Singular semilinear elliptic inequalities in the exterior of a compact set

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We study the semilinear elliptic inequality $-\Delta u \geq \varphi(\delta_K(x))f(u)$ in $\mathbb{R}^N \setminus K$, where φ, f are positive and nonincreasing continuous functions. Here $K \subset \mathbb{R}^N$ $(N \geq 3)$ is a compact set with finitely many components each of which is either the closure of a C^2 domain or an isolated point and $\delta_K(x) = \mathrm{dist}(x, \partial K)$. We obtain optimal conditions in terms of φ and f for the existence of C^2 positive solutions. Under these conditions we prove the existence of a minimal solution and we investigate its behavior around ∂K as well as the removability of the (possible) isolated singularities.

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1. Introduction

In this paper we study the existence and non-existence of C^2 positive solutions $u(x)$ of the following semilinear elliptic inequality

$$-\Delta u \geq \varphi(\delta_K(x))f(u) \quad \text{in } \mathbb{R}^N \setminus K, \tag{1.1}$$

where K is a compact set in $\mathbb{R}^N$ $(N \geq 3)$ and $\delta_K(x) :=\mathrm{dist}(x, \partial K)$. We assume that K has finitely many connected components each of which is either the closure of a C^2 domain or a singleton. We shall write $K = K_1 \cup K_2$ where K_1 is the union of all components of K which are the closure of a C^2 domain and K_2 is the set of all isolated points of K.

We also assume that

$(A1)$ $\quad f \in C^1(0,\infty)$ is a positive and decreasing function;

$(A2)$ $\quad \varphi \in C^{0,\gamma}(0,\infty)$ $(0 < \gamma < 1)$ is a positive and nonincreasing.

Elliptic equations or inequalities in unbounded domains have been subject to extensive study recently (see, e.g., [6, 7, 11, 13, 14, 16, 18, 19] and the references therein). In [6, 7] the authors are concerned with elliptic problems with superlinear nonlinearities f in exterior domains. Large classes of elliptic inequalities in exterior or cone-like domains involving various types of differential operators are considered in [13, 14, 16, 18, 19]. In [20, 21, 22, 23, 24] elliptic inequalities are studied in a

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punctured neighborhood of the origin and asymptotic radial symmetry of solutions is investigated.

The main novelty of the present paper is the presence of the distance function $\delta_K(x)$ to the boundary of the compact set K which, as we shall see, will play a significant role in the qualitative study of (1.1). Whenever (1.1) has solutions we show that it has a minimal solution $\tilde{u}$ and we are interested in further properties of $\tilde{u}$ such as removability of possible singularities at isolated points of K_2 as well as boundary behavior around K_1.

In our approach to (1.1) we shall distinguish between the case where K is *non-degenerate*, that is, $K_1 \neq \emptyset$, and the case where K is *degenerate*, that is $K_1 = \emptyset$, which means K reduces to a finite set of points.

We start first with the non-degenerate case $K_1 \neq \emptyset$. Our first result in this sense is the following:

Theorem 1.1. *Assume (A1), (A2) and $K_1 \neq \emptyset$. Then, inequality (1.1) has C^2 positive solutions if and only if*

$$\int_0^\infty r\varphi(r)dr < \infty. \tag{1.2}$$

If (1.2) holds, then we prove that (1.1) has a minimal C^2 positive solution $\tilde{u}$ (in the sense of the usual order relation) which achieves the equality in (1.1) and $\tilde{u}$ is a *ground-state* of (1.1) in the sense that $\tilde{u}(x) \to 0$ as $|x| \to \infty$. Furthermore, we prove that all (possible) singularities of $\tilde{u}$ at isolated points in K_2 are removable and that $\tilde{u}$ can be continuously extended by zero at ∂K_1. We also determine the rate at which $\tilde{u}$ vanishes around the boundary of K_1. All these results are precisely described in the following theorem.

Theorem 1.2. *Assume (A1), (A2), $K_1 \neq \emptyset$ and condition (1.2) is satisfied. Then there exists a minimal solution $\tilde{u}$ of (1.1) that satisfies*

$$\tilde{u} \in C^2(\mathbb{R}^N \setminus K) \cap C(\mathbb{R}^N \setminus int(K_1))$$

and

$$\begin{cases} -\Delta \tilde{u} = \varphi(\delta_K(x)) f(\tilde{u}),\ \tilde{u} > 0 & \text{in } \mathbb{R}^N \setminus K, \\ \tilde{u} = 0 & \text{on } \partial K_1, \\ \tilde{u} > 0 & \text{on } K_2, \\ \tilde{u}(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.3}$$

In addition, there exist positive constants c_1, c_2, and r_0 such that $\tilde{u}$ satisfies

$$c_1 \le \frac{\tilde{u}(x)}{H(\delta_{K_1}(x))} \le c_2 \quad \text{in } \{x \in \mathbb{R}^N \setminus K : 0 < \delta_{K_1}(x) < r_0\}, \tag{1.4}$$

where $H : [0,1] \to [0,\infty)$ is the unique solution of

$$\begin{cases} -H''(t) = \varphi(t) f(H(t)),\ H(t) > 0 & 0 < t < 1, \\ H(0) = H(1) = 0. \end{cases} \tag{1.5}$$

The existence of a solution to (1.5) follows from [1, Theorem 2.1]. By Theorem 1.1 we have that condition (1.2) is both necessary and sufficient for the existence of a solution to (1.1). If that is the case, the minimal solution $\tilde{u}$ of (1.1) can be continuously extended to ∂K, so that all isolated singularities of $\tilde{u}$ at K_2 are removable. If $\varphi(r) = r^\alpha$ and $f(u) = u^{-p}$, $p > 0$, the behavior of H in (1.5) was studied in [8, Theorem 3.5]. In this case we have:

Corollary 1.3. *Assume (A2), $K_1 \neq \emptyset$, $f(u) = u^{-p}, p > 0$, and*

$$\varphi(r) \sim r^\alpha \quad \text{as } r \to 0 \quad \text{and} \quad \varphi(r) \sim r^\beta \quad \text{as } r \to \infty\,,$$

for some $\alpha, \beta < 0$. Then (1.1) has solutions if and only if $0 > \alpha > -2 > \beta$. In this case (1.1) has a minimal solution $\tilde{u}$ which satisfies (1.3) and there exist positive constants c_1, c_2, and r_0 such that $\tilde{u}$ satisfies (1.4) where

$$H(t) = \begin{cases} t & \text{if } p - \alpha < 1, \\ t\left(\log \dfrac{1}{t}\right)^{\frac{1}{2+\alpha}} & \text{if } p - \alpha = 1, \\ t^{\frac{2+\alpha}{1+p}} & \text{if } p - \alpha > 1. \end{cases}$$

We are next concerned with the degenerate case $K_1 = \emptyset$. In this setting the existence of a solution to (1.1) depends on both φ and f. Our result in this case is:

Theorem 1.4. *Assume (A1), $K_1 = \emptyset$ and that $\varphi \in C^{0,\gamma}(0,\infty)$ $(0 < \gamma < 1)$ is a positive function which is nonincreasing in a neighborhood of zero and of infinity. Then, (1.1) has solutions if and only if*

$$\int_1^\infty r\varphi(r)dr < \infty \tag{1.6}$$

and there exists $a > 0$ such that

$$\int_0^1 r^{N-1}\varphi(r)f(ar^{2-N})dr < \infty. \tag{1.7}$$

Furthermore, if (1.6)-(1.7) hold, then (1.1) has a minimal solution $\tilde{u}$ which satisfies

$$\begin{cases} -\Delta\tilde{u} = \varphi(\delta_K(x))f(\tilde{u}),\ \tilde{u} > 0 & \text{in } \mathbb{R}^N \setminus K, \\ \tilde{u}(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.8}$$

In addition, $\tilde{u}$ has removable singularities at K if and only if $\int_0^1 r\varphi(r)dr < \infty$.

From Theorem 1.1 and Theorem 1.4 we have the following result regarding the inequality

$$-\Delta u \geq \delta_K^\alpha(x)u^{-p} \quad \text{in } \mathbb{R}^N \setminus K, \quad \alpha < 0 < p. \tag{1.9}$$

Corollary 1.5. *Let $K = K_1 \cup K_2$ be as in the statement of Theorem 1.1.*

(i) If K_1 is nonempty, then (1.9) has no positive C^2 solutions;

(ii) If $K_1 = \emptyset$ then (1.9) has solutions if and only if

$$N + \alpha + p(N-2) > 0 \quad \text{and} \quad \alpha < -2, \tag{1.10}$$

and all solutions of (1.9) are singular at points of K_2.

