for nonnegative solutions of (1.1), (1.2) also hold for nonnegative solutions of the initial value problem consisting of (1.2) and the *equation*

$$(\partial_t - \Delta)^\alpha u = u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

According to our results in Section 3 there are essentially only three possibilities for the solutions of (1.1), (1.2) depending on X, Y, λ, and α:

(i) The only solution is $u \equiv 0$ in $\mathbb{R}^n \times \mathbb{R}$;

(ii) There exist sharp nonzero pointwise bounds for solutions as $t \to 0^+$ and as $t \to \infty$;

(iii) There do not exist pointwise bounds for solutions as $t \to 0^+$ and as $t \to \infty$.

All possiblities can occur. For the precise statements of possibilities (i), (ii), and (iii) see Theorem 3.1, Theorems 3.2–3.4, and Theorems 3.5 and 3.6, respectively.

The operator (1.3) is a fully fractional heat operator as opposed to time fractional heat operators in which the fractional derivatives are only with respect to t, and space fractional heat operators, in which the fractional derivatives are only with respect to x.

Some recent results for nonlinear PDEs containing time (resp. space) fractional heat operators can be found in [2, 4, 5, 10, 15, 16, 17, 21, 28, 32, 33] (resp. [1, 3, 7, 8, 9, 11, 12, 14, 18, 22, 29, 30, 31]). We know of no results for nonlinear PDEs containing the fully fractional heat operator (1.3). However results for linear PDEs containing (1.3), including in particular

$$(\partial_t - \Delta)^\alpha u = f,$$

where f is a given function, can be found in [6, 20, 24, 27].

2 Definition and properties of fully fractional heat operators

In this section we give a well-motivated definition of the fully fractional heat operator (1.3), suitable for our study of the initial value problem (1.1), (1.2), and then give some of its properties.

Some of the material in this section is inspired by—and can be viewed as the parabolic analog of—the material in [26, Sec. 5.1] concerning the fractional Laplacian.

Since for functions $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$, which are sufficiently smooth and small at infinity we have

$$((\partial_t - \Delta)u)\widehat{\ }(y, s) = (|y|^2 - is)\widehat{u}(y, s),$$

where $\widehat{\ }$ is the Fourier transform operator on $\mathbb{R}^n \times \mathbb{R}$ given by

$$\widehat{u}(y, s) = \iint_{\mathbb{R}^n \times \mathbb{R}} e^{i(y,s)\cdot(x,t)} u(x, t)\, dx\, dt,$$

the fractional heat operator $(\partial_t - \Delta)^\alpha$, $\alpha > 0$, is formally defined in [25, Chapter 2] by

$$((\partial_t - \Delta u)^\alpha u)\widehat{\ }(y, s) = (|y|^2 - is)^\alpha \widehat{u}(y, s). \tag{2.1}$$

If $f = (\partial_t - \Delta)^\alpha u$ then from (2.1) and the fact (see [25, Theorem 2.2] and Theorem 2.1(i) below) that

$$\widehat{\Phi}_\alpha(y,s) = (|y|^2 - is)^{-\alpha} \quad \text{for } 0 < \alpha < (n+2)/2$$

in the sense of tempered distributions where

$$\Phi_\alpha(x,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \chi_{(0,\infty)}(t), \tag{2.2}$$

we formally get

$$\widehat{u} = \widehat{\Phi}_\alpha \widehat{f}.$$

Hence by the convolution theorem we formally find that

$$u = J_\alpha f := \Phi_\alpha * f \tag{2.3}$$

where $*$ is the convolution operation in $\mathbb{R}^n \times \mathbb{R}$. Since $\Phi_\alpha(x,t) = 0$ for $t \le 0$ we have

$$J_\alpha f(x,t) = \iint_{\mathbb{R}^n \times (-\infty,t)} \Phi_\alpha(x-\xi, t-\tau) f(\xi,\tau)\, d\xi\, d\tau. \tag{2.4}$$

By part (ii) of the following theorem, equations (2.1) and (2.3) are equivalent in the sense that

$$(J_\alpha f)\widehat{\ } = (|y|^2 - is)^{-\alpha} \widehat{f} \quad \text{for } f \in L^1(\mathbb{R}^n \times \mathbb{R}) \text{ and } 0 < \alpha < (n+2)/2$$

in the sense of tempered distributions.

Theorem 2.1. *Suppose* $0 < \alpha < (n+2)/2$.

(i) The Fourier transform of $\Phi_\alpha(x,t)$ *is the function* $(|y|^2 - is)^{-\alpha}$ *in the sense that*

$$\iint_{\mathbb{R}^n \times \mathbb{R}} \Phi_\alpha(x,t) \widehat{\varphi}(x,t)\, dx\, dt = \iint_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \varphi(y,s)\, dy\, ds$$

for all $\varphi \in S$ *where* S *is the Schwarz class of rapidly decreasing functions.*

(ii) The identity $(J_\alpha f)\widehat{\ }(y,t) = (|y|^2 - is)^{-\alpha} \widehat{f}(y,s)$ *holds in the sense that*

$$\iint_{\mathbb{R}^n \times \mathbb{R}} J_\alpha f(x,t) \widehat{g}(x,t)\, dx\, dt = \iint_{\mathbb{R}^n \times \mathbb{R}} (|y|^2 - is)^{-\alpha} \widehat{f}(y,s) g(y,s)\, dy\, ds \tag{2.5}$$

for all $f \in L^1(\mathbb{R}^n \times \mathbb{R})$ *and all* $g \in S$.

Motivated by these formal calculations, we will now define the operator $(\partial_t - \Delta)^\alpha$ as the inverse of a linear operator

$$J_\alpha : X \to Y \tag{2.6}$$

where J_α is defined by (2.4) and (2.2) and X and Y are linear spaces whose elements are functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that the operator (2.6) has the following properties:

(P1) it makes sense because the integral in (2.4) defines a real valued measurable function on $\mathbb{R}^n \times \mathbb{R}$ for all $f \in X$,

(P2) it is one-to-one and onto, and

(P3) if $u = J_\alpha f$ then $f = 0$ in $\mathbb{R}^n \times (-\infty, 0)$ if and only if $u = 0$ in $\mathbb{R}^n \times (-\infty, 0)$.

Property (P3) will be needed to handle the initial condition (1.2). The domain of J_α is usually taken to be $L^p(\mathbb{R}^n \times \mathbb{R})$, $1 \le p < \frac{n+2}{2\alpha}$ (see [24, Section 9.2]). However since the region of integration for the integral (2.4) is not $\mathbb{R}^n \times \mathbb{R}$ but rather $\mathbb{R}^n \times (-\infty, t)$, we see that more natural and less restrictive choices for the domain and range of J_α are

$$X^p := \bigcap_{T\in\mathbb{R}} L^p(\mathbb{R}^n \times \mathbb{R}_T) \tag{2.7}$$

$$Y^p_\alpha := J_\alpha(X^p) \tag{2.8}$$

respectively, where $\mathbb{R}_T = (-\infty, T)$. By (2.7) we mean X^p is the set of all measurable functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}_T)} < \infty \quad \text{for all } T \in \mathbb{R}.$$

The notation in (2.7) should be interpreted similarly elsewhere in this paper.

According to the following two theorems the formal operator

$$J_\alpha : X^p \to Y^p_\alpha, \tag{2.9}$$

where X^p and Y^p_α are defined in (2.7) and (2.8), satisfies properties (P1)–(P3) provided either

$$\left(p > 1 \text{ and } 0 < \alpha < \frac{n+2}{2p}\right) \quad \text{or} \quad \left(p = 1 \text{ and } 0 < \alpha \le \frac{n+2}{2p}\right). \tag{2.10}$$

When p and α satisfy (2.10), part (i) of the following theorem shows that the operator (2.9) satisfies (P1) and parts (ii) and (iii) give some of its properties.

Theorem 2.2. *Suppose p and α are real numbers satisfying* (2.10) *and $f \in X^p$. Then*

(i) $J_\alpha f$, $J_\alpha|f| \in L^p_{loc}(\mathbb{R}^n \times \mathbb{R})$ and

(ii) $J_\beta(J_\gamma f) = J_\alpha f$ in $L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$ whenever $\beta > 0$, $\gamma > 0$, and $\beta + \gamma = \alpha$.

If in addition $\alpha > 1$ then

(iii) $HJ_\alpha f = J_{\alpha-1} f$ in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ where $H = \partial_t - \Delta$ is the heat operator.

Remark 2.1. Theorem 2.2(i) can be improved to $J_\alpha f \in L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$ when

$$1 < p < \frac{n+2}{2\alpha} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n+2}.$$

This can be seen by applying Gopala Rao [13, Theorem 3.1] to the function f_T defined in the proof of Theorem 2.2 in Section 6.

According to the following theorem, if p and α satisfy (2.10) then the operator (2.9) satisfies properties (P2) and (P3) where X^p and Y^p_α are defined by (2.7) and (2.8).

Theorem 2.3. *Suppose p and α are real numbers satisfying* (2.10)*. Then*

(i) the operator (2.9) *is one-to-one and onto, and*

(ii) if

$$f \in X^p \text{ and } T \in \mathbb{R} \tag{2.11}$$

then

$$f|_{\mathbb{R}^n\times\mathbb{R}_T} = 0 \quad \text{if and only if} \quad (J_\alpha f)|_{\mathbb{R}^n\times\mathbb{R}_T} = 0.$$

By the results in this section, the following definition is natural and makes sense.

Definition 2.1. Suppose p and α are real numbers satisfying (2.10) and X^p and Y^p_α are defined by (2.7) and (2.8). Then the operator

$$(\partial_t - \Delta)^\alpha : Y^p_\alpha \to X^p \tag{2.12}$$

is defined to be the inverse of the operator (2.9).

Remark 2.2. The functions $\mu_T : X^p \to \mathbb{R}$, $T \in \mathbb{R}$, defined by $\mu_T(f) = \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}_T)}$, form a separating family of seminorms on X^p which turns X^p into a locally convex topological vector space (see for example [23, Theorem 1.37]). Thus assuming (2.10) and defining a subset O' of Y^p_α to be open if $O' = J_\alpha(O)$ for some open set $O \in X^p$, we see by Theorem 2.3(i) that Y^p_α is also a locally convex topological vector space and the operator (2.12) is a homeomorphism.

We conclude this section by investigating

$$\lim_{a\to 0^+}(\partial_t - a^2\Delta)^\alpha \quad \text{and} \quad \lim_{b\to 0^+}(b\partial_t - \Delta)^\alpha$$

where $\alpha > 0$.

To do this we first repeat the above procedure with $\partial_t - \Delta$ replaced with $b\partial_t - a^2\Delta$ where a and b are positive constants. The end result after defining

$$J_{\alpha,a,b} : X^p \to Y^p_{\alpha,a,b} := J_{\alpha,a,b}(X^p) \tag{2.13}$$

by

$$J_{\alpha,a,b}f = \Phi_{\alpha,a,b} * f,$$

where a, b, α, p are positive constants satisfying (2.10) and

$$\Phi_{\alpha,a,b}(x,t) = \frac{1}{a^n b}\Phi_\alpha\left(\frac{x}{a}, \frac{t}{b}\right),$$

is the following modified version of Definition 2.1.

Definition 2.2. Suppose a, b, p and α are positive constants satsfying (2.10) and X^p and $Y^p_{\alpha,a,b}$ are defined in (2.7) and (2.13). Then the operator

$$(b\partial_t - a^2\Delta)^\alpha : Y^p_{\alpha,a,b} \to X^p$$

is defined to be the inverse of the operator (2.13).

The following theorem states in what sense

$$(\partial_t - a^2\Delta)^\alpha \to \partial_t^\alpha \quad \text{as } a \to 0^+$$

where we formally define the equation

$$\partial_t^\alpha u = f$$

to mean

$$u = J_{\alpha,0,1}f$$

where

$$(J_{\alpha,0,1}f)(x,t) := \int_{-\infty}^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(x,\tau)\,d\tau$$

is the Riemann-Liouville integral of f with respect to t of order α with base point $-\infty$.

Theorem 2.4. *Suppose $\alpha > 0$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support. Then*

$$J_{\alpha,a,1} f \to J_{\alpha,0,1} f \quad \text{as } a \to 0^+$$

uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}$.

The following theorem states in what sense

$$(b\partial_t - \Delta)^\alpha \to (-\Delta)^\alpha \quad \text{as } b \to 0^+$$

where we formally define the equation

$$(-\Delta)^\alpha u = f$$

to mean

$$u = J_{\alpha,1,0} f$$

where

$$(J_{\alpha,1,0} f)(x,t) := \frac{1}{\gamma(n,\alpha)} \int_{\mathbb{R}^n} \frac{f(y,t)\,dy}{|x-y|^{n-2\alpha}}$$

is the Riesz potential of f with respect to x of order α. Here

$$\gamma(n,\alpha) = \frac{4^\alpha \pi^{n/2} \Gamma(\alpha)}{\Gamma(n/2 - \alpha)}. \tag{2.14}$$

Theorem 2.5. *Suppose $0 < 2\alpha < n$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support. Then*

$$J_{\alpha,1,b} f \to J_{\alpha,1,0} f \quad \text{as } b \to 0^+$$

uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}$.

3 Results for fully fractional initial value problems

In this section we state our results concerning pointwise bounds for nonnegative solutions

$$u \in Y^p_\alpha \tag{3.1}$$

of the fully fractional initial value problem

$$0 \le (\partial_t - \Delta)^\alpha u \le u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R},\ n \ge 1, \tag{3.2}$$

$$u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0) \tag{3.3}$$

where $\lambda > 0$ and, as in the Definition 2.1 of the operator (2.12), α and p satisfy (2.10).

Remark 3.1. If α and p satisfy (2.10) and u satisfies (3.1) and the first inequality in (3.2) then

$$f := (\partial_t - \Delta)^\alpha u \ge 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

and hence $u = J_\alpha f \ge 0$ in $\mathbb{R}^n \times \mathbb{R}$ by (2.4). Thus the assumption that u be nonnegative can be omitted when studying (3.1)–(3.3).