Finally, we consider the special case $K = \{0\}$ and describe the solution set of

$$-\Delta u = \varphi(|x|)f(u) \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1.11}$$

For a large class of functions φ, we show that any C^2 positive solution of (1.11) (if exists) is radially symmetric. Furthermore, the solution set of (1.11) consists of a two-parameter family of radially symmetric functions.

Theorem 1.6. *Suppose that f and φ are as in Theorem 1.4 and that φ satisfies (1.6)-(1.7) for all $a > 0$. Then :*

(i) for any $a, b \geq 0$ there exists a radially symmetric positive solution $u_{a,b}$ of (1.11) such that

$$\lim_{|x|\to 0} |x|^{N-2} u_{a,b}(x) = a \quad \text{and} \quad \lim_{|x|\to\infty} u_{a,b}(x) = b. \tag{1.12}$$

(ii) the set of positive solutions of equation (1.11) consists only of $\{u_{a,b} : a, b \geq 0\}$. In particular, any C^2 positive solution of (1.11) is radially symmetric.

We point out that if $N = 2$ then (1.11) has no C^2 positive solutions. More precisely, if $u \in C^2(\mathbb{R}^2 \setminus \{0\})$ satisfies $-\Delta u \geq 0$, $u \geq 0$ in $\mathbb{R}^2 \setminus \{0\}$, then u is constant (see [17, Theorem 29, page 130]). A direct consequence of Theorem 1.6 is the following:

Corollary 1.7. *Let $\alpha \in \mathbb{R}$, $p > 0$. Then, the equation*

$$-\Delta u = |x|^\alpha u^{-p} \quad \text{in } \mathbb{R}^N \setminus \{0\},\ N \geq 3, \tag{1.13}$$

has positive solutions if and only if (1.10) holds. In this case, we have the same conclusion as in Theorem 1.6 and the function

$$\xi(x) := \left[\frac{-(1+p)^2}{(\alpha+2)(p(N-2)+N+\alpha)}\right]^{1/(1+p)} |x|^{(2+\alpha)/(1+p)}, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.14}$$

is the minimal solution of (1.13).

Using Theorem 1.6 we also obtain:

Corollary 1.8. *Let $\alpha \in \mathbb{R}$, $\beta, p > 0$. Then, the equation*

$$-\Delta u = |x|^\alpha \log^\beta(1+|x|) u^{-p} \quad \text{in } \mathbb{R}^N \setminus \{0\},\ N \geq 3, \tag{1.15}$$

has solutions if and only if

$$N + \alpha + \beta + p(N-2) > 0 \quad \text{and} \quad \alpha < -2. \tag{1.16}$$

Furthermore, if (1.16) holds, then:

(i) the set of positive solutions of (1.15) consists of a two-parameter family of radially symmetric functions as described in Theorem 1.6;

(ii) the minimal solution of (1.15) has a removable singularity at the origin if and only if $\alpha+\beta>-2$.

The outline of the paper is as follows. In the next section we collect some preliminary results concerning elliptic boundary value problems in bounded domains involving the distance function up to the boundary. The last four sections of the paper are devoted to the proofs of Theorems 1.1, 1.2, 1.4 and 1.6 respectively.

2. Preliminary results

In this part we obtain some results for related elliptic problems in bounded domains that will be further used in the sequel. We start with the following comparison result.

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ be a nonempty open set and $g:\Omega\times(0,\infty)\to(0,\infty)$ be a continuous function such that $g(x,\cdot)$ is decreasing for all $x\in\Omega$. Assume that u,v are C^2 positive functions that satisfy*

$$\Delta u+g(x,u)\leq 0\leq \Delta v+g(x,v) \qquad \text{in } \Omega,$$

$$\lim_{x\in\Omega,\, x\to y}(v(x)-u(x))\leq 0 \qquad \text{for all } y\in\partial^\infty\Omega.$$

Then $u\geq v$ in Ω. (Here $\partial^\infty\Omega$ stands for the Euclidean boundary $\partial\Omega$ if Ω is bounded and for $\partial\Omega\cup\{\infty\}$ if Ω is unbounded)

Proof. Assume by contradiction that the set $\omega:=\{x\in\Omega: u(x)<v(x)\}$ is not empty and let $w:=v-u$. Since $\lim_{x\in\Omega,\, x\to y}w(x)\leq 0$ for all $y\in\partial^\infty\Omega$, it follows that w is bounded from above and it achieves its maximum on Ω at a point that belongs to ω. At that point, say x_0, we have

$$0\leq -\Delta w(x_0)\leq g(x_0,v(x_0))-g(x_0,u(x_0))<0,$$

which is a contradiction. Therefore, $\omega=\emptyset$, that is, $u\geq v$ in Ω. □

Lemma 2.2. *Let $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ be a bounded domain with C^2 boundary and let $g:\overline{\Omega}\times(0,\infty)\to(0,\infty)$ be a Hölder continuous function such that for all $x\in\overline{\Omega}$ we have $g(x,\cdot)\in C^1(0,\infty)$ and $g(x,\cdot)$ is decreasing. Then, for any $\phi\in C(\partial\Omega)$, $\phi\geq 0$, the problem*

$$\begin{cases} -\Delta u=g(x,u),\ u>0 & \text{in } \Omega,\\ \quad u=\phi(x) & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a unique solution $u\in C^2(\Omega)\cap C(\overline{\Omega})$.

Proof. For all $n\geq 1$ consider the following perturbed problem

$$\begin{cases} -\Delta u=g\Big(x,u+\dfrac{1}{n}\Big),\ u>0 & \text{in } \Omega,\\ \quad u=\phi(x) & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

It is easy to see that $\underline{u} \equiv 0$ is a sub-solution. To construct a super-solution, let w be the solution of

$$\begin{cases} -\Delta w = 1, \ w > 0 & \text{in } \Omega, \\ \ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\overline{u} = Mw + ||\phi||_\infty + 1$ is a super-solution of (2.2) provided $M > 1$ is large enough. Thus, by sub and super-solution method and Lemma 2.1, there exists a unique solution $u_n \in C^2(\Omega) \cap C(\overline{\Omega})$ of (2.2). Furthermore, since $g(x, \cdot)$ is decreasing, by Lemma 2.1 we deduce

$$u_1 \le u_2 \le \cdots \le u_n \le \cdots \le \overline{u} \quad \text{in } \Omega, \tag{2.3}$$

$$u_n + \frac{1}{n} \ge u_{n+1} + \frac{1}{n+1} \quad \text{in } \Omega. \tag{2.4}$$

Hence $\{u_n(x)\}$ is increasing and bounded for all $x \in \Omega$. Letting $u(x) := \lim_{n\to\infty} u_n(x)$, a standard bootstrap argument (see [5], [12]) implies $u_n \to u$ in $C^2_{loc}(\Omega)$ so that passing to the limit in (2.2) we deduce $-\Delta u = g(x, u)$ in Ω. From (2.3) and (2.4) we obtain $u_n + 1/n \ge u \ge u_n$ in Ω, for all $n \ge 1$. This yields $u \in C(\overline{\Omega})$ and $u = \phi(x)$ on $\partial\Omega$. Therefore $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of (2.1). The uniqueness follows from Lemma 2.1. □

Lemma 2.3 and Lemma 2.4 below extend the existence results obtained in [4, 8, 9].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded domain with C^2 boundary. Also let $\varphi \in C^{0,\gamma}(0, \infty)$ $(0 < \gamma < 1)$ and $f \in C^1(0, \infty)$ be positive functions such that:*

(i) f is decreasing;

(ii) φ is nonincreasing and $\int_0^1 r\phi(r)dr < \infty$.

Then, the problem

$$\begin{cases} -\Delta u = \varphi(\delta_\Omega(x)) f(u), \ u > 0 & \text{in } \Omega, \\ \ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Furthermore, there exist $c_1, c_2 > 0$ and $0 < r_0 < 1$ such that the unique solution u of (2.5) satisfies

$$c_1 \le \frac{u(x)}{H(\delta_\Omega(x))} \le c_2 \quad \text{in } \{x \in \Omega : 0 < \delta_\Omega(x) < r_0\}, \tag{2.6}$$

where $H : [0, 1] \to (0, \infty)$ is the unique solution of (1.5).