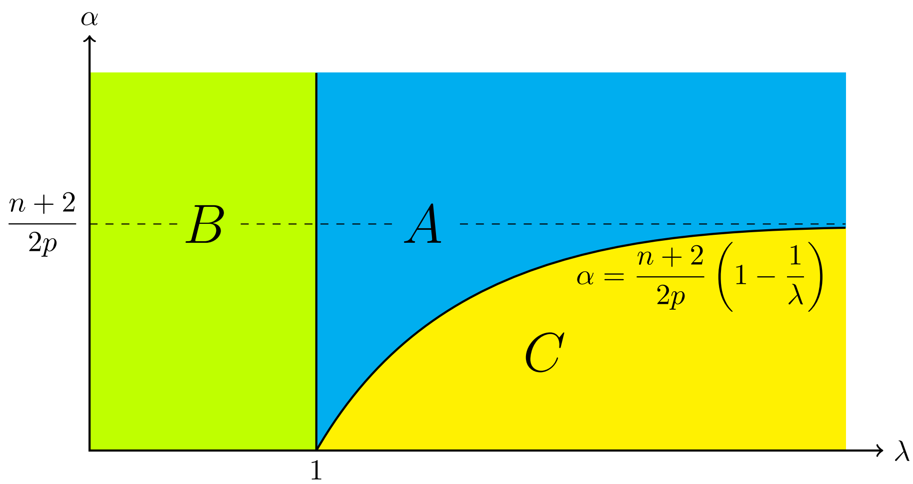

Figure 1: Graphs of the regions A, B, and C.

In order to state our results we first note that for each fixed $p \geq 1$ the open first quadrant of the $\lambda\alpha$-plane is the union of the following pairwise disjoint sets.

$$
\begin{aligned}
A &= \left\{(\lambda,\alpha) : \lambda \geq 1 \text{ and } \alpha > \frac{n+2}{2p}\left(1-\frac{1}{\lambda}\right)\right\}\\
B &= \{(\lambda,\alpha) : 0 < \lambda < 1 \text{ and } \alpha > 0\}\\
C &= \left\{(\lambda,\alpha) : \lambda > 1 \text{ and } 0 < \alpha < \frac{n+2}{2p}\left(1-\frac{1}{\lambda}\right)\right\}\\
D &= \left\{(\lambda,\alpha) : \lambda > 1 \text{ and } \alpha = \frac{n+2}{2p}\left(1-\frac{1}{\lambda}\right)\right\}.
\end{aligned}
$$

Note that A, B, and C are two dimensional regions in the $\lambda\alpha$-plane whereas D is the curve separating A and C. (See Figure 1.) Our results in this section deal with solutions of (3.1)–(3.3) when (λ, α) is in A, B, or C. We have no results when $(\lambda, \alpha) \in D$.

The following theorem deals with the case that $(\lambda, \alpha) \in A$.

Theorem 3.1. *Suppose α and p satisfy* (2.10)*, $(\lambda, \alpha) \in A$, and u satisfies* (3.1)–(3.3)*. Then*

$$u = (\partial_t - \Delta)^\alpha u = 0 \quad \textit{almost everywhere in } \mathbb{R}^n \times \mathbb{R}.$$

The following three theorems deal with the case $(\lambda, \alpha) \in B$.

Theorem 3.2. *Suppose α and p satisfy* (2.10)*, $(\lambda, \alpha) \in B$, and u satisfies* (3.1)–(3.3)*. Then for all $T > 0$ we have*

$$\|u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq (MT^\alpha)^{\frac{1}{1-\lambda}} \tag{3.4}$$

and

$$\|(\partial_t - \Delta)^\alpha u\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq (MT^\alpha)^{\frac{\lambda}{1-\lambda}} \tag{3.5}$$

where

$$M = M(\alpha,\lambda) = \frac{\Gamma(\frac{\alpha\lambda}{1-\lambda}+1)}{\Gamma(\alpha + \frac{\alpha\lambda}{1-\lambda}+1)} \tag{3.6}$$

where Γ is the Gamma function.

By the following theorem the bounds (3.4) and (3.5) in Theorem 3.2 are optimal.

Theorem 3.3. *Suppose* α *and* p *satisfy* (2.10), $(\lambda, \alpha) \in B$, $T > 0$, *and* $N < M$ *where* M *is given by* (3.6). *Then there exists a solution*

$$u \in Y_\alpha^p \cap C(\mathbb{R}^n \times \mathbb{R})$$

of (3.2), (3.3) *such that*

$$(\partial_t - \Delta)^\alpha u \in L^p(\mathbb{R}^n \times \mathbb{R}) \cap C(\mathbb{R}^n \times \mathbb{R}),$$

$$u(0,t) \geq (Nt^\alpha)^{\frac{1}{1-\lambda}} \quad \textit{for } 0 < t < T$$

and

$$(\partial_t - \Delta)^\alpha u(0,t) = (Nt^\alpha)^{\frac{\lambda}{1-\lambda}} \quad \textit{for } 0 < t < T.$$

Although the estimates (3.4) and (3.5) are optimal there still remains the question as to whether there is a *single* solution which has the same size as these estimates as $t \to \infty$. By the following theorem there is such a solution.

Theorem 3.4. *Suppose* α *and* p *satisfy* (2.10) *and* $(\lambda, \alpha) \in B$. *Then there exists* $N > 0$ *and* $u \in Y_\alpha^p$ *satisfying* (3.2), (3.3) *such that*

$$u(x,t) \geq (Nt^\alpha)^{\frac{1}{1-\lambda}} \quad \textit{for } (x,t) \in \Omega$$

and

$$(\partial_t - \Delta)^\alpha u(x,t) \geq (Nt^\alpha)^{\frac{\lambda}{1-\lambda}} \quad \textit{for } (x,t) \in \Omega$$

where $\Omega = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t\}$.

According to the following theorem, if $(\lambda, \alpha) \in C$ then there exist bounds as $t \to 0^+$ for solutions of (3.1)–(3.3) in neither the pointwise (i.e. L^∞) sense nor in the L^q sense when $q > p$.

Moreover by Theorem 3.6 the same is true as $t \to \infty$ provided $q \in [q_0, \infty]$ for some $q_0 = q_0(n, \alpha, \lambda) > p$.

Theorem 3.5. *Suppose* α *and* p *satisfy* (2.10)

$$(\lambda, \alpha) \in C \qquad \textit{and} \qquad q \in (p, \infty].$$

Then there exists a solution $u \in Y_\alpha^p$ *of* (3.2), (3.3) *and a sequence* $\{t_j\} \subset (0,1)$ *such that*

$$\lim_{j \to \infty} t_j = 0$$

and

$$\|u^\lambda\|_{L^q(R_j)} = \|(\partial_t - \Delta)^\alpha u\|_{L^q(R_j)} = \infty \quad \textit{for } j = 1, 2, ...,$$

where

$$R_j = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x| < \sqrt{t_j} \textit{ and } t_j < t < 2t_j\}. \tag{3.7}$$

Theorem 3.6. *Suppose* α *and* p *satisfy* (2.10),

$$(\lambda, \alpha) \in C \qquad \textit{and} \qquad q \in \left[\frac{n+2}{2\alpha}\left(1 - \frac{1}{\lambda}\right), \infty\right].$$

Then there exists a solution $u \in Y_\alpha^p$ *of* (3.2), (3.3) *and a sequence* $\{t_j\} \subset (1, \infty)$ *such that*

$$\lim_{j \to \infty} t_j = \infty$$

and

$$\|u^\lambda\|_{L^q(R_j)} = \|(\partial_t - \Delta)^\alpha u\|_{L^q(R_j)} = \infty \quad \textit{for } j = 1, 2, ...,$$

where R_j *is given in* (3.7).

4 J_α version of fully fractional initial value problems

In order to prove our results stated in Section 3, we will first reformulate them in terms of the inverse J_α of the fractional heat operator (2.12) as follows.

Suppose that $\lambda > 0$ and, as assumed in Definition 2.1 and Theorems 3.1–3.6, that p and α satisfy (2.10). Then, by Theorem 2.3, u satisfies (3.1)–(3.3) if and only if $f := (\partial_t - \Delta)^\alpha u$ satisfies

$$f \in X^p \tag{4.1}$$

$$0 \le f \le (J_\alpha f)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{4.2}$$

$$f = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0). \tag{4.3}$$

Thus the two problems (3.1)–(3.3) and (4.1)–(4.3) are equivalent under the transformation $u = J_\alpha f$ when p and α satisfy (2.10). This restriction on p and α was imposed so that $J_\alpha f$ would be defined pointwise in $\mathbb{R}^n \times \mathbb{R}$ for all $f \in X^p$. If $p \ge 1$ and $\alpha > 0$ do not satisfy (2.10), that is, if

$$\left(p > 1 \text{ and } \alpha \ge \frac{n+2}{2p}\right) \quad \text{or} \quad \left(p = 1 \text{ and } \alpha > \frac{n+2}{2p}\right) \tag{4.4}$$

then $J_\alpha f$ is generally not defined pointwise as an extended real valued function for $f \in X^p$. (However it can be defined for all f in the subspace $L^p(\mathbb{R}^n \times \mathbb{R})$ of X^p as a distribution on a certain subspace of the Schwarz space S (see [24, Sec 9.2.5]).

Even though $J_\alpha f$ is generally not defined pointwise as and extended real valued function for $f \in X^p$ when p and α satisfy (4.4), it is defined pointwise as a nonnegative extended real value function for all *nonnegative* functions $f \in X^p$ for all $p \ge 1$ and $\alpha > 0$ because then the integrand of $J_\alpha f$ is a nonnegative function. Hence, since f is nonnegative in the problem (4.1)–(4.3), we see that the problem (4.1)–(4.3) makes sense for all $p \ge 1$ and $\alpha > 0$ when J_α is defined in the pointwise sense, which is the sense in which we will define it in this section. However J_α, when restricted to the set X^p_+ of all nonnegative functions $f \in X^p$, is not one-to-one when p and α satisfy (4.4). Thus our results in this section for the problem (4.1)–(4.3) when $p \ge 1$ and $\alpha > 0$ will yield corresponding results for the problem (3.1)–(3.3) only when p and α satisfy (2.10).

In view of these remarks, we will consider in this section solutions

$$f \in X^p \tag{4.5}$$

of the following J_α version of the fully fractional initial value problem (3.2), (3.3):

$$0 \le f \le K(J_\alpha f)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R},\ n \ge 1 \tag{4.6}$$

$$f = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0) \tag{4.7}$$

where

$$p \in [1, \infty) \quad \text{and} \quad K, \lambda, \alpha \in (0, \infty) \tag{4.8}$$

are constants, X^p is defined by (2.7), and J_α is given by (2.4).

Under the equivalence of problems (3.1)–(3.3) and (4.1)–(4.3) discussed above, the following Theorems 4.1–4.6, when restricted to the case that p and α satisfy (2.10) and $K = 1$, clearly imply Theorems 3.1–3.6 in Section 3. We will prove Theorems 4.1–4.6 in Section 8.

Theorem 4.1. *Suppose* $(\lambda, \alpha) \in A$ *and* $f, p,$ *and* K *satisfy* (4.5)–(4.8). *Then*

$$f = J_\alpha f = 0 \quad \textit{almost everywhere in } \mathbb{R}^n \times \mathbb{R}. \tag{4.9}$$

Theorem 4.2. *Suppose* $(\lambda,\alpha)\in B$ *and* $f,p,$ *and* K *satisfy* (4.5)–(4.8). *Then for all* $b>0$ *we have*

$$\|f\|_{L^\infty(\mathbb{R}^n\times(0,b))}\le K^{\frac{1}{1-\lambda}}(Mb^\alpha)^{\frac{\lambda}{1-\lambda}} \tag{4.10}$$

and

$$\|J_\alpha f\|_{L^\infty(\mathbb{R}^n\times(0,b))}\le K^{\frac{1}{1-\lambda}}(Mb^\alpha)^{\frac{1}{1-\lambda}} \tag{4.11}$$

where

$$M=M(\alpha,\lambda)=\frac{\Gamma(\frac{\alpha\lambda}{1-\lambda}+1)}{\Gamma(\alpha+\frac{\alpha\lambda}{1-\lambda}+1)}. \tag{4.12}$$

Theorem 4.3. *Suppose* p *and* K *satisfy* (4.8), $(\lambda,\alpha)\in B$, $T>0$, *and* $0<N<M$ *where* M *is given by* (4.12). *Then there exists a solution*

$$f\in L^p(\mathbb{R}^n\times\mathbb{R})\cap C(\mathbb{R}^n\times\mathbb{R}) \tag{4.13}$$

of (4.6), (4.7) *such that*

$$J_\alpha f\in C(\mathbb{R}^n\times\mathbb{R}) \tag{4.14}$$

$$f(0,t)=K^{\frac{1}{1-\lambda}}(Nt^\alpha)^{\frac{\lambda}{1-\lambda}}\quad \text{for } 0<t<T \tag{4.15}$$

and

$$J_\alpha f(0,t)\ge K^{\frac{1}{1-\lambda}}(Nt^\alpha)^{\frac{1}{1-\lambda}}\quad \text{for } 0<t<T. \tag{4.16}$$

Theorem 4.4. *Suppose* p *and* K *satisfy* (4.8) *and* $(\lambda,\alpha)\in B$. *Then there exists* $N>0$ *and*

$$f\in X^p$$

satisfying (4.6), (4.7) *such that*

$$f(x,t)\ge K^{\frac{1}{1-\lambda}}(Nt^\alpha)^{\frac{\lambda}{1-\lambda}}\quad \text{for } |x|^2<t \tag{4.17}$$

and

$$J_\alpha f(x,t)\ge K^{\frac{1}{1-\lambda}}(Nt^\alpha)^{\frac{1}{1-\lambda}}\quad \text{for } |x|^2<t. \tag{4.18}$$

Theorem 4.5. *Suppose* p *and* K *satisfy* (4.8),

$$(\lambda,\alpha)\in C\qquad \text{and}\qquad q\in(p,\infty]. \tag{4.19}$$

Then there exists a solution

$$f\in L^p(\mathbb{R}^n\times\mathbb{R}) \tag{4.20}$$

of (4.6), (4.7) *and a sequence* $\{t_j\}\subset(0,1)$ *such that*

$$\lim_{j\to\infty}t_j=0$$

and

$$\|f\|_{L^q(R_j)}=\infty\quad \text{for } j=1,2,..., \tag{4.21}$$

where

$$R_j=\{(x,t)\in\mathbb{R}^n\times\mathbb{R}:|x|<\sqrt{t_j}\text{ and } t_j<t<2t_j\}. \tag{4.22}$$

Theorem 4.6. *Suppose* p *and* K *satisfy* (4.8),

$$(\lambda, \alpha) \in C \quad \text{and} \quad \frac{n+2}{2\alpha}(1 - \frac{1}{\lambda}) \le q \le \infty. \tag{4.23}$$

Then there exists a solution

$$f \in X^p \tag{4.24}$$

of (4.6), (4.7) *and a sequence* $\{t_j\} \subset (1, \infty)$ *such that*

$$\lim_{j \to \infty} t_j = \infty$$

and

$$\|f\|_{L^q(R_j)} = \infty \quad \text{for } j = 1, 2, ..., \tag{4.25}$$

where R_j *is given in* (4.22).