Proof. Let (λ_1, e_1) be the first eigenvalue and the first eigenfunction of $-\Delta$ in Ω subject to Dirichlet boundary condition. It is well known that e_1 has constant sign in Ω so that normalizing, we may assume that $e_1 > 0$ in Ω. Also, since Ω has a C^2 boundary, we have $\partial e_1/\partial\nu < 0$ on $\partial\Omega$ and

$$C_1 \delta_\Omega(x) \le e_1(x) \le C_2 \delta_\Omega(x) \quad \text{in } \Omega, \tag{2.7}$$

where ν is the outward unit normal at $\partial\Omega$ and C_1, C_2 are two positive constants. We claim that there exist $M > 1$ and $c > 0$ such that $\overline{u} = MH(ce_1)$ is a super-solution of (2.5). First, since the solution H of (1.5) is positive and concave, we can find $0 < a < 1$ such that $H' > 0$ on $(0, a]$. Let $c > 0$ be such that

$$ce_1(x) \le \min\{a, \delta_\Omega(x)\} \quad \text{in } \Omega.$$

Then

$$\begin{aligned} -\Delta\overline{u} &= -Mc^2H''(ce_1)|\nabla e_1|^2 + Mc\lambda_1 e_1 H'(ce_1) \\ &= Mc^2\varphi(ce_1)f(H(ce_1))|\nabla e_1|^2 + Mc\lambda_1 e_1 H'(ce_1) \\ &\ge Mc^2\varphi(\delta_\Omega(x))f(\overline{u})|\nabla e_1|^2 + Mc\lambda_1 e_1 H'(ce_1) \quad \text{in } \Omega. \end{aligned} \tag{2.8}$$

Since $e_1 > 0$ in Ω and $\partial e_1/\partial\nu < 0$ on $\partial\Omega$, we can find $d > 0$ and a subdomain $\omega \subset\subset \Omega$ such that

$$|\nabla e_1| > d \quad \text{in } \Omega \setminus \omega.$$

Therefore, from (2.8) we obtain

$$-\Delta\overline{u} \ge Mc^2d^2\varphi(\delta_\Omega(x))f(\overline{u}) \quad \text{in } \Omega\backslash\omega, \qquad -\Delta\overline{u} \ge Mc\lambda_1 e_1 H'(ce_1) \quad \text{in } \omega. \tag{2.9}$$

Now, we choose $M > 0$ large enough such that

$$Mc^2d^2 > 1 \quad \text{and} \quad Mc\lambda_1 e_1 H'(ce_1) \ge \varphi(\delta_\Omega(x))f(\overline{u}) \quad \text{in } \omega. \tag{2.10}$$

Note that the last relation in (2.10) is possible since in ω the right side of the inequality is bounded and the left side is bounded away from zero. Thus, from (2.9) and (2.10), $\overline{u}$ is a super-solution for (2.5). Similarly, we can choose $m > 0$ small enough such that $\underline{u} = mH(ce_1)$ is a sub-solution of (2.5). Therefore, by the sub and super-solution method we find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\underline{u} \le u \le \overline{u}$ in Ω. The uniqueness follows from Lemma 2.1. In order to prove the boundary estimate (2.6), note first that $ce_1 \le \delta_\Omega(x)$ in Ω so

$$u(x) \le \overline{u}(x) \le MH(\delta_\Omega(x)) \quad \text{in } \{x \in \Omega : 0 < \delta_\Omega(x) < a\}.$$

On the other hand, since H is concave and $H(0) = 0$, we easily derive that $t \to H(t)/t$ is decreasing on $(0, 1)$. Also we can assume $cC_1 < 1$. Thus,

$$u(x) \ge mH(ce_1) \ge mH(cC_1\delta_\Omega(x)) \ge mcC_1H(\delta_\Omega(x)),$$

for all $x \in \Omega$ with $0 < \delta_\Omega(x) < 1$. The proof of Lemma 2.3 is now complete. □

Lemma 2.4. *Let $K \subset \mathbb{R}^N$ $(N \ge 2)$ be a compact set, $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $K \subset \Omega$ and $\Omega \setminus K$ is connected and has C^2 boundary. Let φ and f be as in Lemma 2.3. Then, there exists a unique solution $u \in C^2(\Omega \setminus K) \cap C(\overline{\Omega} \setminus int(K))$ of the problem*

$$\begin{cases} -\Delta u = \varphi(\delta_K(x))f(u), \ u > 0 & \text{in } \Omega \setminus K, \\ u = 0 & \text{on } \partial(\Omega \setminus K). \end{cases} \tag{2.11}$$

Furthermore, there exist $c_1, c_2 > 0$ and $0 < r_0 < 1$ such that the unique solution u of (2.11) satisfies

$$c_1 \le \frac{u(x)}{H(\delta_K(x))} \le c_2 \quad \text{in } \{x \in \Omega \setminus K : 0 < \delta_K(x) < r_0\}, \tag{2.12}$$

where H is the unique solution of (1.5).

Proof. According to Lemma 2.3 there exists $v \in C^2(\Omega \setminus K) \cap C(\overline{\Omega \setminus K})$ such that

$$\begin{cases} -\Delta v = \varphi(\delta_{\Omega\setminus K}(x))f(v), \ v > 0 & \text{in } \Omega \setminus K, \\ v = 0 & \text{on } \partial(\Omega \setminus K), \end{cases}$$

which further satisfies

$$c_1 \le \frac{v(x)}{H(\delta_{\Omega\setminus K}(x))} \le c_2 \quad \text{in } \{x \in \Omega \setminus K : 0 < \delta_{\Omega\setminus K}(x) < \rho_0\}, \tag{2.13}$$

for some $0 < \rho_0 < 1$ and $c_1, c_2 > 0$. Since $\delta_K(x) \ge \delta_{\Omega\setminus K}(x)$ for all $x \in \Omega \setminus K$ and φ is nonincreasing, it is easy to see that $\overline{u} = v$ is a super-solution of (2.11). Also it is not difficult to see that $\underline{u} = mw$ is a sub-solution to (2.11) for $m > 0$ sufficiently small, where w satisfies

$$\begin{cases} -\Delta w = 1, \ w > 0 & \text{in } \Omega \setminus K, \\ w = 0 & \text{on } \partial(\Omega \setminus K). \end{cases}$$

Using Lemma 2.1 we have $\underline{u} \le \overline{u}$ in $\Omega \setminus K$. Therefore, there exists a solution $u \in C^2(\Omega \setminus K) \cap C(\overline{\Omega} \setminus int(K))$ of (2.11). As before, the uniqueness follows from Lemma 2.1. In order to prove (2.12), let $0 < r_0 < \rho_0$ be small such that

$$\omega := \{x \in \Omega\setminus K : 0 < \delta_K(x) < r_0\} \subset\subset \Omega \quad \text{and} \quad \delta_{\Omega\setminus K}(x) = \delta_K(x) \quad \text{for all } x \in \omega.$$

Then, from (2.13) we have

$$u \le \overline{u} \le c_2 H(\delta_K(x)) \quad \text{in } \omega.$$

For the remaining part of (2.12), let $M > 1$ be such that $Mu \ge v$ on $\partial\omega \setminus \partial K$. Also

$$-\Delta(Mu) = M\varphi(\delta_K(x))f(u) \ge \varphi(\delta_K(x))f(Mu) \quad \text{in } \omega.$$

By Lemma 2.1 we have $Mu \ge v$ in ω and from (2.13) we obtain the first inequality in (2.12). This concludes the proof. $\square$

The following result is a direct consequence of Lemma 2.4.

Lemma 2.5. *Let $K_1, K_2, L \subset \mathbb{R}^N$ $(N \ge 2)$ be three compact sets (see Figure 1) such that*

$$K_1 \cap L = \emptyset, \quad K_2 \subset int(L), \quad K_1, L \textit{ are the closure of } C^2 \textit{ domains.}$$

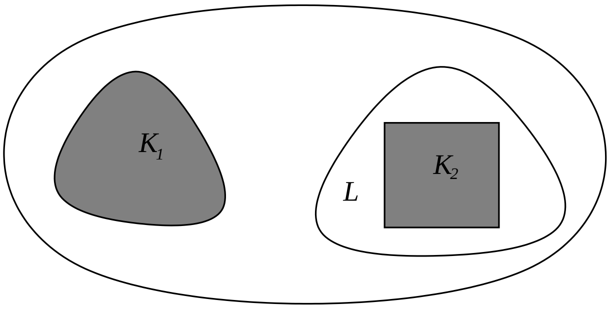

Figure 1. The compact sets K_1, K_2 and L.