5 Preliminary results for fully fractional heat operators

In this section we provide some lemmas needed for the proofs of our results in Section 2 concerning the fully fractional heat operator (2.12).

The following lemma is needed for the proof of Theorem 2.2.

Lemma 5.1. *Suppose* $\alpha, \beta > 0$. *Then*

$$\Phi_{\alpha+\beta} = \Phi_\alpha * \Phi_\beta \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{5.1}$$

where Φ_α *is defined in* (2.2).

Proof. Since

$$\begin{aligned} \Phi_\alpha * \Phi_\beta(x,t) &= \int_{-\infty}^{\infty} \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x-\xi, t-\tau)\Phi_\beta(\xi,\tau)\, d\xi\, d\tau \\ &= \begin{cases} 0 & \text{for } (x,t) \in \mathbb{R}^n \times (-\infty, 0] \\ \int_0^t \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x-\xi, t-\tau)\Phi_\beta(\xi,\tau)\, d\xi\, d\tau & \text{for } (x,t) \in \mathbb{R}^n \times (0, \infty), \end{cases} \end{aligned} \tag{5.2}$$

we have (5.1) holds in $\mathbb{R}^n \times (-\infty, 0]$.

Using the well-known facts that

$$\widehat{\Phi}_\alpha(\cdot, t)(y) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t|y|^2} \quad \text{for } t > 0 \text{ and } y \in \mathbb{R}^n \tag{5.3}$$

and

$$\int_0^t \frac{(t-\tau)^{\alpha-1}\tau^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} d\tau = \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \quad \text{for } t, \alpha, \beta > 0, \tag{5.4}$$

and assuming we can interchange the order of integration in the following calculation (we will justify this after the calculation) we obtain for $t > 0$ and $y \in \mathbb{R}^n$ that

$$
\begin{aligned}
&(\Phi_\alpha * \Phi_\beta)\widehat{\ }(\cdot,t)(y)\\
&= \int_{x\in\mathbb{R}^n} e^{ix\cdot y} \int_0^t \left(\int_{\xi\in\mathbb{R}^n} \Phi_\alpha(x-\xi,t-\tau)\Phi_\beta(\xi,\tau)\,d\xi \right) d\tau\, dx \qquad (5.5)\\
&= \int_0^t \left(\int_{x\in\mathbb{R}^n} e^{ix\cdot y} \left(\int_{\xi\in\mathbb{R}^n} \Phi_\alpha(x-\xi,t-\tau)\Phi_\beta(\xi,\tau)\,d\xi \right) dx \right) d\tau\\
&= \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-|y|^2(t-\tau)} \frac{\tau^{\beta-1}}{\Gamma(\beta)} e^{-|y|^2\tau} d\tau \quad \text{(by the convolution theorem)}\\
&= e^{-|y|^2 t} \int_0^t \frac{(t-\tau)^{\alpha-1}\tau^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} d\tau\\
&= e^{-t|y|^2} \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = \widehat{\Phi}_{\alpha+\beta}(\cdot,t)(y). \qquad (5.6)
\end{aligned}
$$

This calculation is justified by Fubini's theorem and the fact that the integral (5.5) with $e^{ix\cdot y}$ replaced with 1 is, by Fubini's theorem for nonnegative functions and (5.4), equal to

$$
\begin{aligned}
&\int_0^t \int_{\xi\in\mathbb{R}^n} \left(\int_{x\in\mathbb{R}^n} \Phi_\alpha(x-\xi,t-\tau)dx \right) \Phi_\beta(\xi,\tau)\,d\xi\,d\tau\\
&= \int_0^t \int_{\xi\in\mathbb{R}^n} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \Phi_\beta(\xi,\tau)\,d\xi\,d\tau\\
&= \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \quad \text{for } t>0 \text{ and } y\in\mathbb{R}^n.
\end{aligned}
$$

It follows now from (5.6) that (5.1) holds in $\mathbb{R}^n \times (0,\infty)$. □

The following lemma is needed for the proof of Lemma 5.3 which in turn is needed for the proof of Theorem 2.3.

Lemma 5.2. *Suppose $f \in L^1(-\infty,0)$ and $0 < \alpha \le 1$. Then*

$$
g(t) := \int_{-\infty}^t (t-\tau)^{\alpha-1}|f(\tau)|d\tau < \infty \qquad \textit{for almost all } t \in (-\infty,0).
$$

Proof. The lemma is clearly true if $\alpha = 1$. Hence we can assume $0 < \alpha < 1$. Since

$$
\begin{aligned}
\int_{-\infty}^0 (-t)^{-\alpha} g(t)\,dt &= \int_{-\infty}^0 (-t)^{-\alpha} \int_{-\infty}^t (t-\tau)^{\alpha-1}|f(\tau)|\,d\tau\,dt\\
&= \int_{-\infty}^0 |f(\tau)| \left(\int_\tau^0 (-t)^{(1-\alpha)-1}(t-\tau)^{\alpha-1}dt \right) d\tau\\
&= \Gamma(1-\alpha)\Gamma(\alpha) \int_{-\infty}^0 |f(\tau)|\,d\tau < \infty,
\end{aligned}
$$

where we have used (5.4), we see that $g(t) < \infty$ for almost all $t \in (-\infty,0)$. □

Lemma 5.3. *Suppose $f \in L^1(\mathbb{R}^n \times (-\infty, 0))$, $\alpha \in (0,1]$, and $y \in \mathbb{R}^n$. Then for almost all $t \in (-\infty, 0)$ we have*

$$\widehat{J_\alpha f}(\cdot, t)(y) = \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-|y|^2(t-\tau)} \widehat{f}(\cdot, \tau)(y)\, d\tau.$$

Proof. By Fubini's theorem for nonnegative functions and Lemma 5.2 we find for almost all $t \in (-\infty, 0)$ that

$$\int_{x\in\mathbb{R}^n} |e^{ix\cdot y}| \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \int_{\xi\in\mathbb{R}^n} \Phi_1(x-\xi, t-\tau)|f(\xi,\tau)|\, d\xi\, d\tau\, dx$$
$$= \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{\xi\in\mathbb{R}^n} |f(\xi,\tau)| d\xi \right) d\tau < \infty.$$

Hence by Fubini's theorem, the convolution theorem for Fourier transforms, and (5.3), we see for almost all $t \in (-\infty, 0)$ that

$$\widehat{J_\alpha f}(\cdot, t)(y) = \int_{-\infty}^{t} \int_{x\in\mathbb{R}^n} e^{ix\cdot y} \int_{\xi\in\mathbb{R}^n} \Phi_\alpha(x-\xi, t-\tau) f(\xi,\tau)\, d\xi\, dx\, d\tau$$
$$= \int_{-\infty}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-|y|^2(t-\tau)} \widehat{f}(\cdot, \tau)(y)\, d\tau.$$

□

6 Fully fractional heat operator proofs

In this section we prove our fully fractional heat operator results which we stated in Section 2.

Proof of Theorem 2.1. Part (i) was proved by Sampson [25, Theorem 2.2]. We prove part (ii) in two steps.

Step 1. Suppose $f, g \in S$. Let $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ be momentarily fixed and define $\varphi \in S$ by

$$\varphi(y,s) = f(x+y, t+s).$$

Then

$$\widehat{\widehat{\varphi}}(y,s) = (2\pi)^{n+1}\varphi(-y,-s) = (2\pi)^{n+1} f(x-y, t-s)$$

and

$$\widehat{\varphi}(y,s) = e^{-ix\cdot y - its} \widehat{f}(y,s).$$

Thus by part (i) with φ replaced with $\widehat{\varphi}$ we get

$$\begin{aligned}(2\pi)^{n+1} J_\alpha f(x,t) &= (2\pi)^{n+1} \iint_{\mathbb{R}^n\times\mathbb{R}} \Phi_\alpha(y,s) f(x-y, t-s)\, dy\, ds \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}} \Phi_\alpha(y,s) \widehat{\widehat{\varphi}}(y,s)\, dy\, ds \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}} (|y|^2 - is)^{-\alpha} \widehat{\varphi}(y,s)\, dy\, ds \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}} (|y|^2 - is)^{-\alpha} \widehat{f}(y,s) e^{-ix\cdot y - its}\, dy\, ds. \qquad (6.1)\end{aligned}$$

Multiplying (6.1) by $\widehat{g}(x,t)/(2\pi)^{n+1}$, integrating the resulting equation with respect to (x,t), and interchanging the order of integration in the resulting integral on the RHS, which is allowed by Fubini's theorem and the fact that

$$\iint_{||y|^2-is|\leq 1} ||y|^2 - is|^{-\alpha} dy\, ds < \infty \quad \text{for } 0 < \alpha < (n+2)/2, \tag{6.2}$$

we get (2.5).

Step 2. Suppose $f \in L^1(\mathbb{R}^n \times \mathbb{R})$ and $g \in S$. Then $\widehat{g} \in S$ and $\widehat{f} \in C(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty(\mathbb{R}^n \times \mathbb{R})$. Since S is dense in $L^1(\mathbb{R}^n \times \mathbb{R})$ there exists $\{f_j\} \subset S$ such that $f_j \to f$ in $L^1(\mathbb{R}^n \times \mathbb{R})$ and by Step 1

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J_\alpha f_j(x,t)\widehat{g}(x,t)\, dx\, dt = \iint_{\mathbb{R}^n\times\mathbb{R}} (|y|^2 - is)^{-\alpha} \widehat{f_j}(y,s) g(y,s)\, dy\, ds. \tag{6.3}$$

Since

$$\|\widehat{f_j} - \widehat{f}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R})} \leq \|f_j - f\|_{L^1(\mathbb{R}^n\times\mathbb{R})} \to 0 \quad \text{as } j \to \infty$$

we have

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^n\times\mathbb{R}} (\widehat{f_j}(y,s) - \widehat{f}(y,s))(|y|^2 - is)^{-\alpha} g(y,s)\, dy\, ds \right| \\
&\leq \|\widehat{f_j} - \widehat{f}\|_{L^\infty(\mathbb{R}^n\times\mathbb{R})} \iint_{\mathbb{R}^n\times\mathbb{R}} ||y|^2 - is|^{-\alpha} |g(y,s)|\, dy\, ds \\
&\to 0 \quad \text{as } j \to \infty
\end{aligned}$$

by (6.2). Thus the RHS of (6.3) tends to the RHS of (2.5) as $j \to \infty$.

Also, defining $h(x,t) = |\widehat{g}(-x,-t)|$ we have

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^n\times\mathbb{R}} J_\alpha (f_j - f)(x,t)\widehat{g}(x,t)\, dx\, dt \right| \\
&\qquad \leq \iint_{\mathbb{R}^n\times\mathbb{R}} \iint_{\mathbb{R}^n\times\mathbb{R}} \Phi_\alpha(x-y, t-s)|(f_j - f)(y,s)|\, dy\, ds\, |\widehat{g}(x,t)|\, dx\, dt \\
&\qquad = \iint_{\mathbb{R}^n\times\mathbb{R}} |(f_j - f)(y,s)|(\Phi_\alpha * h)(-y,-s)\, dy\, ds \\
&\qquad \to 0 \quad \text{as } j \to \infty
\end{aligned}$$

because noting that $h \in L^1(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty(\mathbb{R}^n \times \mathbb{R})$,

$$\begin{aligned}
\|\Phi_\alpha \chi_{\mathbb{R}^n\times(0,1)}\|_{L^1(\mathbb{R}^n\times\mathbb{R})} &= \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_{x\in\mathbb{R}^n} \Phi_1(x,t)\, dx\, dt \\
&= \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt < \infty \quad \text{for } \alpha > 0,
\end{aligned} \tag{6.4}$$

and $\Phi_\alpha \chi_{\mathbb{R}^n\times(1,\infty)} \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ for $\alpha < (n+2)/2$ we find that

$$\Phi_\alpha * h = \Phi_\alpha \chi_{\mathbb{R}^n\times(0,1)} * h + \Phi_\alpha \chi_{\mathbb{R}^n\times(1,\infty)} * h \in L^\infty(\mathbb{R}^n \times \mathbb{R})$$

by Young's inequality. Thus the LHS of (6.3) tends to the LHS of (2.5) as $j \to \infty$. □

Proof of Theorem 2.2. Since

$$\bigcap_{T\in\mathbb{R}} L^p_{\text{loc}}(\mathbb{R}^n\times\mathbb{R}_T) = L^p_{\text{loc}}(\mathbb{R}^n\times\mathbb{R})$$

and since $(J_\alpha f_T)|_{\mathbb{R}^n\times\mathbb{R}_T} = (J_\alpha f)|_{\mathbb{R}^n\times\mathbb{R}_T}$, where $f_T = f\chi_{\mathbb{R}^n\times\mathbb{R}_T}$ to prove (i), (ii) and (iii) it suffices to prove for all $T\in\mathbb{R}$ that

(i)′ $J_\alpha f_T,\ J_\alpha|f_T| \in L^p_{\text{loc}}(\mathbb{R}^n\times\mathbb{R}_T)$

(ii)′ $J_\beta J_\gamma f_T = J_\alpha f_T$ in $L^p_{\text{loc}}(\mathbb{R}^n\times\mathbb{R}_T)$ whenever $\beta>0$, $\gamma>0$, and $\beta+\gamma=\alpha$ and

(iii)′ $HJ_\alpha f_T = J_{\alpha-1} f_T$ in $D'(\mathbb{R}^n\times\mathbb{R}_T)$ when $\alpha>1$.

To do this, let $T\in\mathbb{R}$ be fixed. Since $f\in X^p\subset L^p(\mathbb{R}^n\times\mathbb{R}_T)$ we have

$$f_T \in L^p(\mathbb{R}^n\times\mathbb{R}). \tag{6.5}$$

Proof of (i)′. Since $|J_\alpha f_T| \le J_\alpha|f_T|$, to prove (i)′ it suffices to prove only that

$$J_\alpha|f_T| \in L^p_{\text{loc}}(\mathbb{R}^n\times\mathbb{R}_T). \tag{6.6}$$

By (2.3) we have

$$J_\alpha|f_T| = u_1+u_2, \tag{6.7}$$

where

$$u_1 = (\Phi_\alpha\chi_{\mathbb{R}^n\times(0,1)})*|f_T| \quad\text{and}\quad u_2 = (\Phi_\alpha\chi_{\mathbb{R}^n\times(1,\infty)})*|f_T|.$$

It follows from (6.4), (6.5), and Young's inequality that

$$u_1 \in L^p(\mathbb{R}^n\times\mathbb{R}).$$

Thus to complete the proof of (6.6) and hence of (i)′ it suffices to show

$$u_2 \in L^\infty(\mathbb{R}^n\times\mathbb{R}). \tag{6.8}$$

To do this we consider two cases.

Case I. Suppose $1<p<\frac{n+2}{2\alpha}$. Let q be the conjugate Hölder exponent for p. Then

$$\frac{1}{q} = 1-\frac{1}{p} < 1-\frac{2\alpha}{n+2} = \frac{n+2-2\alpha}{n+2}$$

and thus making the change of variables $\sqrt{\frac{q}{4s}}\,y = z$ we obtain

$$\begin{aligned}\|\Phi_\alpha\chi_{\mathbb{R}^n\times(1,\infty)}\|^q_{L^q(\mathbb{R}^n\times\mathbb{R})} &= C(n,\alpha,q)\int_1^\infty\int_{y\in\mathbb{R}^n} s^{(\alpha-1-n/2)q}e^{-\frac{q}{4s}|y|^2}\,dy\,ds\\ &= C(n,\alpha,q)\int_1^\infty s^{(\alpha-1-n/2)q+n/2}\int_{z\in\mathbb{R}^n} e^{-|z|^2}dz\,ds<\infty.\end{aligned}$$

Hence (6.8) follows from (6.5) and Young's inequality.

Case II. Suppose $1 = p \le \frac{n+2}{2\alpha}$. Then

$$\begin{aligned}\Phi_\alpha \chi_{\mathbb{R}^n\times(1,\infty)}(y,s) &\le C(n,\alpha) s^{\alpha-1-n/2}\chi_{\mathbb{R}^n\times(1,\infty)}(y,s)\\ &\le C(n,\alpha) \quad \text{for } (y,s)\in\mathbb{R}^n\times\mathbb{R}.\end{aligned}$$

Thus (6.8) follows from (6.5) and so the proof of (i)′ is complete.

Proof of (ii)′. Using Fubini's theorem for nonnegative functions and Lemma 5.1 we have

$$\begin{aligned}J_\beta(J_\gamma|f_T|)(x,t) &= \iint_{\mathbb{R}^n\times\mathbb{R}} \Phi_\beta(x-\xi,t-\tau)\iint_{\mathbb{R}^n\times\mathbb{R}} \Phi_\gamma(\xi-\eta,\tau-\zeta)|f_T(\eta,\zeta)|\,d\eta\,d\zeta\,d\xi\,d\tau\\ &= \iint_{\mathbb{R}^n\times\mathbb{R}} \Phi_{\beta+\gamma}(x-\eta,t-\zeta)|f_T(\eta,\zeta)|\,d\eta\,d\zeta\\ &= (J_\alpha|f_T|)(x,t) < \infty \quad \text{a.e. in } \mathbb{R}^n\times\mathbb{R}\end{aligned}$$

by part (i)′. Hence by Fubini's theorem the above calculation can be repeated with $|f_T|$ replaced with f_T which gives (ii)′.

Proof of (iii)′. By (i)′ we have

$$J_\alpha|f_T|,\ J_{\alpha-1}|f_T| \in L^p_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{R}_T) \subset D'(\mathbb{R}^n\times\mathbb{R}_T). \tag{6.9}$$

Let $\varphi\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}_T)$. Then noting that

$$\iint_{\mathbb{R}^n\times\mathbb{R}_T} \Phi_1(x-\eta,t-\zeta)H^*\varphi(x,t)\,dx\,dt = \varphi(\eta,\zeta) \quad \text{for } (\eta,\zeta)\in\mathbb{R}^n\times\mathbb{R}_T \tag{6.10}$$

where $H^* = -\partial_t - \Delta$ and assuming we can interchange the order of integration in the following calculation (we will justify this after the calculation) it follows from Lemma 5.1 that

$$(H(J_\alpha f_T))(\varphi) = (J_\alpha f_T)(H^*\varphi) \tag{6.11}$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}_T}\left(\iint_{\mathbb{R}^n\times\mathbb{R}_T}\Phi_\alpha(x-\xi,t-\tau)f_T(\xi,\tau)\,d\xi\,d\tau\right)H^*\varphi(x,t)\,dx\,dt$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}_T}\iint_{\mathbb{R}^n\times\mathbb{R}_T}\left(\iint_{\mathbb{R}^n\times\mathbb{R}_T}\Phi_1(x-\eta,t-\zeta)\Phi_{\alpha-1}(\eta-\xi,\zeta-\tau)\,d\eta\,d\zeta\right)$$
$$\times f_T(\xi,\tau)\,d\xi\,d\tau H^*\varphi(x,t)\,dx\,dt$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}_T}\left(\iint_{\mathbb{R}^n\times\mathbb{R}_T}\left(\iint_{\mathbb{R}^n\times\mathbb{R}_T}\Phi_1(x-\eta,t-\zeta)H^*\varphi(x,t)\,dx\,dt\right)\Phi_{\alpha-1}(\eta-\xi,\zeta-\tau)\,d\eta\,d\zeta\right)$$
$$\times f_T(\xi,\tau)\,d\xi\,d\tau \tag{6.12}$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}_T}\left(\iint_{\mathbb{R}^n\times\mathbb{R}_T}\Phi_{\alpha-1}(\eta-\xi,\zeta-\tau)f_T(\xi,\tau)\,d\xi\,d\tau\right)\varphi(\eta,\zeta)\,d\eta\,d\zeta$$
$$= (J_{\alpha-1}f_T)(\varphi).$$

To justify this calculation, it suffices by Fubini's theorem to show the integral (6.12), with f_T and $H^*\varphi$ replaced with $|f_T|$ and $|H^*\varphi|$, is finite. However in the same way that (6.12) was obtained from (6.11), we see that this modified integral equals

$$\iint_{\mathbb{R}^n\times\mathbb{R}_T} (J_\alpha|f_T|)(x,t)|H^*\varphi|(x,t)\,dx\,dt < \infty$$

by (6.9). □

Proof of Theorem 2.3. Clearly (ii) implies (i). We now prove (ii). Suppose (2.11). It follows from (2.4) that

$$f|_{\mathbb{R}^n\times\mathbb{R}_T} = 0 \text{ implies } (J_\alpha f)|_{\mathbb{R}^n\times\mathbb{R}_T} = 0.$$

Conversely suppose

$$(J_\alpha f)|_{\mathbb{R}^n\times\mathbb{R}_T} = 0. \tag{6.13}$$

The complete the proof of (ii) it suffices to prove

$$f|_{\mathbb{R}^n\times\mathbb{R}_T} = 0. \tag{6.14}$$

By Theorem 2.2(iii) and mathematical induction, we can, without loss of generality, assume for the proof (6.14) that

$$0 < \alpha \le 1. \tag{6.15}$$

Moreover, by translating we can assume

$$T = 0. \tag{6.16}$$

We divide the proof of (6.14) into two cases.

Case I. Suppose $(2.10)_2$ holds. Then

$$1 = p \le \frac{n+1}{2\alpha}. \tag{6.17}$$

Let

$$F(y,t) = \widehat{f}(\cdot,t)(y) \quad \text{for } (y,t) \in \mathbb{R}^n \times (-\infty, 0). \tag{6.18}$$

By (2.11) and (6.17) we have

$$f \in L^1(\mathbb{R}^n \times (-\infty,0)) \tag{6.19}$$

and thus

$$f(\cdot,t) \in L^1(\mathbb{R}^n) \quad \text{for almost all } t \in (-\infty,0)$$

which implies

$$F(\cdot,t) \in C(\mathbb{R}^n) \quad \text{for almost all } t \in (-\infty,0).$$

Also, by (6.19)

$$\begin{aligned} \|F(y,\cdot)\|_{L^1(-\infty,0)} &= \int_{-\infty}^0 \left| \int_{\mathbb{R}^n} e^{ix\cdot y} f(x,t)\,dx \right| dt \\ &\le \|f\|_{L^1(\mathbb{R}^n\times(-\infty,0))} < \infty \quad \text{for all } y \in \mathbb{R}^n. \end{aligned} \tag{6.20}$$

Case I(a). Suppose $\alpha = 1$. Then by (6.19), (6.13), and Lemma 5.3 we have for each $y \in \mathbb{R}^n$ that

$$\int_{-\infty}^{t} e^{|y|^2\tau} F(y,\tau)\, d\tau = e^{|y|^2 t} \int_{-\infty}^{t} e^{-|y|^2(t-\tau)} F(y,\tau)\, d\tau = 0$$

for almost all $t \in (-\infty, 0)$. Hence, by (6.20) and the measure theoretic fundamental theorem of calculus, we get $F = 0$ in $L^1(\mathbb{R}^n \times (-\infty, 0))$ which together with (6.18) implies (6.14).

Case I(b). Suppose $0 < \alpha < 1$. To handle this case we hold $y \in \mathbb{R}^n \backslash \{0\}$ fixed and define

$$F_0(t) := F(y,t). \tag{6.21}$$

Then by (6.20)

$$F_0 \in L^1(-\infty, 0). \tag{6.22}$$

From (6.19), (6.13), and Lemma 5.3 we have

$$g(t) := \int_{-\infty}^{t} (t-\tau)^{\alpha-1} e^{|y|^2\tau} F_0(\tau) d\tau = 0 \tag{6.23}$$

for almost all $t \in (-\infty, 0)$. On the other hand, assuming we can interchange the order of integration in the following calculation (we will justify this after the calculation), we find for $b \in \mathbb{R}$ that

$$\begin{aligned}
&\int_{-\infty}^{0} \left(\int_{t}^{0} (\zeta - t)^{-\alpha} \cos b\zeta \, d\zeta \right) g(t)\, dt \\
&= \int_{-\infty}^{0} e^{|y|^2\tau} F_0(\tau) \left(\int_{\tau}^{0} \cos b\zeta \left(\int_{\tau}^{\zeta} (t-\tau)^{\alpha-1} (\zeta - t)^{-\alpha} dt \right) d\zeta \right) d\tau \\
&= C(\alpha) \int_{-\infty}^{0} e^{|y|^2\tau} F_0(\tau) \left(\int_{\tau}^{0} \cos b\zeta \, d\zeta \right) d\tau
\end{aligned} \tag{6.24}$$

because making the change of variables $t = \zeta - (\zeta - \tau)s$ we see that

$$\int_{\tau}^{\zeta} (t-\tau)^{\alpha-1} (\zeta - t)^{-\alpha} dt = \int_{0}^{1} (1-s)^{\alpha-1} s^{-\alpha} ds = C(\alpha).$$

The calculation (6.24) is justified by Fubini's theorem and the fact that if we replace $\cos b\zeta$ and $g(t)$ with $|\cos b\zeta|$ and

$$g_0(t) = \int_{-\infty}^{t} (t-\tau)^{\alpha-1} e^{|y|^2\tau} |F_0(\tau)|\, d\tau$$

respectively in the above calculation we get by Fubini's theorem for nonnegative functions that

$$\begin{aligned}
&\int_{-\infty}^{0} \int_{t}^{0} (\zeta - t)^{-\alpha} |\cos b\zeta| \, d\zeta \, g_0(t)\, dt \\
&\leq C(\alpha) \int_{-\infty}^{0} e^{|y|^2\tau} |F_0(\tau)| \left(\int_{\tau}^{0} |\cos b\zeta| \, d\zeta \right) d\tau \\
&\leq C(\alpha) \int_{-\infty}^{0} (-\tau) e^{|y|^2\tau} |F_0(\tau)|\, d\tau < \infty
\end{aligned}$$

by (6.22)

It follows now from (6.23), (6.24) and (6.21) that

$$0=\int_{-\infty}^{0} e^{|y|^2\tau} F(y,\tau)\sin b\tau\, d\tau$$

for all $y\in\mathbb{R}^n\backslash\{0\}$ and all $b\in\mathbb{R}$. Thus since the Fourier sine transform is one to one on $L^1(-\infty,0)$ we have $F(y,\cdot)=0$ in $L^1(-\infty,0)$ for all $y\in\mathbb{R}^n\backslash\{0\}$. Hence by Fubini's theorem, $F=0$ in $L^1(\mathbb{R}^n\times(-\infty,0))$, which together with (6.18) and (6.16) implies (6.14).

Case II. Suppose $(2.10)_1$ holds. Let $f_T=f\chi_{\mathbb{R}^n\times\mathbb{R}_T}$ and $u=J_\alpha f_T$. Then by (2.11) we have

$$f_T\in L^p(\mathbb{R}^n\times\mathbb{R}),$$

and by (2.4) and (6.13) we have

$$u=0 \quad \text{in } \mathbb{R}^n\times\mathbb{R}_T. \tag{6.25}$$

Let $J_\varepsilon^{-\alpha}u$ be as defined in Theorem A.1. By (6.25) we have for $l>\alpha$ that $(\Delta^l_{y,\tau}u)(x,t)=0$ for $(x,t)\in\mathbb{R}^n\times\mathbb{R}_T$ and $(y,\tau)\in\mathbb{R}^n\times(0,\infty)$. Thus for $\varepsilon>0$ we have

$$J_\varepsilon^{-\alpha}u=0 \quad \text{in } \mathbb{R}^n\times\mathbb{R}_T.$$

Hence (6.14) follows from Theorem A.1. □

Proof of Theorem 2.4. For $a,\tau>0$ and $\delta\geq 0$ we have

$$\begin{aligned}\int_{|\xi|>\delta}\Phi_{\alpha,a,1}(\xi,\tau)\,d\xi &= \int_{|\xi|>\delta}\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{(4\pi a^2\tau)^{n/2}}e^{-\frac{|\xi|^2}{4a^2\tau}}\,d\xi\\ &=\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{\pi^{n/2}}\int_{|\eta|>\frac{\delta}{\sqrt{4a^2\tau}}}e^{-|\eta|^2}\,d\eta.\end{aligned} \tag{6.26}$$

In particular, taking $\delta=0$ we find that

$$\int_{\mathbb{R}^n}\Phi_{\alpha,a,1}(\xi,\tau)\,d\xi=\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} \quad \text{for } a,\tau>0. \tag{6.27}$$

Let Ω be a compact subset of $\mathbb{R}^n\times\mathbb{R}$. Choose $T>0$ such that

$$f=0 \quad \text{on } \mathbb{R}^n\times\mathbb{R}_{-T} \tag{6.28}$$

and

$$\Omega\subset\mathbb{R}^n\times\mathbb{R}_T. \tag{6.29}$$

Let $\varepsilon>0$. Since f is uniformly continuous on $\mathbb{R}^n\times\mathbb{R}$ there exists $\delta>0$ such that

$$|f(x-\xi,\zeta)-f(x,\zeta)|<\varepsilon \tag{6.30}$$

whenever $x,\xi\in\mathbb{R}^n,\zeta\in\mathbb{R}$, and $|\xi|<\delta$.