Also let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary such that $K_1 \cup L \subset \Omega$ and $\Omega \setminus (K_1 \cup L)$ is connected. Let φ, f be as in Lemma 2.3. Then, there exists a unique solution

$$u \in C^2(\Omega \setminus (K_1 \cup L)) \cap C(\overline{\Omega} \setminus int(K_1 \cup L))$$

of the problem

$$\begin{cases} -\Delta u = \varphi(\delta_{K_1 \cup K_2}(x)) f(u), \ u > 0 & \text{in } \Omega \setminus (K_1 \cup L), \\ u = 0 & \text{on } \partial(\Omega \setminus (K_1 \cup L)). \end{cases} \tag{2.14}$$

Furthermore, there exist $c_1, c_2 > 0$ and $0 < r_0 < 1$ such that the unique solution u of problem (2.14) satisfies

$$c_1 \le \frac{u(x)}{H(\delta_{K_1}(x))} \le c_2 \quad \text{in } \{x \in \Omega \setminus (K_1 \cup L) : 0 < \delta_{K_1}(x) < r_0\}, \tag{2.15}$$

where H is the unique solution of (1.5).

Proof. By Lemma 2.4 there exists a unique $v \in C^2(\Omega \setminus (K_1 \cup L)) \cap C(\overline{\Omega} \setminus int(K_1 \cup L))$ such that

$$\begin{cases} -\Delta v = \varphi(\delta_{K_1 \cup L}(x)) f(v), \ v > 0 & \text{in } \Omega \setminus (K_1 \cup L), \\ v = 0 & \text{on } \partial(\Omega \setminus (K_1 \cup L)). \end{cases}$$

Since $\delta_{K_1 \cup L}(x) \le \delta_{K_1 \cup K_2}(x)$ in $\Omega \setminus (K_1 \cup L)$ and φ is nonincreasing, we derive that $\overline{u} = v$ is a super-solution of (2.14). As a sub-solution we use $\underline{u} = mw$ where m is sufficiently small and w satisfies

$$\begin{cases} -\Delta w = 1, \ w > 0 & \text{in } \Omega \setminus (K_1 \cup L), \\ w = 0 & \text{on } \partial(\Omega \setminus (K_1 \cup L)). \end{cases}$$

Therefore, problem (2.14) has a solution u. The uniqueness follows from Lemma 2.1 while the asymptotic behavior of u around K_1 is obtained in the same manner as in Lemma 2.4. This ends the proof. □

Several times in this paper we shall use the following elementary results that provide an equivalent integral condition to (1.2).

Lemma 2.6. *Let $N \geq 3$ and $\varphi : (0,\infty) \to [0,\infty)$ be a continuous function.*

(i) $\int_0^1 r\varphi(r)dr < \infty$ *if and only if* $\int_0^1 t^{1-N}\int_0^t s^{N-1}\varphi(s)dsdt < \infty$;

(ii) $\int_1^\infty r\varphi(r)dr < \infty$ *if and only if* $\int_1^\infty t^{1-N}\int_1^t s^{N-1}\varphi(s)dsdt < \infty$;

(iii) $\int_0^\infty r\varphi(r)dr < \infty$ *if and only if* $\int_0^\infty t^{1-N}\int_0^t s^{N-1}\varphi(s)dsdt < \infty$;

Proof. We prove only (i). The proof of (ii) is similar, while (iii) follows from (i)-(ii).

Assume first that $\int_0^1 r\varphi(r)dr < \infty$. Integrating by parts we have

$$
\begin{aligned}
\int_0^1 t^{1-N}\int_0^t s^{N-1}\varphi(s)dsdt &= -\frac{1}{N-2}\int_0^1 \left(t^{2-N}\right)'\int_0^t s^{N-1}\varphi(s)dsdt\\
&= \frac{1}{N-2}\left(\int_0^1 t\varphi(t)dt - \int_0^1 t^{N-1}\varphi(t)dt\right)\\
&\leq \frac{1}{N-2}\int_0^1 t\varphi(t)dt < \infty.
\end{aligned}
$$

Conversely, for $0 < \varepsilon < 1/2$ we have

$$
\begin{aligned}
\int_\varepsilon^1 t^{1-N}\int_0^t s^{N-1}\varphi(s)dsdt &= \frac{1}{N-2}\left(\int_\varepsilon^1 t\varphi(t)dt - \int_0^1 t^{N-1}\varphi(t)dt + \varepsilon^{2-N}\int_0^\varepsilon t^{N-1}\varphi(t)dt\right)\\
&\geq \frac{1}{N-2}\left(\int_\varepsilon^1 t\varphi(t)dt - \int_\varepsilon^1 t^{N-1}\varphi(t)dt\right)\\
&= \frac{1}{N-2}\int_\varepsilon^1 (1-t^{N-2})t\varphi(t)dt\\
&\geq \frac{1}{N-2}\left(1-\left(\frac{1}{2}\right)^{N-2}\right)\int_\varepsilon^{1/2} t\varphi(t)dt.
\end{aligned}
$$

Passing to the limit with $\varepsilon \searrow 0$ we deduce $\int_0^1 t\varphi(t)dt < \infty$. This concludes the proof of Lemma 2.6. □

3. Proof of Theorem 1.1

We start first with two nonexistence results that will help us to prove the necessity part in Theorem 1.1.

Proposition 3.1. *Let $\varphi : (0,\infty) \to [0,\infty)$ and $f : (0,\infty) \to (0,\infty)$ be continuous functions such that:*

(i) $\liminf_{t\searrow 0} f(t) > 0$;

(ii) $\varphi(r)$ *is monotone for r large;*

(iii) $\int_1^\infty r\varphi(r)\,dr = \infty$;

Then, for any compact set $K \subset \mathbb{R}^N$ $(N \geq 3)$ there does not exist a C^2 positive solution $u(x)$ of (1.1).

Proof. It is easy to construct a C^1 function $g : [0,\infty) \to (0,\infty)$ such that $g < f$ in $(0,\infty)$ and g' is negative and nondecreasing. Therefore, we may assume $f : [0,\infty) \to (0,\infty)$ is of class C^1 and f' is negative and nondecreasing.

Suppose for contradiction that $u(x)$ is a C^2 positive solution of (1.1). By translation, we may assume that $0 \in K$. Choose $r_0 > 0$ such that

$$K \subset B_{r_0/2}(0), \quad \varphi(r_0/2) > 0, \quad \text{and} \quad \varphi \text{ is monotone on } [r_0/2,\infty).$$

Define $\psi : [r_0/2,\infty) \to (0,\infty)$ by

$$\psi(r) = \min_{r_0/2 \leq \rho \leq r} \varphi(\rho) = \begin{cases} \varphi(r) & \text{if } \varphi \text{ is nonincreasing for } r \geq r_0/2, \\ \varphi(r_0/2) & \text{if } \varphi \text{ is nondecreasing for } r \geq r_0/2. \end{cases}$$

Then $\int_{r_0}^\infty r\psi(r)dr = \infty$. Also, since $r_0/2 \leq \delta_K(x) \leq |x|$ for all $x \in \mathbb{R}^N \setminus B_{r_0}(0)$, we have

$$\varphi(\delta_K(x)) \geq \psi(|x|) \quad \text{for all } x \in \mathbb{R}^N \setminus B_{r_0}(0).$$

Thus, the solution u of (1.1) satisfies

$$-\Delta u \geq \psi(|x|)f(u) \quad \text{in } \mathbb{R}^N \setminus B_{r_0}(0). \tag{3.1}$$

Averaging (3.1) and using Jensen's inequality, we get

$$-\Big(\bar{u}''(r) + \frac{n-1}{r}\bar{u}'(r)\Big) \geq \psi(r)f(\bar{u}(r)) \quad \text{for all } r \geq r_0. \tag{3.2}$$

Here $\bar{u}$ is the spherical average of u, that is

$$\bar{u}(r) = \frac{1}{\sigma_N r^{N-1}} \int_{\partial B_r(0)} u(x)\,d\sigma(x), \tag{3.3}$$

where σ denotes the surface area measure in $\mathbb{R}^N$ and $\sigma_N = \sigma(\partial B_1(0))$.

Making in (3.2) the change of variables $\bar{u}(r) = v(\rho)$, $\rho = r^{2-N}$ we get

$$-v''(\rho) \geq \frac{1}{(N-2)^2}\rho^{2(N-1)/(2-N)}\psi(\rho^{1/(2-N)})f(v(\rho)) \quad \text{for all } 0 < \rho \leq \rho_0,$$

where $\rho_0 = r_0^{2-N}$. Since v is concave down and positive, v is bounded for $0 < \rho \leq \rho_0$. Hence $f(v(\rho)) \geq (N-2)^2 C$ for some positive constant C. Consequently

$$-v''(\rho) \geq C\rho^{2(N-1)/(2-N)}\psi(\rho^{1/(2-N)}) \quad \text{for all } 0 < \rho \leq \rho_0.$$

Integrating this inequality twice we get

$$\begin{aligned}\infty &> \int_0^{\rho_0} v'(\rho)\,d\rho - \rho_0 v'(\rho_0)\\ &\geq C\int_0^{\rho_0}\int_\rho^{\rho_0} \bar\rho^{2(N-1)/(2-N)}\psi(\bar\rho^{1/(2-N)})\,d\bar\rho\,d\rho\\ &= C\int_0^{\rho_0} \bar\rho^{1+2(N-1)/(2-N)}\psi(\bar\rho^{1/(2-N)})\,d\bar\rho\\ &= (N-2)C\int_{r_0}^\infty r\psi(r)\,dr = \infty.\end{aligned}$$

This contradiction completes the proof. □

Proposition 3.2. *Let $\varphi : (0,\infty) \to [0,\infty)$ and $f : (0,\infty) \to (0,\infty)$ be continuous functions such that*

(i) $\liminf_{t\searrow 0} f(t) > 0$;

(ii) $\int_0^1 r\varphi(r)\,dr = \infty$.

Then there does not exist a C^2 positive solution $u(x)$ of

$$-\Delta u \geq \varphi(\delta_\Omega(x))f(u) \quad \text{in } \{x \in \mathbb{R}^N \setminus \overline{\Omega} : 0 < \delta_\Omega(x) < 2r_0\}, \quad N \geq 2, \tag{3.4}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N$ and $r_0 > 0$.

For the proof of Proposition 3.2 we shall use the following lemma concerning the geometry of a C^2 bounded domain. One can prove it using the methods from [13, page 96].

Lemma 3.3. *Let Ω be a C^2 bounded domain in $\mathbb{R}^N$, $N \geq 2$, such that $\mathbb{R}^N \setminus \Omega$ is connected. Then, there exists $r_0 > 0$ such that*

(i) $\Omega_r := \{x \in \mathbb{R}^N : \operatorname{dist}(x,\Omega) < r\}$ *is a C^1 domain for each $0 < r \leq r_0$;*

(ii) for $0 \leq r \leq r_0$ the function $T(\cdot,r) : \partial\Omega \to \mathbb{R}^N$ defined by $T(\xi,r) = \xi + r\eta_\xi$, where η_ξ is the outward unit normal to $\partial\Omega$ at ξ, is a C^1 diffeomorphism from $\partial\Omega$ onto $\partial\Omega_r$ (onto $\partial\Omega$ if $r = 0$) whose volume magnification factor (i.e., the absolute value of its Jacobian determinant) $J(\cdot,r) : \partial\Omega \to (0,\infty)$ is continuous on $\partial\Omega$ and C^∞ with respect to r;

(iii) if $\eta_{T(\xi,r)}$ is the unit outward normal to $\partial\Omega_r$ at $T(\xi,r)$ then $\eta_{T(\xi,r)}$ and η_ξ are equal (but have different base points) for $\xi \in \partial\Omega$ and $0 \leq r \leq r_0$.

Proof. [Proof of Proposition 3.2] Without loss of generality we can assume $\mathbb{R}^N \setminus \Omega$ is connected. Suppose for contradiction that $u(x)$ is a C^2 positive solution of (3.4). By decreasing r_0 if necessary, the conclusion of Lemma 3.3 holds.

Lemma 3.4. *The function*

$$g(r) = \int_{\partial\Omega_r} u(x)\,d\sigma(x), \quad 0 < r \leq r_0,$$

is continuously differentiable and there exists a positive constant C such that

$$\left| g'(r) - \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma(x) \right| \le C g(r) \quad \textit{for all } 0 < r \le r_0,$$

where η is the outward unit normal to $\partial\Omega_r$.

Proof. [Proof of Lemma 3.4] By Lemma 3.3 we have

$$g(r) = \int_{\partial\Omega} u(\xi + r\eta_\xi) J(\xi, r)\, d\sigma(\xi) \quad \text{for all } 0 < r \le r_0,$$

and thus

$$\begin{aligned} g'(r) &= \int_{\partial\Omega} \left[\frac{\partial}{\partial r} \Big(u(\xi + r\eta_\xi) \Big) \right] J(\xi, r)\, d\sigma(\xi) + \int_{\partial\Omega} u(\xi + r\eta_\xi) J_r(\xi, r)\, d\sigma(\xi) \\ &= \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}(x)\, d\sigma(x) + \int_{\partial\Omega_r} u(x) \frac{J_r(\xi, r)}{J(\xi, r)}\, d\sigma(x), \end{aligned} \tag{3.5}$$

for all $0 < r \le r_0$, where in the last integral $\xi = x - r\eta_\xi \in \partial\Omega$. Since, by Lemma 3.3, $J(\xi, r)$ is positive and continuous for $\xi \in \partial\Omega$ and $0 \le r \le r_0$ and $J_r(\xi, r)$ is continuous there, we see that Lemma 3.4 follows from (3.5). □

We now come back to the proof of Proposition 3.2. For $0 < r \le r_0$ we have

$$\begin{aligned} 0 &\le \int_{\Omega_{r_0} \setminus \Omega_r} -\Delta u(x)\, dx = \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma(x) - \int_{\partial\Omega_{r_0}} \frac{\partial u}{\partial \eta}\, d\sigma(x) \\ &\le g'(r) + C g(r) + C \end{aligned} \tag{3.6}$$

for some positive constant C by Lemma 3.4. Hence

$$\Big(e^{Cr} (g(r) + 1) \Big)' \ge 0 \quad \text{for all } 0 < r \le r_0,$$

and integrating this inequality over $[r, r_0]$ we obtain

$$g(r) \le e^{C(r_0 - r)} (g(r_0) + 1) - 1 \le C_1 \quad \text{for all } 0 < r \le r_0 \tag{3.7}$$

and for some $C_1 > 0$. Thus

$$U(r) := \frac{1}{|\partial\Omega_r|} \int_{\partial\Omega_r} u(x)\, d\sigma(x) = \frac{g(r)}{|\partial\Omega_r|}$$

is bounded for $0 < r \le r_0$. Consequently, by the assumption (i) of f, it follows that

$$|\partial\Omega_\rho| f(U(\rho)) \ge \varepsilon > 0 \quad \text{for all } 0 < \rho \le r_0. \tag{3.8}$$

As in the proof of Proposition 3.1, we may assume that $f : [0, \infty) \to (0, \infty)$ is of class C^1 and f' is negative and nondecreasing. From (3.4), (3.6)-(3.8) and Jensen's inequality we now obtain

$$\begin{aligned}
g'(r) + C_2 &\geq \int_{\Omega_{r_0}\setminus\Omega_r} -\Delta u\, dx \\
&\geq \int_r^{r_0} \varphi(\rho) \int_{\partial\Omega_\rho} f(u(x))\, d\sigma(x)\, d\rho \\
&\geq \int_r^{r_0} \varphi(\rho)|\partial\Omega_\rho| f(U(\rho))\, d\rho \\
&\geq \varepsilon \int_r^{r_0} \varphi(\rho)\, d\rho \quad \text{ for all } 0 < r \leq r_0.
\end{aligned}$$

Integrating over $[r, r_0]$ in the above estimate we find

$$\begin{aligned}
g(r_0) - g(r) + C_2 r_0 &\geq \varepsilon \int_r^{r_0} \int_s^{r_0} \varphi(\rho)\, d\rho\, ds \\
&= \varepsilon \int_r^{r_0} (\rho - r)\varphi(\rho)\, d\rho \to \varepsilon \int_0^{r_0} \rho\varphi(\rho)\, d\rho = \infty \quad \text{ as } r \searrow 0,
\end{aligned}$$

which contradicts $g > 0$ and completes the proof. □

Proof of Theorem 1.1 The necessity of (1.2) follows from Propositions 3.1 and 3.2. To prove the sufficiency part we shall separately analyse the cases $K_2 = \emptyset$ and $K_2 \neq \emptyset$.