Let $(x,t)\in\Omega$. Then $t<T$ and thus for $\tau\geq 2T$ we have

$$t-\tau<T-2T=-T.$$

Hence for $a > 0$ we have by (6.28) and (6.27) that

$$\begin{aligned}
&|(J_{\alpha,a,1}f - J_{\alpha,0,1}f)(x,t)| \\
&\le \int_0^{2T} \left| \int_{\mathbb{R}^n} \Phi_{\alpha,a,1}(\xi,\tau) f(x-\xi,t-\tau)\,d\xi - \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} f(x,t-\tau) \right| d\tau \\
&= \int_0^{2T} \left| \int_{\mathbb{R}^n} \Phi_{\alpha,a,1}(\xi,\tau)(f(x-\xi,t-\tau) - f(x,t-\tau))\,d\xi \right| d\tau \\
&\le K_1(x,t) + K_2(x,t) \qquad (6.31)
\end{aligned}$$

where

$$K_1(x,t) = \int_0^{2T} \int_{|\xi|<\delta} \Phi_{\alpha,a,1}(\xi,\tau)|f(x-\xi,t-\tau) - f(x,t-\tau)|\,d\xi\,d\tau$$

and

$$K_2(x,t) = \int_0^{2T} \int_{|\xi|>\delta} \Phi_{\alpha,a,1}(\xi,\tau)|f(x-\xi,t-\tau) - f(x,t-\tau)|\,d\xi\,d\tau.$$

From (6.30) and (6.27) we conclude that

$$K_1(x,t) \le \varepsilon \int_0^{2T} \left(\int_{\mathbb{R}^n} \Phi_{\alpha,a,1}(\xi,\tau)\,d\xi \right) d\tau = \varepsilon \int_0^{2T} \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} d\tau$$

and letting $M = 2\|f\|_{L^\infty(\mathbb{R}^n\times\mathbb{R})}$ and using (6.26) we obtain

$$K_2(x,t) \le M \int_0^{2T} \left(\int_{|\xi|>\delta} \Phi_{\alpha,a,1}(\xi,\tau)\,d\xi \right) d\tau \le M \left(\int_0^{2T} \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} d\tau \right) C(n,a,\delta,T)$$

where

$$C(n,a,\delta,T) = \frac{1}{\pi^{n/2}} \int_{|\eta|>\frac{\delta}{\sqrt{8a^2T}}} e^{-|\eta|^2} d\eta \to 0 \quad \text{as } a \to 0^+.$$

The theorem therefore follows from (6.31). □

Proof of Theorem 2.5. For $b > 0, \delta > 0$, and $\xi \in \mathbb{R}^n \backslash \{0\}$ we have

$$\begin{aligned}
\int_\delta^\infty \Phi_{\alpha,1,b}(\xi,\tau)\,d\tau &= \int_\delta^\infty \frac{(\tau/b)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{(4\pi\tau/b)^{n/2}} e^{-\frac{b|\xi|^2}{4\tau}}\,d\tau/b \\
&= \int_\delta^\infty \frac{1}{\Gamma(\alpha)(4\pi)^{n/2}} \left(\frac{\tau}{b}\right)^{\alpha-1-n/2} e^{-\frac{b|\xi|^2}{4\tau}} \frac{1}{b}\,d\tau \\
&= \int_0^{\frac{b|\xi|^2}{4\delta}} \frac{1}{\Gamma(\alpha)(4\pi)^{n/2}} \left(\frac{|\xi|^2}{4\zeta}\right)^{\alpha-1-n/2} e^{-\zeta} \frac{|\xi|^2}{4\zeta^2}\,d\zeta \\
&= \frac{(|\xi|^2/4)^{\alpha-n/2}}{\Gamma(\alpha)(4\pi)^{n/2}} \int_0^{\frac{b|\xi|^2}{4\delta}} \zeta^{n/2-\alpha-1} e^{-\zeta} d\zeta \\
&= \frac{|\xi|^{2\alpha-n}}{4^\alpha \pi^{n/2}\Gamma(\alpha)} \int_0^{\frac{b|\xi|^2}{4\delta}} \zeta^{n/2-\alpha-1} e^{-\zeta} d\zeta \qquad (6.32) \\
&\le \left(\frac{|\xi|^{2\alpha-n}}{4^\alpha \pi^{n/2}\Gamma(\alpha)} \right) \frac{1}{n/2-\alpha} \left(\frac{b|\xi|^2}{4\delta} \right)^{n/2-\alpha} = C(n,\alpha) \left(\frac{b}{\delta}\right)^{n/2-\alpha}. \qquad (6.33)
\end{aligned}$$

Moreover, letting $\delta \to 0^+$ in (6.32) we obtain

$$\int_0^\infty \Phi_{\alpha,1,b}(\xi,\tau)\,d\tau = \frac{|\xi|^{2\alpha-n}}{\gamma(n,\alpha)} \quad \text{for } b>0 \text{ and } \xi \neq 0, \tag{6.34}$$

where γ is given in (2.14).

Let Ω be a compact subset of $\mathbb{R}^n \times \mathbb{R}$. Choose $R>0$ such that

$$f = 0 \quad \text{on } (\mathbb{R}^n \backslash B_R(0)) \times \mathbb{R} \tag{6.35}$$

and

$$\Omega \subset B_R(0) \times \mathbb{R}. \tag{6.36}$$

Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathbb{R}^n \times \mathbb{R}$ there exists $\delta > 0$ such that

$$|f(\eta, t-\tau) - f(\eta,t)| < \varepsilon \tag{6.37}$$

whenever $\eta \in \mathbb{R}^n$, $t,\tau \in \mathbb{R}$, and $|\tau| < \delta$.

Let $(x,t) \in \Omega$. Then $|x| < R$ and thus for $|\xi| \geq 2R$ we have

$$|x-\xi| \geq |\xi| - |x| > 2R - R = R.$$

Hence for $b>0$ we find by (6.35) and (6.34) that

$$\begin{aligned}
&|(J_{\alpha,1,b}f - J_{\alpha,1,0}f)(x,t)| \\
&\leq \int_{|\xi|<2R} \left| \int_0^\infty \Phi_{\alpha,1,b}(\xi,\tau) f(x-\xi,t-\tau)\,d\tau - \frac{f(x-\xi,t)}{\gamma(n,\alpha)|\xi|^{n-2\alpha}} \right| d\xi \\
&= \int_{|\xi|<2R} \left| \int_0^\infty \Phi_{\alpha,1,b}(\xi,\tau)(f(x-\xi,t-\tau) - f(x-\xi,t))\,d\tau \right| d\xi \\
&\leq K_1(x,t) + K_2(x,t)
\end{aligned} \tag{6.38}$$

where

$$K_1(x,t) = \int_{|\xi|<2R} \int_0^\delta \Phi_{\alpha,1,b}(\xi,\tau)|f(x-\xi,t-\tau) - f(x-\xi,t)|\,d\tau\,d\xi$$

and

$$K_2(x,t) = \int_{|\xi|<2R} \int_\delta^\infty \Phi_{\alpha,1,b}(\xi,\tau)|f(x-\xi,t-\tau) - f(x-\xi,t)|\,d\tau\,d\xi.$$

From (6.37) and (6.34) we conclude

$$K_1(x,t) \leq \varepsilon \int_{|\xi|<2R} \left(\int_0^\infty \Phi_{\alpha,1,b}(\xi,\tau)\,d\tau \right) d\xi = \varepsilon \int_{|\xi|<2R} \frac{d\xi}{\gamma(n,\alpha)|\xi|^{n-2\alpha}}$$

and letting $M = 2\|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R})}$ and using (6.33) we obtain

$$\begin{aligned}
K_2(x,t) &\leq M \int_{|\xi|<2R} \left(\int_\delta^\infty \Phi_{\alpha,1,b}(\xi,\tau)\,d\tau \right) d\xi \\
&\leq MC(n,\alpha) \left(\frac{b}{\delta} \right)^{n/2-\alpha} |B_{2R}(0)| \to 0 \quad \text{as } b \to 0^+.
\end{aligned}$$

The theorem therefore follows from (6.38). $\square$

7 Preliminary results for J_α problems

In this section we provide some lemmas needed for the proofs of our results in Section 4 dealing with solutions of the J_α problem (4.5)–(4.8).

Let $\Omega = \mathbb{R}^n \times (a,b)$ where $n \geq 1$ and $a < b$. Lemmas 7.1 and 7.2 give estimates for the convolution

$$(V_{\alpha,\Omega} f)(x,t) = \iint_\Omega \Phi_\alpha(x-\xi, t-\tau) f(\xi,\tau)\, d\xi\, d\tau \tag{7.1}$$

where $\alpha > 0$ and Φ_α is defined in (2.2).

Remark 7.1. Note that if $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a nonnegative measurable function such that $\|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}_a)} = 0$ then

$$V_{\alpha,\Omega} f = J_\alpha f \quad \text{in } \Omega := \mathbb{R}^n \times (a,b).$$

Lemma 7.1. *For $\alpha > 0$, $\Omega = \mathbb{R}^n \times (a,b)$ and $f \in L^\infty(\Omega)$ we have*

$$\|V_{\alpha,\Omega} f\|_{L^\infty(\Omega)} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^\infty(\Omega)}.$$

Proof. The lemma is obvious if $\|f\|_{L^\infty(\Omega)} = 0$. Hence we can assume $\|f\|_{L^\infty(\Omega)} > 0$. Then for $(x,t) \in \Omega$

$$\begin{aligned}
\frac{|(V_{\alpha,\Omega} f)(x,t)|}{\|f\|_{L^\infty(\Omega)}} &\leq \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \overbrace{\left(\int_{\xi \in \mathbb{R}^n} \Phi_1(x-\xi, t-\tau) d\xi \right)}^{=1} d\tau \\
&= -\frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} \bigg|_{\tau=a}^{\tau=t} = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \\
&\leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

$\square$

Lemma 7.2. *Let $p, q \in [1,\infty]$, α, and δ satisfy*

$$0 \leq \delta := \frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{n+2} < 1. \tag{7.2}$$

Then $V_{\alpha,\Omega}$ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ and for $f \in L^p(\Omega)$ we have

$$\|V_{\alpha,\Omega} f\|_{L^q(\Omega)} \leq M \|f\|_{L^p(\Omega)}$$

where

$$M = C(b-a)^{\frac{2\alpha-(n+2)\delta}{2}} \text{ for some constant } C = C(n,\alpha,\delta).$$

Proof. Define $r \in [1,\infty)$ by

$$1 - \frac{1}{r} = \delta \tag{7.3}$$

and define $P_\alpha, \bar{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$P_\alpha(x,t) = \Phi_\alpha(x,t) \chi_{(0,b-a)}(t)$$

and

$$\bar{f}(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in \Omega \\ 0 & \text{elsewhere.} \end{cases}$$

Since for $t \in (a,b)$ and $\tau \in (a,t)$ we have $t - \tau \in (0, b-a)$ we see for $(x,t) \in \Omega$ that

$$\begin{aligned} V_{\alpha,\Omega} f(x,t) &= \int_a^t \int_{\xi \in \mathbb{R}^n} P_\alpha(x-\xi, t-\tau) f(\xi,\tau)\, d\xi\, d\tau \\ &= \iint_\Omega P_\alpha(x-\xi, t-\tau) f(\xi,\tau)\, d\xi\, d\tau \\ &= (P_\alpha * \bar{f})(x,t) \end{aligned} \tag{7.4}$$

where $*$ is the convolution operation in $\mathbb{R}^n \times \mathbb{R}$.

Also since

$$\int_{\mathbb{R}^n} e^{-r|x|^2/(4t)} dx = \left(\frac{4\pi t}{r} \right)^{n/2}$$

we have by (7.2) and (7.3) that

$$\begin{aligned} \|P_\alpha\|_{L^r(\mathbb{R}^n \times \mathbb{R})} &= \frac{1}{\Gamma(\alpha)(4\pi)^{n/2}} \left(\int_0^{b-a} t^{r(\alpha-1-n/2)} \left(\int_{x \in \mathbb{R}^n} e^{-r|x|^2/(4t)} dx \right) dt \right)^{1/r} \\ &= C(n,\alpha,r) \left(\int_0^{b-a} t^{r(\alpha-1-n/2)+\frac{n}{2}} dt \right)^{1/r} \\ &= C(n,\alpha,r)(b-a)^{\frac{2\alpha-(n+2)\delta}{2}}. \end{aligned}$$

Thus by (7.4), (7.2), (7.3), and Young's inequality we have

$$\begin{aligned} \|V_{\alpha,\Omega} f\|_{L^q(\Omega)} &= \|P_\alpha * \bar{f}\|_{L^q(\Omega)} \le \|P_\alpha * \bar{f}\|_{L^q(\mathbb{R}^n \times \mathbb{R})} \\ &\le \|P_\alpha\|_{L^r(\mathbb{R}^n \times \mathbb{R})} \|\bar{f}\|_{L^p(\mathbb{R}^n \times \mathbb{R})} \\ &\le C(b-a)^{\frac{2\alpha-(n+2)\delta}{2}} \|f\|_{L^p(\Omega)}. \end{aligned}$$

□

Lemma 7.3. *Suppose f, p, and K satisfy* (4.5)–(4.8) *and* $(\lambda, \alpha) \in A \cup B$. *Then*

$$f \in X^\infty.$$

Proof. Let $T > 0$ be fixed. Then $f \in L^p(\mathbb{R}^n \times \mathbb{R}_T)$ and to complete the proof it suffices to show

$$f \in L^\infty(\mathbb{R}^n \times (0,T)). \tag{7.5}$$

We consider two cases.