3.1. Case $K_2 = \emptyset$

Assume first that $\mathbb{R}^N \setminus K$ is connected and let $0 < \rho < R$ be such that $K \subset B_\rho(0)$. By Lemma 2.4 there exists

$$u \in C^2(B_\rho(0) \setminus K) \cap C(\overline{B}_\rho(0) \setminus int(K))$$

such that

$$\begin{cases} -\Delta u = \varphi(\delta_K(x)) f(u),\ u > 0 & \text{in } B_\rho(0) \setminus K, \\ u = 0 & \text{on } \partial(B_\rho(0) \setminus K). \end{cases} \tag{3.9}$$

We next construct a solution v of (1.1) in a neighborhood of infinity. To this aim, let

$$A(r) := \int_r^\infty t^{1-N} \int_R^t s^{N-1}\varphi(s-\rho) ds dt \quad \text{ for all } r \geq R.$$

Since $\int_R^\infty r\varphi(r-\rho)dr < \infty$, by Lemma 2.6 we have that A is well defined for all $r \geq R$. Also, it is easy to check that

$$-\Delta A(|x|) = \varphi(|x| - \rho) \quad \text{ in } \mathbb{R}^N \setminus B_R(0).$$

Since the mapping

$$[0, \infty) \ni t \longmapsto \int_0^t \frac{1}{f(s)} ds \in [0, \infty)$$

is bijective, we can define $v : \mathbb{R}^N \setminus B_R(0) \to (0,\infty)$ implicitly as the unique solution of

$$\int_0^{v(x)} \frac{1}{f(t)} dt = A(|x|) \quad \text{for all } x \in \mathbb{R}^N \setminus B_R(0). \tag{3.10}$$

Then, using the properties of A we deduce that $v \in C^2(\mathbb{R}^N \setminus B_R(0))$, $v > 0$ and $v(x) \to 0$ as $|x| \to \infty$. Further from (3.10) we obtain

$$\nabla A(|x|) = \frac{1}{f(v)} \nabla v \quad \text{in } \mathbb{R}^N \setminus B_R(0),$$

and

$$\varphi(|x| - \rho) = -\Delta A(|x|) = \frac{f'(v)}{f^2(v)} |\nabla v|^2 - \frac{1}{f(v)} \Delta v \quad \text{in } \mathbb{R}^N \setminus B_R(0).$$

Since f is decreasing, we have $f' \leq 0$ which implies

$$-\Delta v \geq \varphi(|x| - \rho) f(v) \quad \text{in } \mathbb{R}^N \setminus B_R(0).$$

Therefore, $v \in C^2(\mathbb{R}^N \setminus B_R(0))$ satisfies

$$\begin{cases} -\Delta v \geq \varphi(\delta_K(x)) f(v),\ v > 0 & \text{in } \mathbb{R}^N \setminus B_R(0), \\ \ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.11}$$

Let now $0 < \rho_0 < \rho$ be such that $K \subset B_{\rho_0}(0)$ and let u, v be the solutions of (3.9) and (3.11) respectively. Consider

$$w : (B_{\rho_0}(0) \setminus int(K)) \cup (\mathbb{R}^N \setminus B_R(0)) \to [0, \infty),$$

defined by

$$w(x) = u(x) \text{ if } x \in B_{\rho_0}(0) \setminus int(K), \quad w(x) = v(x) \text{ if } x \in \mathbb{R}^N \setminus B_R(0).$$

Let W be a positive C^2 extension of w to $\mathbb{R}^N \setminus K$. We claim that there exists $M > 0$ large enough such that

$$U(x) = W(x) + M(1 + |x|^2)^{(2-N)/2}, \quad x \in \mathbb{R}^N \setminus int(K) \tag{3.12}$$

satisfies (1.1). Indeed, since $(1 + |x|^2)^{(2-N)/2}$ is superharmonic, this condition is already satisfied in $B_{\rho_0}(0) \setminus K$ and $\mathbb{R}^N \setminus B_R(0)$. In $B_R(0) \setminus B_{\rho_0}(0)$ we use the fact that $-\Delta(1 + |x|^2)^{(2-N)/2}$ is positive and bounded away from zero. Therefore we have constructed a solution $U \in C^2(\mathbb{R}^N \setminus K) \cap C(\mathbb{R}^N \setminus int(K))$ of (1.1) that tends to zero at infinity.

Assume now that $\mathbb{R}^N \setminus K$ is not connected. We shall construct a solution to (1.1) by considering each component of $\mathbb{R}^N \setminus K$. Note that since K is compact, $\mathbb{R}^N \setminus K$ has only one unbounded component on which we proceed as above. Since φ satisfies (1.2), by Lemma 2.3, on each bounded component of $\mathbb{R}^N \setminus K$ we construct a solution of $-\Delta u = \varphi(\delta_K(x)) f(u)$ that vanishes continuously on the boundary and has the behavior described by (1.4).

3.2. Case $K_2 \neq \emptyset$

Proceeding in the same manner as above (see (3.12)) we can find a function

$$U \in C^2(\mathbb{R}^N \setminus K_1) \cap C(\mathbb{R}^N \setminus int(K_1))$$

such that

$$\begin{cases} -\Delta U \geq \varphi(\delta_{K_1}(x))f(U),\ U > 0 & \text{in } \mathbb{R}^N \setminus K_1, \\ U(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.13}$$

We next construct a function $V \in C^2(\mathbb{R}^N \setminus K_2) \cap C(\mathbb{R}^N)$ such that

$$\begin{cases} -\Delta V \geq \varphi(\delta_{K_2}(x))f(V),\ V > 0 & \text{in } \mathbb{R}^N \setminus K_2, \\ V(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.14}$$

Using (1.2) and Lemma 2.6(iii), the function

$$D(r) := \int_r^\infty t^{1-N} \int_0^t s^{N-1}\varphi(s)dsdt \quad \text{for all } r \geq 0,$$

is well defined and $-\Delta D(|x|) = \varphi(|x|)$ in $\mathbb{R}^N \setminus \{0\}$. We next define $v : \mathbb{R}^N \to (0,\infty)$ by

$$\int_0^{v(x)} \frac{1}{f(t)}dt = D(|x|) \quad \text{for all } x \in \mathbb{R}^N.$$

Using the same arguments as in the previous case we have $v \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$ and

$$\begin{cases} -\Delta v \geq \varphi(|x|)f(v),\ v > 0 & \text{in } \mathbb{R}^N \setminus \{0\}, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.15}$$

Let now $V : \mathbb{R}^N \to (0,\infty)$ defined by

$$V(x) = \sum_{a \in K_2} v(x-a).$$

By (3.15) we have $V \in C^2(\mathbb{R}^N \setminus K_2) \cap C(\mathbb{R}^N)$, $V(x) \to 0$ as $|x| \to \infty$ and

$$\begin{aligned} -\Delta V(x) &= -\sum_{a \in K_2} \Delta v(x-a) \geq \sum_{a \in K_2} \varphi(|x-a|)f(v(x-a)) \\ &\geq \Big(\sum_{a \in K_2} \varphi(|x-a|)\Big)f(V(x)) \geq \varphi(\min_{a \in K_2} |x-a|)f(V(x)) \\ &= \varphi(\delta_{K_2}(x))f(V(x)) \quad \text{for all } x \in \mathbb{R}^N \setminus K_2. \end{aligned}$$

Therefore, V fulfills (3.14). Now

$$W := U + V : \mathbb{R}^N \setminus K \to \mathbb{R} \tag{3.16}$$

satisfies $W(x) \to 0$ as $|x| \to \infty$ and

$$\begin{aligned} -\Delta W(x) &\geq \varphi(\delta_{K_1}(x))f(U) + \varphi(\delta_{K_2}(x))f(V) \\ &\geq (\varphi(\delta_{K_1}(x)) + \varphi(\delta_{K_2}(x)))f(W) \\ &\geq \varphi\Big(\min\{\delta_{K_1}(x), \delta_{K_2}(x)\}\Big)f(W) \\ &= \varphi(\delta_K(x))f(W) \quad \text{for all } x \in \mathbb{R}^N \setminus K. \end{aligned}$$

Thus, W is a solution of (1.1) and the proof of Theorem 1.1 is now complete. □

Remark. The approach in Theorem 1.1 can be used to study the inequality (1.1) in some cases where the compact set K consists of infinitely many components all of them with C^2 boundary. For instance, it is easy to see that the same arguments apply for compact sets K of the form

$$K = B_1(0) \cup \bigcup_{n\geq 1} \left\{x \in \mathbb{R}^N : 1 + \frac{1}{2n+1} < |x| < 1 + \frac{1}{2n}\right\}$$

or

$$K = \partial B_1(0) \cup \bigcup_{n\geq 1} \partial B_{1+1/n}(0).$$

4. Proof of Theorem 1.2

4.1. Case $K_2 = \emptyset$

We shall assume that $\mathbb{R}^N \setminus K$ is connected as using the arguments in the proof of Theorem 1.1 on any bounded component of $\mathbb{R}^N \setminus K$ we can construct a solution of $-\Delta u = \varphi(\delta_K(x))f(u)$ that vanishes continuously on its boundary and has the behavior described by (1.4).

According to Lemma 2.4, for any $n \geq 1$ there exists a unique

$$u_n \in C^2(B_{R+n}(0) \setminus K) \cap C(\overline{B}_{R+n}(0) \setminus int(K))$$

such that

$$\begin{cases} -\Delta u_n = \varphi(\delta_K(x))f(u_n),\ u_n > 0 & \text{in } B_{R+n}(0) \setminus K, \\ \ \ u_n = 0 & \text{on } \partial(B_{R+n}(0) \setminus K). \end{cases} \tag{4.1}$$

We extend $u_n = 0$ on $\mathbb{R}^N \setminus B_{R+n}(0)$ and by Lemma 2.1 we have that $\{u_n\}$ is a nondecreasing sequence of functions and $u_n \leq U$ in $\mathbb{R}^N \setminus K$. Let

$$\tilde{u}(x) = \lim_{n\to\infty} u_n(x) \quad \text{for all } x \in \mathbb{R}^N \setminus int(K).$$

By standard elliptic arguments, we have $\tilde{u} \in C^2(\mathbb{R}^N \setminus K)$ and

$$-\Delta \tilde{u} = \varphi(\delta_K(x))f(\tilde{u}) \quad \text{in } \mathbb{R}^N \setminus K.$$

We next prove that $\tilde{u}$ vanishes continuously on ∂K.

Let u_1 be the unique solution of (4.1) with $n = 1$ and $\omega := \{x \in \mathbb{R}^N \setminus K : 0 < \delta_K(x) < 1\}$. Since both u_1 and $\tilde{u}$ are continuous and positive on $\partial\omega \setminus K$, one can find $M > 1$ such that $Mu_1 \geq \tilde{u}$ on $\partial\omega \setminus K$. Now, using the fact that the sequence $\{u_n\}$ is nondecreasing, this also yields $Mu_1 \geq u_n$ on $\partial\omega \setminus K$, for all $n \geq 1$. The above inequality also holds true on ∂K (since u_1 and u_n are zero there). Therefore $Mu_1 \geq u_n$ on $\partial\omega$ for all $n \geq 1$ which by the comparison result in Lemma 2.1 (note that the function Mu_1 satisfies (1.1) in ω) gives

$$Mu_1 \geq u_n \quad \text{in } \omega,$$

for all $n \geq 1$. Passing to the limit with $n \to \infty$ in the above estimate, we obtain $Mu_1 \geq \tilde{u}$ in ω and since u_1 vanishes continuously on ∂K, so does $\tilde{u}$.

The boundary behavior of $\tilde{u}$ near K follows from the fact that $u_1 \leq \tilde{u} \leq Mu_1$ in ω and u_1 satisfies (2.12). From Lemma 2.1 we obtain that any solution u of (1.1) satisfies $u \geq u_n$ in $\mathbb{R}^N \setminus K$ which implies $u \geq \tilde{u}$. Hence, $\tilde{u}$ is the minimal solution of (1.1).

4.2. Case $K_2 \neq \emptyset$

Using, if necessary, a dilation argument, we can assume that $\mathrm{dist}(K_1, K_2) > 2$ and the distance between any two distinct points of K_2 is greater than 2. We fix $R > 0$ large enough such that

$$K_1 \cup \bigcup_{a \in K_2} \overline{B}_1(a) \subset B_R(0).$$

We now apply Lemma 2.5 for $L = \bigcup_{a \in K_2} \overline{B}_{1/n}(a)$ and $\Omega = B_{R+n}(a)$. Thus, there exists a unique solution u_n of

$$\begin{cases} -\Delta u_n = \varphi(\delta_K(x)) f(u_n), \ u_n > 0 & \text{in } B_{R+n}(0) \setminus \Big(K_1 \cup \bigcup_{a \in K_2} \overline{B}_{1/n}(a)\Big), \\ u_n = 0 & \text{on } \partial B_{R+n}(0) \cup \partial K_1 \cup \bigcup_{a \in K_2} \partial B_{1/n}(a). \end{cases} \tag{4.2}$$

Extending $u_n = 0$ outside of $\overline{B}_{R+n}(0) \setminus \bigcup_{a \in K_2} \overline{B}_{1/n}(a)$, by Lemma 2.1 we obtain

$$0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \quad \text{in } \mathbb{R}^N \setminus K.$$

By Lemma 2.1 we obtain $u_n \leq W$ in $\mathbb{R}^N \setminus K$, where W is defined by (3.16). Thus, passing to the limit in (4.2) and by elliptic arguments, we obtain that $\tilde{u} := \lim_{n \to \infty} u_n$ satisfies

$$-\Delta \tilde{u} = \varphi(\delta_K(x)) f(\tilde{u}) \quad \text{in } \mathbb{R}^N \setminus K.$$

The fact that $\tilde{u}$ is minimal, vanishes continuously on ∂K_1 and has the behavior near ∂K_1 as described by (1.4) follows exactly in the same way as in the proof of Theorem 1.1.

It remains to prove that $\tilde{u}$ can be continuously extended at any point of K_2 and $\tilde{u} > 0$ on K_2. To this aim, we state and prove the following auxiliary results.

Lemma 4.1. *Let $r > 0$ and $x \in \mathbb{R}^N \setminus \partial B_r(0)$, $N \geq 3$. Then*

$$\frac{1}{\sigma_N r^{N-1}} \int_{\partial B_r(0)} \frac{1}{|x-y|^{N-2}} d\sigma(y) = \begin{cases} \dfrac{1}{|x|^{N-2}} & \text{if } |x| > r, \\ \dfrac{1}{r^{N-2}} & \text{if } |x| < r. \end{cases}$$

Proof. [Proof of Lemma 4.1] Suppose first $|x| > r$. Then $u(y) = |y - x|^{2-N}$ is harmonic in $B_{r+\varepsilon}(0)$, for $\varepsilon > 0$ small. By the mean value theorem we have

$$\frac{1}{\sigma_N r^{N-1}} \int_{\partial B_r(0)} \frac{1}{|x-y|^{N-2}} d\sigma(y) = u(0) = \frac{1}{|x|^{N-2}}.$$

Assume now $|x| < r$. Since

$$v(x) := \frac{1}{\sigma_N r^{N-1}} \int_{\partial B_r(0)} \frac{1}{|x-y|^{N-2}} d\sigma(y)$$

is harmonic and radially symmetric, it follows that v is constant in $B_r(0)$. Thus $v(x) = v(0) = r^{2-N}$ for $x \in B_r(0)$. $\square$

Lemma 4.2. *Let u be a C^2 positive solution of*

$$-\Delta u \geq 0 \quad \text{in } B_{2r_1}(0) \setminus \{0\}, \; N \geq 2.$$

Then

$$u(x) \geq m := \min_{|y|=r_1} u(y) \quad \text{for all } x \in \overline{B}_{r_1}(0) \setminus \{0\}.$$

Proof. [Proof of Lemma 4.2] For $0 < r_0 < r_1$ define $v_{r_0} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ by

$$v_{r_0}(x) = \frac{m(\Phi(r_0) - \Phi(|x|))}{\Phi(r_0) - \Phi(r_1)},$$

where

$$\Phi(r) = \begin{cases} \log \dfrac{1}{r} & \text{if } N = 2, \\ \dfrac{1}{r^{N-2}} & \text{if } N \geq 3. \end{cases}$$

Then v_{r_0} is harmonic in $\mathbb{R}^N \setminus \{0\}$ and $v_{r_0} \leq u$ on $\partial B_{r_1}(0) \cup \partial B_{r_0}(0)$. Thus, by the maximum principle, $v_{r_0} \leq u$ in $\overline{B}_{r_1}(0) \setminus B_{r_0}(0)$. Fix $x \in \overline{B}_{r_1}(0) \setminus \{0\}$. Then, for $0 < r_0 < |x|$ we have $u(x) \geq v_{r_0}(x) \to m$ as $r_0 \searrow 0$. This concludes the proof. $\square$

Lemma 4.3. *Let $\varphi, f : (0, \infty) \to [0, \infty)$ be continuous functions such that $\int_0^1 r\varphi(r)dr < \infty$. Suppose that u is a C^2 positive bounded solution of $-\Delta u = \varphi(|x|)f(u)$ in a punctured neighborhood of the origin in $\mathbb{R}^N$, $N \geq 3$. Then, for some $L > 0$ we have $u(x) \to L$ as $x \to 0$.*