<u>Case I.</u> Suppose $0 < \alpha < \frac{n+2}{2p}$. Then

$$0 < \lambda < \frac{n+2}{n+2-2\alpha p}$$

and thus there exists $\varepsilon = \varepsilon(n, \lambda, \alpha, p) > 0$ such that

$$\varepsilon < 2\alpha p, \qquad 2\varepsilon < n + 2 - 2\alpha p, \quad \text{and} \quad \lambda < \frac{n+2}{n+2-2\alpha p + 2\varepsilon}.$$

Suppose

$$f \in L^{p_0}(\mathbb{R}^n \times (0,T)) \quad \text{for some } p_0 \in \left[p, \frac{n+2}{2\alpha}\right). \tag{7.6}$$

Then letting

$$q = \frac{(n+2)p_0}{n+2-2\alpha p_0 + \varepsilon}$$

we have

$$\frac{1}{p_0} - \frac{1}{q} = \frac{2\alpha}{n+2} - \frac{\varepsilon}{(n+2)p_0} \in \left(0, \frac{2\alpha}{n+2}\right).$$

Hence by (4.7), Remark 7.1, and Lemma 7.2 we see that

$$J_\alpha f \in L^q(\mathbb{R}^n \times (0,T)).$$

Thus by (4.6) we find that

$$0 \le f \le K(J_\alpha f)^\lambda \in L^{q/\lambda}(\mathbb{R}^n \times (0,T)). \tag{7.7}$$

Since

$$\begin{aligned} \frac{q/\lambda}{p_0} &= \frac{n+2}{\lambda(n+2-2\alpha p_0+\varepsilon)} \ge \frac{n+2-2\alpha p + 2\varepsilon}{n+2-2\alpha p_0 + \varepsilon} \\ &\ge \frac{n+2-2\alpha p + 2\varepsilon}{n+2-2\alpha p + \varepsilon} = C(n,\lambda,\alpha,p) > 1 \end{aligned}$$

we see that starting with $p_0 = p$ and iterating a finite number of times the process of going from (7.6) to (7.7) yields

$$f \in L^{p_0}(\mathbb{R}^n \times (0,T)) \quad \text{for some } p_0 > \frac{n+2}{2\alpha}.$$

Hence (7.5) follows from (4.6) and Lemma 7.2.

Case II. Suppose $\alpha \ge \frac{n+2}{2p}$. Clearly there exists $\widehat{\alpha} \in (0, \frac{n+2}{2p})$ such that $(\lambda, \widehat{\alpha}) \in A \cup B$. Then for $(x,t), (\xi,\tau) \in \mathbb{R}^n \times (0,T)$ we have

$$\begin{aligned} \frac{\Phi_\alpha(x-\xi, t-\tau)}{\Phi_{\widehat{\alpha}}(x-\xi,t-\tau)} &= (t-\tau)^{\alpha-\widehat{\alpha}}\Gamma(\widehat{\alpha})/\Gamma(\alpha) \\ &\le T^{\alpha-\widehat{\alpha}}\Gamma(\widehat{\alpha})/\Gamma(\alpha) \\ &= C(T,\alpha,\widehat{\alpha}). \end{aligned}$$

Thus for $(x,t) \in \mathbb{R}^n \times (0,T)$ we have

$$J_\alpha f(x,t) \le C(T,\alpha,\widehat{\alpha}) J_{\widehat{\alpha}} f(x,t)$$

and hence by (4.6) we see that

$$0 \le f \le KC(T,\alpha,\widehat{\alpha})^\lambda (J_{\widehat{\alpha}} f)^\lambda \quad \text{almost everywhere in } \mathbb{R}^n \times (0,T).$$

It follows therefore from Case I that f satisfies (7.5). $\square$

Lemma 7.4. *Suppose $x \in \mathbb{R}^n$ and $t, \tau \in (0, \infty)$ satisfy*

$$|x|^2 < t \quad \textit{and} \quad \frac{t}{4} < \tau < \frac{3t}{4}. \tag{7.8}$$

Then

$$\int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau)\, d\xi \geq C(n) > 0$$

where Φ_α is defined by (2.2).

Proof. Making the change of variables $z = \frac{x-\xi}{\sqrt{4(t-\tau)}}$, letting $e_1 = (1, 0, ..., 0)$, and using (7.8) and (2.2) we find that

$$\begin{aligned}\int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau)\, d\xi &= \frac{1}{\pi^{n/2}} \int_{|z - \frac{x}{\sqrt{4(t-\tau)}}| < \frac{\sqrt{\tau}}{\sqrt{4(t-\tau)}}} e^{-|z|^2} dz \\ &\geq \frac{1}{\pi^{n/2}} \int_{|z - \frac{\sqrt{t}}{\sqrt{4(t-\tau)}} e_1| < \frac{\sqrt{\tau}}{\sqrt{4(t-\tau)}}} e^{-|z|^2} dz \\ &\geq \frac{1}{\pi^{n/2}} \int_{|z - e_1| < \frac{1}{2\sqrt{3}}} e^{-|z|^2} dz \\ &= C(n) > 0\end{aligned}$$

where in this calculation we used the fact that the integral of $e^{-|z|^2}$ over a ball is decreased if the absolute value of the center of the ball is increased or the radius of the ball is decreased. $\square$

Lemma 7.5. *For $\tau < t \leq T$ and $|x| \leq \sqrt{T-t}$ we have*

$$\int_{|\xi| < \sqrt{T-\tau}} \Phi_1(x - \xi, t - \tau)\, d\xi \geq C$$

where $C = C(n)$ is a positive constant.

Proof. Making the change of variables $z = \frac{x-\xi}{\sqrt{t-\tau}}$ and letting $e_1 = (1, 0, ..., 0)$ we get

$$\begin{aligned}\int_{|\xi| < \sqrt{T-\tau}} \Phi_1(x - \xi, t - \tau)\, d\xi &= \frac{1}{(4\pi)^{n/2}} \frac{1}{(t-\tau)^{n/2}} \int_{|\xi| < \sqrt{T-\tau}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} d\xi \\ &= \frac{1}{(4\pi)^{n/2}} \int_{|z - \frac{x}{\sqrt{t-\tau}}| < \frac{\sqrt{T-\tau}}{\sqrt{t-\tau}}} e^{-|z|^2/4} dz \qquad (7.9) \\ &\geq \frac{1}{(4\pi)^{n/2}} \int_{|z - \frac{\sqrt{T-\tau}}{\sqrt{t-\tau}} e_1| < \frac{\sqrt{T-\tau}}{\sqrt{t-\tau}}} e^{-|z|^2/4} dz \qquad (7.10) \\ &\geq \frac{1}{(4\pi)^{n/2}} \int_{|z - e_1| < 1} e^{-|z|^2/4} dz, \qquad (7.11)\end{aligned}$$

where the last two inequalities need some explanation. Since $|x| \leq \sqrt{T-t} < \sqrt{T-\tau}$, the center of the ball of integration in (7.9) is closer to the origin than the center of the ball of integration in (7.10). Thus, since the integrand is a decreasing function of $|z|$, we obtain (7.10). Since $\sqrt{T-\tau} \geq \sqrt{t-\tau}$, the ball of integration in (7.10) contains the ball of integration in (7.11) and hence (7.11) holds. $\square$

Lemma 7.6. *Suppose $\alpha > 0$, $\gamma > 0$, $p \geq 1$, and*

$$f_0(x,t) = \left(\frac{1}{t}\right)^{\frac{n+2}{2p}-\gamma} \chi_{\Omega_0}(x,t) \qquad \textit{where } \Omega_0 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t\}.$$

Then $f_0 \in X^p$ and

$$C_1 \left(\frac{1}{t}\right)^{\frac{n+2}{2p}-\gamma-\alpha} \leq J_\alpha f_0(x,t) \leq C_2 \left(\frac{1}{t}\right)^{\frac{n+2}{2p}-\gamma-\alpha} \qquad \textit{for } (x,t) \in \Omega_0$$

where C_1 and C_2 are positive constants depending only on n, α, γ, and p.

Proof. For $T > 0$ we have

$$\begin{aligned} \|f_0\|^p_{L^p(\mathbb{R}^n \times \mathbb{R}_T)} &= \int_0^T \int_{|x|<\sqrt{t}} \left(\frac{1}{t}\right)^{\frac{n+2}{2}-\gamma p} dx\, dt \\ &= C(n) \int_0^T t^{\gamma p - 1} dt < \infty \end{aligned}$$

because $\gamma p > 0$. Hence $f_0 \in X^p$.

Also for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ we have

$$\begin{aligned} J_\alpha f_0(x,t) &= \int_{-\infty}^t \int_{\xi \in \mathbb{R}^n} \Phi_\alpha(x-\xi, t-\tau) f_0(\xi,\tau)\, d\xi\, d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{1}{\tau}\right)^{\frac{n+2}{2p}-\gamma} \left(\int_{|\xi|^2<\tau} \Phi_1(x-\xi, t-\tau)\, d\xi\right) d\tau. \end{aligned} \tag{7.12}$$

Hence by Lemma 7.4 we see for $(x,t) \in \Omega_0$ that

$$\begin{aligned} J_\alpha f_0(x,t) &\geq C(n,\alpha) \int_{t/4}^{3t/4} (t-\tau)^{\alpha-1} \left(\frac{1}{\tau}\right)^{\frac{n+2}{2p}-\gamma} d\tau \\ &= C(n,\alpha) t^{\alpha - \frac{n+2}{2p} + \gamma} \int_{1/4}^{3/4} (1-s)^{\alpha-1} \left(\frac{1}{s}\right)^{\frac{n+2}{2p}-\gamma} ds \quad \text{where } \tau = ts \\ &= C(n,\alpha,\gamma,p) t^{\alpha - \frac{n+2}{2p} + \gamma}. \end{aligned}$$

Moreover for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $0 < \tau < t/2$ we have

$$\begin{aligned} \int_{|\xi|^2<\tau} \Phi_1(x-\xi, t-\tau)\, d\xi &= \frac{1}{\pi^{n/2}} \int_{|z - \frac{x}{\sqrt{4(t-\tau)}}| < \frac{\sqrt{\tau}}{\sqrt{4(t-\tau)}}} e^{-|z|^2} dz \quad \text{where } z = \frac{x-\xi}{\sqrt{4(t-\tau)}} \\ &\leq \frac{|B_1(0)|}{\pi^{n/2}} \left(\frac{\sqrt{\tau}}{\sqrt{4(t-\tau)}}\right)^n \end{aligned}$$

and for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $t/2 < \tau < t$ we have

$$\int_{|\xi|^2<\tau} \Phi_1(x-\xi, t-\tau)\, d\xi \leq \int_{\mathbb{R}^n} \Phi_1(x-\xi, t-\tau)\, d\xi = 1.$$

Thus by (7.12) for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ we have

$$\begin{aligned}
J_\alpha f_0(x,t) &\le C(n,\alpha)\Bigg[\int_0^{t/2} (t-\tau)^{\alpha-1}\left(\frac{1}{\tau}\right)^{\frac{n+2}{2p}-\gamma}\left(\frac{\tau}{t-\tau}\right)^{n/2} d\tau \\
&\qquad + \int_{t/2}^{t} (t-\tau)^{\alpha-1}\left(\frac{1}{\tau}\right)^{\frac{n+2}{2p}-\gamma} d\tau\Bigg] \\
&= C(n,\alpha)t^{\alpha-\frac{n+2}{2p}+\gamma}\Bigg[\int_0^{1/2} (1-s)^{\alpha-1}\left(\frac{1}{s}\right)^{\frac{n+2}{2p}-\gamma}\left(\frac{s}{1-s}\right)^{n/2} ds \\
&\qquad + \int_{1/2}^{1} (1-s)^{\alpha-1}\left(\frac{1}{s}\right)^{\frac{n+2}{2p}-\gamma} ds\Bigg] \\
&= C(n,\alpha,\gamma,p)t^{\alpha-\frac{n+2}{2p}+\gamma}
\end{aligned}$$

because α and γ are positive. □

Lemma 7.7. *Suppose $\alpha > 0$, $\gamma \in \mathbb{R}$, $0 \le t_0 < T$, $p \in [1,\infty)$, and*

$$f(x,t) = \left(\frac{1}{T-t}\right)^{\frac{n+2}{2p}-\gamma} \chi_\Omega(x,t)$$

where

$$\Omega = \{(x,t) \in \mathbb{R}^n \times (t_0,T) : |x| < \sqrt{T-t}\}.$$

Then

$$J_\alpha f(x,t) \ge C\left(\frac{1}{T-t}\right)^{\frac{n+2}{2p}-\gamma-\alpha}$$

for $(x,t) \in \Omega^+ := \{(x,t) \in \Omega : \frac{T+t_0}{2} < t < T\}$ where $C = C(n,\alpha,\gamma,p) > 0$. Moreover,

$$f \in L^p(\mathbb{R}^n \times \mathbb{R}) \text{ if and only if } \gamma > 0 \tag{7.13}$$

and in this case

$$\|f\|^p_{L^p(\mathbb{R}^n\times\mathbb{R})} = C(n)\int_0^{T-t_0} s^{\gamma p-1} ds. \tag{7.14}$$

Proof. Since

$$\begin{aligned}
\|f\|^p_{L^p(\mathbb{R}^n\times\mathbb{R})} &= \int_{t_0}^{T}\int_{|x|<\sqrt{T-t}} (T-t)^{\gamma p-\frac{n+2}{2}}\, dx\, dt \\
&= C(n)\int_{t_0}^{T} (T-t)^{\gamma p-1} dt = C(n)\int_0^{T-t_0} s^{\gamma p-1} ds
\end{aligned}$$

we see that (7.13) and (7.14) hold.

Let $r = \frac{n+2}{2p} - \gamma - \alpha$. Then for $(x,t) \in \Omega$ we have

$$\begin{aligned} J_\alpha f(x,t) &= \int_{t_0}^{t} (T-\tau)^{-r-\alpha} \int_{|\xi|<\sqrt{T-\tau}} \Phi_\alpha(x-\xi, t-\tau)\, d\xi\, d\tau \\ &= C \int_{t_0}^{t} (T-\tau)^{-r-\alpha} (t-\tau)^{\alpha-1} \left(\int_{|\xi|<\sqrt{T-\tau}} \Phi_1(x-\xi, t-\tau)\, d\xi \right) d\tau \\ &\geq C \int_{t_0}^{t} (T-\tau)^{-r-\alpha} (t-\tau)^{\alpha-1} d\tau, \quad \text{by Lemma 7.5,} \\ &= C (T-t)^{-r} g\left(\frac{t-t_0}{T-t} \right) \end{aligned}$$

where $g(z) = \int_0^z (\zeta+1)^{-r-\alpha} \zeta^{\alpha-1} d\zeta$ and where we made the change of variables $t - \tau = (T-t)\zeta$. Thus

$$J_\alpha f(x,t) \geq C(T-t)^{-r} \quad \text{for } (x,t) \in \Omega^+$$

because $\frac{t-t_0}{T-t} > 1$ in Ω^+. □

8 Proofs of results for J_α problems

In this section we prove our results stated in Section 4 concerning pointwise bounds for nonnegative solutions f of (4.5)–(4.8). As explained in Section 4, these results immediately imply Theorems 3.1–3.6 in Section 3.

Remark 8.1. The function $g : \mathbb{R}^n \times \mathbb{R} \to [0,\infty)$ defined by

$$g(x,t) = g(t) = \begin{cases} (Mt^\alpha)^{\frac{\lambda}{1-\lambda}} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0, \end{cases}$$

where $\alpha > 0$, $0 < \lambda < 1$, and $M = M(\alpha, \lambda)$ is defined in (4.12), satisfies

$$g = (J_\alpha g)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{8.1}$$

which can be verified using (5.4). Even though $g \notin X^p$ for all $p \geq 1$, it will be useful in our analysis of solutions of (4.6), (4.7) which are in X^p for some $p \geq 1$.

Remark 8.2. It will be convenient to scale (4.6) as follows. Suppose $K, \lambda, \alpha, T \in (0,\infty)$, $\lambda \neq 1$, and $f, \bar{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are nonnegative measurable functions such that $f = \bar{f} = 0$ in $\mathbb{R}^n \times (-\infty, 0)$ and

$$f(x,t) = K^{\frac{1}{1-\lambda}} T^{\frac{\alpha\lambda}{1-\lambda}} \bar{f}(\bar{x}, \bar{t})$$

where

$$x = T^{1/2}\bar{x} \quad \text{and} \quad t = T\bar{t}.$$

Then f satisfies

$$0 \leq f \leq K(J_\alpha f)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

if and only if $\bar{f}$ satisfies

$$0 \leq \bar{f} \leq (J_\alpha \bar{f})^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

Moreover

$$\frac{f(x,t)}{K^{\frac{1}{1-\lambda}} t^{\frac{\alpha\lambda}{1-\lambda}}} = \frac{\bar{f}(\bar{x}, \bar{t})}{\bar{t}^{\frac{\alpha\lambda}{1-\lambda}}} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty)$$

and

$$\frac{J_\alpha f(x,t)}{K^{\frac{1}{1-\lambda}} t^{\frac{\alpha}{1-\lambda}}} = \frac{J_\alpha \bar{f}(\bar{x},\bar{t})}{\bar{t}^{\frac{\alpha}{1-\lambda}}} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Proof of Theorem 4.1. Suppose for contradiction that (4.9) is false. Then there exists $T > 0$ such that

$$\|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}_T)} > 0.$$

Hence by (4.7) there exists $t_0 \in [0,T)$ such that

$$\|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}_t)} \begin{cases} = 0 & \text{for } t \le t_0 \\ > 0 & \text{for } t > t_0. \end{cases}$$

Thus by Remark 7.1, we have for all $b > t_0$ that

$$J_\alpha f = V_{\alpha,\Omega_b} f \quad \text{in } \Omega_b$$

where $\Omega_b = \mathbb{R}^n \times (t_0, b)$ and $V_{\alpha,\Omega}$ is defined by (7.1). Also, by Lemma 7.3,

$$\|f\|_{L^\infty(\Omega_b)} \le \|f\|_{L^\infty(\Omega_T)} < \infty \quad \text{for } t_0 < b < T.$$

It follows therefore from (4.6) and Lemma 7.1 that for $t_0 < b < T$ we have

$$0 < K^{-1} \le \frac{\|V_{\alpha,\Omega_b} f\|^\lambda_{L^\infty(\Omega_b)}}{\|f\|_{L^\infty(\Omega_b)}} \le \left(\frac{(b-t_0)^\alpha}{\Gamma(\alpha+1)}\right)^\lambda \|f\|^{\lambda-1}_{L^\infty(\Omega_b)} \to 0 \quad \text{as } b \to t_0^+$$

because $\lambda \ge 1$. This contradiction proves Theorem 4.1. □

Proof of Theorem 4.2. By Remark 8.2 with $T = 1$ we can assume $K = 1$. For $b > 0$ we have by Lemma 7.3 that

$$f \in L^\infty(\mathbb{R}^n \times \mathbb{R}_b)$$

and by (4.6), (4.7), Remark 7.1 with $a = 0$, and Lemma 7.1 that

$$\|f\|_{L^\infty(\Omega_b)} \le \|J_\alpha f\|^\lambda_{L^\infty(\Omega_b)} \le \left(\frac{b^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^\infty(\Omega_b)}\right)^\lambda$$

where $\Omega_b = \mathbb{R}^n \times (0,b)$. Thus, since $0 < \lambda < 1$, we see that

$$\|f\|_{L^\infty(\Omega_b)} \le \left(\frac{b^\alpha}{\Gamma(\alpha+1)}\right)^{\frac{\lambda}{1-\lambda}} \quad \text{for all } b > 0. \tag{8.2}$$

Define $\{\gamma_j\} \subset (0,\infty)$ by $\gamma_1 = 1$ and

$$\gamma_{j+1} = (\bar{M}\gamma_j)^\lambda, \ j = 1,2,..., \quad \text{where } \bar{M} = \Gamma(\alpha+1)M. \tag{8.3}$$

Then, since $0 < \lambda < 1$, we see that

$$\gamma_j \to \bar{M}^{\frac{\lambda}{1-\lambda}} \quad \text{as } j \to \infty. \tag{8.4}$$

Suppose for some positive integer j that

$$\|f\|_{L^\infty(\Omega_b)} \le \gamma_j \left(\frac{b^\alpha}{\Gamma(\alpha+1)} \right)^{\frac{\lambda}{1-\lambda}} \quad \text{for all } b > 0. \tag{8.5}$$

Then for $b > 0$ and $(x,t) \in \Omega_b$ we find from (4.6) and (5.4) that

$$\begin{aligned}
f(x,t) &\le (J_\alpha f(x,t))^\lambda \\
&\le \left(\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{\xi\in\mathbb{R}^n} \Phi_1(x-\xi, t-\tau)\, d\xi \right) \|f\|_{L^\infty(\Omega_\tau)} d\tau \right)^\lambda \\
&\le \left(\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \gamma_j \left(\frac{\tau^\alpha}{\Gamma(\alpha+1)} \right)^{\frac{\lambda}{1-\lambda}} d\tau \right)^\lambda \\
&= \left(\gamma_j \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)^{\frac{\lambda}{1-\lambda}}} \int_0^t (t-\tau)^{\alpha-1} \tau^{\frac{\alpha\lambda}{1-\lambda}} d\tau \right)^\lambda \\
&= \left(\gamma_j \frac{\Gamma(\alpha)\Gamma(\frac{\alpha\lambda}{1-\lambda}+1) t^{\alpha+\frac{\alpha\lambda}{1-\lambda}}}{\Gamma(\alpha)\Gamma(\alpha+1)^{\frac{\lambda}{1-\lambda}} \Gamma(\alpha+\frac{\alpha\lambda}{1-\lambda}+1)} \right)^\lambda \\
&= \left(\gamma_j \frac{M t^{\frac{\alpha}{1-\lambda}}}{\Gamma(\alpha+1)^{\frac{\lambda}{1-\lambda}}} \right)^\lambda = \left(\gamma_j \frac{\bar{M} t^{\frac{\alpha}{1-\lambda}}}{\Gamma(\alpha+1)^{\frac{1}{1-\lambda}}} \right)^\lambda \\
&= \gamma_{j+1} \left(\frac{t^\alpha}{\Gamma(\alpha+1)} \right)^{\frac{\lambda}{1-\lambda}}.
\end{aligned} \tag{8.6}$$

Thus

$$\|f\|_{L^\infty(\Omega_b)} \le \gamma_{j+1} \left(\frac{b^\alpha}{\Gamma(\alpha+1)} \right)^{\frac{\lambda}{1-\lambda}} \quad \text{for all } b > 0.$$

Hence (4.10) follows inductively from (8.2)–(8.5).

Finally, repeating the calculation (8.6) with $\gamma_j = \gamma_{j+1} = \bar{M}^{\frac{\lambda}{1-\lambda}}$ we get

$$(J_\alpha f(x,t))^\lambda \le \bar{M}^{\frac{\lambda}{1-\lambda}} \left(\frac{t^\alpha}{\Gamma(\alpha+1)} \right)^{\frac{\lambda}{1-\lambda}} \quad \text{for } (x,t) \in \Omega_b$$

which proves (4.11). □

Proof of Theorem 4.3. By Remark 8.2 we can assume $K = T = 1$. For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $\delta \in (0,1)$ let

$$g_\delta(x,t) = g_\delta(t) = \psi_\delta(t) g(t) \tag{8.7}$$

where g is as in Remark 8.1 and $\psi_\delta \in C^\infty(\mathbb{R} \to [0,1])$ satisfies

$$\psi_\delta(t) = \begin{cases} 1 & \text{if } t \le 1 \\ 0 & \text{if } t \ge 1+\delta. \end{cases}$$

Then for $1 \le t \le 1+\delta$

$$\begin{aligned} J_\alpha g(t) - J_\alpha g_\delta(t) &= \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau)(1-\psi_\delta(\tau))\, d\tau \\ &\le \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau)\, d\tau \le g(1+\delta) \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\, d\tau \\ &= g(1+\delta)\frac{(t-1)^\alpha}{\Gamma(\alpha+1)} \le g(2)\frac{\delta^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

and thus by (8.1) we have for $1 \le t \le 1+\delta$ that

$$\begin{aligned} \frac{J_\alpha g_\delta(t)}{J_\alpha g(t)} &= \frac{J_\alpha g(t) - (J_\alpha g(t) - J_\alpha g_\delta(t))}{g(t)^{1/\lambda}} \\ &\ge 1 - \frac{g(2)\delta^\alpha}{\Gamma(\alpha+1)g(1)^{1/\lambda}} \\ &= 1 - C(\alpha,\lambda)\delta^\alpha \ge \sqrt{\frac{N}{M}} \end{aligned}$$

provided we choose $\delta = \delta(\alpha,\lambda,N) \in (0,1)$ sufficiently small. Hence for $1 \le t \le 1+\delta$ we see from (8.1) that

$$g_\delta(t) \le g(t) = (J_\alpha g(t))^\lambda \le \left(\frac{M}{N}\right)^{\lambda/2} (J_\alpha g_\delta(t))^\lambda \tag{8.8}$$

which by (8.7) and (8.1) holds for all other t as well.

Next let $\varphi(x) = e^{-\psi(x)}$ where $\psi(x) = \sqrt{1+|x|^2} - 1$. Then for $\varepsilon \in (0,1)$, $\gamma > 1$, and $|\xi - x| < \gamma\sqrt{2}$ we have

$$\frac{\varphi(\varepsilon\xi)}{\varphi(\varepsilon x)} = e^{-(\psi(\varepsilon\xi)-\psi(\varepsilon x))} \ge e^{-\varepsilon|\xi-x|} \ge e^{-\varepsilon\gamma\sqrt{2}}.$$

Thus defining $f_\varepsilon : \mathbb{R}^n \times \mathbb{R} \to [0,\infty)$ by

$$f_\varepsilon(x,t) = \varphi(\varepsilon x)\left(\frac{N}{M}\right)^{\frac{\lambda}{1-\lambda}} g_\delta(t)$$

we find for $|\xi - x| < \gamma\sqrt{2}$ and $\tau \in \mathbb{R}$ that

$$f_\varepsilon(\xi,\tau) \ge \varphi(\varepsilon x)e^{-\varepsilon\gamma\sqrt{2}}\left(\frac{N}{M}\right)^{\frac{\lambda}{1-\lambda}} g_\delta(\tau).$$

Thus for $(x,t) \in \mathbb{R}^n \times (0,2)$ we have

$$J_\alpha f_\varepsilon(x,t) \ge \varphi(\varepsilon x)e^{-\varepsilon\gamma\sqrt{2}}\left(\frac{N}{M}\right)^{\frac{\lambda}{1-\lambda}} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g_\delta(\tau) \int_{|\xi-x|<\gamma\sqrt{2}} \Phi_1(x-\xi,t-\tau)\, d\xi\, d\tau. \tag{8.9}$$

But for $x,\xi \in \mathbb{R}^n$ and $0 < \tau < t < 2$ we find making the change of variables $z = \frac{x-\xi}{\sqrt{4(t-\tau)}}$ that

$$\begin{aligned} \int_{|\xi-x|<\gamma\sqrt{2}} \Phi_1(x-\xi,t-\tau)\, d\xi &\ge \int_{|\xi-x|<\gamma\sqrt{t-\tau}} \frac{1}{(4\pi(t-\tau))^{n/2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} d\xi \\ &= \frac{1}{\pi^{n/2}} \int_{|z|<\gamma/2} e^{-|z|^2} dz =: I(\gamma) \to 1 \end{aligned}$$

as $\gamma \to \infty$. Thus by (8.9) and (8.8) we have for $(x,t) \in \mathbb{R}^n \times (0, 1+\delta)$ that

$$\frac{(J_\alpha f_\varepsilon(x,t))^\lambda}{f_\varepsilon(x,t)} \geq \frac{\varphi(\varepsilon x)^\lambda e^{-\varepsilon\gamma\lambda\sqrt{2}} (\frac{N}{M})^{\frac{\lambda^2}{1-\lambda}} I(\gamma)^\lambda (J_\alpha g_\delta(t))^\lambda}{\varphi(\varepsilon x)(\frac{N}{M})^{\frac{\lambda}{1-\lambda}} g_\delta(t)}$$
$$\geq \left(\frac{M}{N}\right)^{\lambda/2} I(\gamma)^\lambda e^{-\varepsilon\gamma\lambda\sqrt{2}}. \tag{8.10}$$

So first choosing γ so large that $(\frac{M}{N})^{\lambda/2} I(\gamma)^\lambda > 1$ and then choosing $\varepsilon > 0$ so small that (8.10) is greater than 1 we see that $f := f_\varepsilon$ satisfies(4.6) in $\mathbb{R}^n \times (0, 1+\delta)$. Thus, since $g_\delta(t)$ and hence $f(x,t)$ is identically zero in $\mathbb{R}^n \times ((-\infty, 0] \cup [1+\delta, \infty))$ see that f satisfies (4.6), (4.7).

From the exponential decay of $\varphi(x)$ as $|x| \to \infty$, we see that f satisfies (4.13). Also since f is uniformly continuous and bounded on $\mathbb{R}^n \times \mathbb{R}$ and

$$\int_a^b \int_{\mathbb{R}^n} \Phi_\alpha(x,t)\, dx\, dt = \frac{1}{\Gamma(\alpha+1)}(b^\alpha - a^\alpha) \quad \text{for } a < b,$$

we easily check that (4.14) holds.

Finally, since

$$f(0,t) = \left(\frac{N}{M}\right)^{\frac{\lambda}{1-\lambda}} g(t) \quad \text{for } 0 \leq t \leq 1$$

we find that (4.15) holds and thus (4.16) follows from (4.6). □

Proof of Theorem 4.4. By Remark 8.2 with $T = 1$ we can assume $K = 1$. Define $\bar{f} : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ by

$$\bar{f}(x,t) = g(t)\chi_{\{|x|^2 < t\}}(x,t) \tag{8.11}$$

where g is defined in Remark 8.1. Then for $(x,t) \in \mathbb{R}^n \times (0, \infty)$ we have

$$J_\alpha \bar{f}(x,t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{|\xi|^2 < \tau} \Phi_1(x-\xi, t-\tau)\, d\xi \right) g(\tau)\, d\tau.$$

Thus by Lemma 7.4 we see for $|x|^2 < t$ that

$$J_\alpha \bar{f}(x,t) \geq C(n,\alpha,\lambda) \int_{t/4}^{3t/4} (t-\tau)^{\alpha-1} \tau^{\frac{\alpha\lambda}{1-\lambda}}\, d\tau$$
$$= C(n,\alpha,\lambda) t^{\frac{\alpha}{1-\lambda}}$$
$$= C(n,\alpha,\lambda) g(x,t)^{1/\lambda}$$
$$= C(n,\alpha,\lambda) \bar{f}(x,t)^{1/\lambda} \tag{8.12}$$

which also holds in $(\mathbb{R}^n \times \mathbb{R}) \backslash \{|x|^2 \leq t\}$ because $\bar{f} = 0$ there. Thus letting $f = L\bar{f}$ where

$$L = C^{\frac{\lambda}{1-\lambda}}$$

where $C = C(n,\alpha,\lambda)$ is as in (8.12) we find that f satisfies (4.5)–(4.7).

It follows from (8.11) and the definitions of g and f that there exists $N > 0$ such that (4.17) holds. Thus, since f solves (4.6) we obtain (4.18). □

Proof of Theorem 4.5. Since $|R_j| < \infty$, to prove Theorem 4.5 it suffices to show for each $\varepsilon \in (0,1)$ that the conclusion of Theorem 4.5 holds for some

$$q \in (p, p+\varepsilon). \tag{8.13}$$

So let $\varepsilon \in (0,1)$. By $(4.19)_1$, there exists q satisfying (8.13) such that

$$\alpha < \frac{n+2}{2q}\left(1 - \frac{1}{\lambda}\right). \tag{8.14}$$

Define $f_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$f_0(x,t) = \left(\frac{1}{t}\right)^r \chi_{\Omega_0}(x,t) \tag{8.15}$$

where

$$\Omega_0 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t < 1\}$$

and

$$r := \frac{n+2}{2q} < \frac{n+2}{2p} \tag{8.16}$$

by (8.13). Then by (8.16) and Lemma 7.6 we have

$$f_0 \in L^p(\mathbb{R}^n \times \mathbb{R}) \tag{8.17}$$

and

$$J_\alpha f_0(x,t) \ge C\left(\frac{1}{t}\right)^{r-\alpha} \quad \text{for } (x,t) \in \Omega_0 \tag{8.18}$$

where, throughout this entire proof, $C = C(n, \lambda, \alpha, p, q)$ is a positive constant whose value may change from line to line.

Let $\{T_j\} \subset (0, 1/2)$ be a sequence such that

$$T_{j+1} < T_j/4 \qquad j = 1, 2, ...$$

and define

$$t_j = T_j/2. \tag{8.19}$$

Then

$$\Omega_j := \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y| < \sqrt{T_j - s} \text{ and } t_j < s < T_j\} \subset R_j \subset \Omega_0 \tag{8.20}$$

and thus defining $f_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$f_j(x,t) = (T_j - t)^{-r} \chi_{\Omega_j}(x,t) \tag{8.21}$$

we obtain from (8.16) and Lemma 7.7 that

$$\|f_j\|^p_{L^p(\mathbb{R}^n \times \mathbb{R})} = C(n) \int_0^{T_j - t_j} s^{(\frac{n+2}{2p} - r)p - 1} ds \to 0 \quad \text{as } j \to \infty, \tag{8.22}$$

$$\|f_j\|_{L^q(R_j)} = \|f_j\|_{L^q(\mathbb{R}^n \times \mathbb{R})} = \infty \quad \text{for } j = 1, 2, ..., \tag{8.23}$$

and

$$J_\alpha f_j(x,t) \ge C\left(\frac{1}{(T_j - t)}\right)^{r-\alpha} \quad \text{for } (x,t) \in \Omega_j^+ \tag{8.24}$$

where

$$\Omega_j^+ = \{(x,t) \in \Omega_j : \frac{3T_j}{4} < t < T_j\}.$$

It follows from (8.15) and (8.18) that

$$\frac{f_0(x,t)}{(J_\alpha f_0(x,t))^\lambda} \le Ct^{(r-\alpha)\lambda - r} \quad \text{for } (x,t) \in \Omega_0$$

and from (8.14) and (8.16) that the exponent

$$(r-\alpha)\lambda - r = \lambda[r(1-1/\lambda) - \alpha] > 0. \tag{8.25}$$

Thus

$$\sup_{\Omega_0} \frac{f_0}{(J_\alpha f_0)^\lambda} \le C \tag{8.26}$$

and by (8.20)

$$\sup_{\Omega_j} \frac{f_0}{(J_\alpha f_0)^\lambda} \le CT_j^{(r-\alpha)\lambda - r} < 1 \tag{8.27}$$

by taking a subsequence.

By (8.21), (8.24), and (8.25) we have

$$\begin{aligned} \sup_{\Omega_j^+} \frac{f_j}{(J_\alpha f_j)^\lambda} &\le C \sup_{(x,t)\in\Omega_j^+} (T_j - t)^{(r-\alpha)\lambda - r} \\ &\le C(T_j - t_j)^{(r-\alpha)\lambda - r} < 1 \end{aligned} \tag{8.28}$$

by taking a subsequence.

It follows from (8.15), (8.21), (8.20), and (8.19) that

$$\sup_{\Omega_j} \frac{f_0}{f_j} = \sup_{(x,t)\in\Omega_j} \frac{(T_j - t)^r}{t^r} \le \frac{(T_j - t_j)^r}{t_j^r} = 1 \tag{8.29}$$

and letting $\Omega_j^- = \Omega_j \backslash \Omega_j^+$ we see from (8.21), (8.18), (8.20), and (8.25) that

$$\begin{aligned} \sup_{\Omega_j^-} \frac{f_j}{(J_\alpha f_0)^\lambda} &\le C \sup_{(x,t)\in\Omega_j^-} \frac{t^{(r-\alpha)\lambda}}{(T_j - t)^r} \le C\frac{T_j^{(r-\alpha)\lambda}}{(T_j/4)^r} \\ &= CT_j^{(r-\alpha)\lambda - r} < \frac{1}{2} \end{aligned} \tag{8.30}$$

by taking a subsequence.

Taking an appropriate subsequence of f_j and letting

$$f = f_0 + \sum_{j=1}^{\infty} f_j$$

we find from (8.17) and (8.22) that f satsfies (4.20).

In Ω_j^+ we have by (8.27) and (8.28) that

$$\begin{aligned} f = f_0 + f_j &\le (J_\alpha f_0)^\lambda + (J_\alpha f_j)^\lambda \\ &\le (J_\alpha(f_0 + f_j))^\lambda \le (J_\alpha f)^\lambda. \end{aligned}$$

In Ω_j^- we have by (8.29) and (8.30) that

$$f = f_0 + f_j \le 2f_j \le (J_\alpha f_0)^\lambda \le (J_\alpha f)^\lambda.$$

In $\Omega_0 \backslash \cup_{j=1}^\infty \Omega_j$ we have by (8.26) that

$$f = f_0 \le C(J_\alpha f_0)^\lambda \le C(J_\alpha f)^\lambda.$$

In $(\mathbb{R}^n \times \mathbb{R})\backslash\Omega_0$, $f = 0 \le (J_\alpha f)^\lambda$. Thus, after scaling f, we see that f is a solution of (4.6), (4.7). Also (4.21) holds by (8.23). □

Proof of Theorem 4.6. By $(4.23)_1$, there exists a unique number $\gamma \in (0, \frac{n+2}{2p} - \alpha)$ such that

$$\lambda = \frac{\frac{n+2}{2p} - \gamma}{\frac{n+2}{2p} - \alpha - \gamma}. \tag{8.31}$$

Let f_0 and Ω_0 be as in Lemma 7.6. Then by (8.31) and Lemma 7.6 we have

$$f_0 \in X^p \tag{8.32}$$

and

$$f_0 \le C(J_\alpha f_0)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{8.33}$$

where in this proof $C = C(n, \lambda, \alpha, p)$ is a positive constant whose value may change from line to line. Let $\{T_j\}$, $\{t_j\} \subset (2, \infty)$ satisfy

$$T_{j+1} \ge 4T_j \quad \text{and} \quad T_j = 2t_j$$

and define $f_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$f_j(x,t) = \left(\frac{1}{T_j - t}\right)^{\frac{n+2}{2p} - \gamma} \chi_{\Omega_j}(x,t) \tag{8.34}$$

where

$$\Omega_j := \{(x,t) \in \mathbb{R}^n \times (T_j/2, T_j) : |x| < \sqrt{T_j - t}\}.$$

Then

$$\Omega_j \subset R_j \subset \Omega_0, \quad \Omega_j \cap \Omega_k = \emptyset \quad \text{for } j \ne k, \tag{8.35}$$

$$\inf \{t : (x,t) \in \Omega_j\} = T_j/2 \to \infty \quad \text{as } j \to \infty, \tag{8.36}$$

and by (8.34), (8.31), and Lemma 7.7 we have

$$f_j \in L^p(\mathbb{R}^n \times \mathbb{R}) \tag{8.37}$$

and

$$f_j \le C(J_\alpha f_j)^\lambda \quad \text{in } \Omega_j^+$$

where

$$\Omega_j^+ = \{(x,t) \in \Omega_j : \frac{3T_j}{4} < t < T_j\}.$$

It follows therefore from (8.33) that

$$f_0 + f_j \le C((J_\alpha f_0)^\lambda + (J_\alpha f_j)^\lambda) \le C(J_\alpha(f_0 + f_j))^\lambda \quad \text{in } \Omega_j^+. \tag{8.38}$$

In $\Omega_j^- := \Omega_j \backslash \Omega_j^+$ we have

$$\frac{f_j}{f_0} = \left(\frac{t}{(T_j - t)} \right)^{\frac{n+2}{2p} - \gamma} \leq \left(\frac{\frac{3}{4} T_j}{\frac{1}{4} T_j} \right)^{\frac{n+2}{2p} - \gamma} = 3^{\frac{n+2}{2p} - \gamma}$$

and thus we obtain from (8.33) that

$$f_0 + f_j \leq C f_0 \leq C (J_\alpha f_0)^\lambda \leq C (J_0 (f_0 + f_j))^\lambda \quad \text{in } \Omega_j^-. \tag{8.39}$$

Let $f = f_0 + \sum_{j=1}^\infty f_j$. Then clearly f satisfies (4.7) and by (8.32), (8.37), and (8.36) we see that f satisfies (4.24).

In Ω_j we have by $(8.35)_2$, (8.38), and (8.39) that

$$f = f_0 + f_j \leq C (J_\alpha (f_0 + f_j))^\lambda \leq C (J_\alpha f)^\lambda$$

and in $(\mathbb{R}^n \times \mathbb{R}) \backslash \cup_{j=1}^\infty \Omega_j$ we have by (8.33) that

$$f = f_0 \leq C (J_\alpha f_0)^\lambda \leq C (J_\alpha f)^\lambda.$$

Thus after scaling f, we find that f satisfies (4.6).

Since $|R_j| < \infty$, we can for the proof of (4.25) assume instead of $(4.23)_2$ that

$$q = \frac{n+2}{2\alpha} (1 - \frac{1}{\lambda})$$

and hence by (8.31) we get

$$\frac{n+2}{2p} - \gamma = \frac{\alpha}{1 - \frac{1}{\lambda}} = \frac{n+2}{2q}.$$

Consequently from $(8.35)_1$, (8.34), and Lemma 7.7 we find that

$$\|f\|_{L^q(R_j)} \geq \|f_j\|_{L^q(\Omega_j)} = \infty \quad \text{for } j = 1, 2, ...$$

which proves (4.25) □

A Appendix

For the proof of Theorem 2.3(ii) we will need the following result due to Nogin and Rubin [19] concerning the inversion of the operator J_α in the framework of the spaces $L^p(\mathbb{R}^n \times \mathbb{R})$. See also [24, Theorem 9.24].

Theorem A.1. *Suppose* $0 < \alpha < \frac{n+2}{2p}$, $1 < p < \infty$, *and* $u = J_\alpha f$ *with* $f \in L^p(\mathbb{R}^n \times \mathbb{R})$. *Then*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon^{-\alpha} u = f \quad \textit{in } L^p(\mathbb{R}^n \times \mathbb{R})$$

where

$$J_\varepsilon^{-\alpha} u(x,t) = C(n, \alpha, l) \iint_{\mathbb{R}^n \times (\varepsilon, \infty)} \frac{(\Delta_{y,\tau}^l u)(x,t)}{\tau^{1+\alpha}} e^{-\frac{|y|^2}{4}} \, dy \, d\tau \tag{A.1}$$

and

$$(\Delta_{y,\tau}^l u)(x,t) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} u(x - y\sqrt{k\tau}, t - k\tau), \quad l > \alpha. \tag{A.2}$$

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