Proof. [Proof of Lemma 4.3] By Lemma 4.2 we can find $r_0 > 0$ small such that u is bounded away from zero in $\overline{B}_{r_0}(0) \setminus \{0\}$. Hence, for some $M > 0$ we have

$$f(u(x)) \leq M \quad \text{in } \overline{B}_{r_0}(0) \setminus \{0\}. \tag{4.3}$$

For $x \in \mathbb{R}^N$ let

$$I(x) := \frac{1}{\sigma_N} \int_{B_{r_0}(0)} \frac{\varphi(y) f(u(y))}{|x-y|^{N-2}} dy.$$

Then,

$$I(x) = \int_0^{r_0} F(x,r)dr, \qquad \text{where} \qquad F(x,r) = \frac{\varphi(r)}{\sigma_N} \int_{\partial B_r(0)} \frac{f(u(y))}{|x-y|^{N-2}} d\sigma(y).$$

Since, by (4.3) and Lemma 4.1 we have

(i) $F(x,r) \le Mr\varphi(r)$ for $x \in \mathbb{R}^N$ and $0 < r < r_0$;

(ii) $\int_0^{r_0} r\varphi(r)dr < \infty$;

(iii) $F(x,r) \to F(0,r)$ as $x \to 0$ pointwise for $0 < r < r_0$,

it follows that I is bounded in $\mathbb{R}^N$ and by the dominated convergence theorem,

$$I(x) \to I(0) \qquad \text{as } x \to 0. \tag{4.4}$$

Since $v := u - \frac{1}{N-2} I$ is harmonic and bounded in $B_{r_0}(0) \setminus \{0\}$, it is well known that $\lim_{x\to 0} v(x)$ exists. Thus, by (4.4), $\lim_{x\to 0} u(x)$ exists and is finite. □

Now, the fact that the minimal solution $\tilde{u}$ can be continuously extended on K_2 and $\tilde{u} > 0$ on K_2 follows by applying Lemma 4.3 for each point of K_2. This finishes the proof of Theorem 1.1. □

Remark. The existence of a positive ground state solution in the exterior of a compact set is a particular feature of the case $N \ge 3$. Such solutions do not exist in dimension $N = 2$. Indeed, suppose that u is a C^2 positive solution of

$$-\Delta u \ge 0 \quad \text{ in } \mathbb{R}^2 \setminus K, \quad u(x) \to 0 \quad \text{ as } |x| \to \infty,$$

where $K \subset \mathbb{R}^2$ is a compact set, not necessarily with smooth boundary. Choose $r_0 > 0$ such that $K \subset B_{r_0}(0)$ and let $m = \min_{|x|=r_0} u(x) > 0$. For each $r_1 > r_0$ define

$$v_{r_1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, \quad v_{r_1}(x) = \frac{m(\log r_1 - \log |x|)}{\log r_1 - \log r_0}.$$

Then

$$v_{r_1} \text{ is harmonic in } \mathbb{R}^2 \setminus \{0\}, \quad v_{r_1} = m \text{ on } \partial B_{r_0}(0), \quad v_{r_1} = 0 \text{ on } \partial B_{r_1}(0).$$

Let $w_{r_1}(x) = u(x) - v_{r_1}(x)$, $x \in \mathbb{R}^N \setminus B_{r_0}(0)$. Thus,

$$-\Delta w_{r_1} = -\Delta u \ge 0 \quad \text{ in } \overline{B}_{r_1}(0) \setminus B_{r_0}(0), \quad w_{r_1} \ge 0 \quad \text{ on } \partial B_{r_1}(0) \cup \partial B_{r_0}(0).$$

By maximum principle it follows that $w_{r_1} \ge 0$ in $\overline{B}_{r_1}(0) \setminus B_{r_0}(0)$, that is $u(x) \ge v_{r_1}(x)$ in $\overline{B}_{r_1}(0) \setminus B_{r_0}(0)$.

Let now $x \in \mathbb{R}^2 \setminus \overline{B}_{r_0}(0)$ be fixed. Then, for $r_1 > |x|$ we have

$$u(x) \ge v_{r_1}(x) \to m \quad \text{ as } r_1 \to \infty,$$

so $u(x) \ge m$ in $\mathbb{R}^2 \setminus \overline{B}_{r_0}(0)$, which contradicts $u(x) \to 0$ as $|x| \to \infty$.

5. Proof of Theorem 1.4

Assume first that (1.1) has a C^2 positive solution u. From Proposition 3.1 it follows that $\int_1^\infty r\varphi(r)dr < \infty$. By translation one may assume that $0 \in K$ and fix $r_0 > 0$ such that $\delta_K(x) = |x|$ for $0 < |x| < r_0$. Let now u_* be the image of u through the Kelvin transform, that is,

$$u_*(x) = |x|^{2-N} u\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^N \setminus B_{1/r_0}(0). \tag{5.1}$$

Then u_* satisfies

$$\begin{aligned} -\Delta u_* &\geq |x|^{-2-N} \varphi\left(\frac{1}{|x|}\right) f\left(u\left(\frac{x}{|x|^2}\right)\right) \\ &= |x|^{-2-N} \varphi\left(\frac{1}{|x|}\right) f(|x|^{N-2} u_*(x)) \quad \text{ in } \mathbb{R}^N \setminus B_{1/r_0}(0). \end{aligned}$$

By taking the spherical average of u and then using the change of variable $\rho = r^{2-N}$ as in the proof of Proposition 3.1 (note that here we do not need $\varphi(r)$ to be monotone for small values of $r > 0$) we deduce

$$\int_1^\infty t^{-1-N} \varphi\left(\frac{1}{t}\right) f(at^{N-2})dt < \infty.$$

Now with the change of variable $r = t^{-1}$, $0 < r \leq 1$ we derive the condition (1.7).

Conversely, assume now that (1.6)-(1.7) hold and let us construct a solution to (1.1) in the case $K = \{0\}$. This will follow from lemma below.

Lemma 5.1. *Let $a > 0$ be such that (1.6) and (1.7) hold. Then for all $b > 0$ there exists a radially symmetric function $v_{a,b} \in C^2(\mathbb{R}^N \setminus \{0\})$ such that*

$$-\Delta v_{a,b} \geq \varphi(|x|) f(v_{a,b}) \quad \textit{in } \mathbb{R}^N \setminus \{0\},$$

and

$$\lim_{|x|\to 0} |x|^{N-2} v_{a,b}(x) = a\,, \quad \lim_{|x|\to\infty} v_{a,b}(x) = b.$$

Proof. Let $u_0(r) = ar^{2-N} + b$, $r > 0$ and for all $n \geq 1$ define inductively the sequence

$$u_n(r) = u_0 + \int_r^\infty t^{1-N} \int_0^t s^{N-1} \varphi(s) f(u_{n-1}(s))dsdt\,, \quad r > 0. \tag{5.2}$$

Remark first that u_n is well defined since $u_{n-1} \geq u_0$ and by Lemma 2.6 we have

$$\begin{aligned} &\int_r^\infty t^{1-N} \int_0^t s^{N-1} \varphi(s) f(u_{n-1}(s))dsdt \leq \int_0^\infty t^{1-N} \int_0^t s^{N-1} \varphi(s) f(u_0(s))dsdt \\ &\quad \leq \int_0^\infty r\varphi(r) f(u_0(r))dr \leq \int_0^1 r^{1-N} \varphi(r) f(ar^{2-N})dr + f(b) \int_1^\infty r\varphi(r)dr < \infty. \end{aligned}$$

A straightforward induction argument yields

$$u_1 \geq u_{2n-1} \geq u_{2n+1} \geq u_{2n} \geq u_{2n-2} \geq u_0\,, \tag{5.3}$$

for all $n \geq 1$. Thus, there exists $u(r) := \lim_{n\to\infty} u_{2n}(r)$ and $v(r) := \lim_{n\to\infty} u_{2n-1}(r)$, $r > 0$. Passing to the limit in (5.2) and (5.3) we find

$$\begin{cases} u(r) = u_0 + \displaystyle\int_r^\infty t^{1-N} \int_0^t s^{N-1}\varphi(s) f(v(s))dsdt\,, & r > 0, \\ v(r) = u_0 + \displaystyle\int_r^\infty t^{1-N} \int_0^t s^{N-1}\varphi(s) f(u(s))dsdt\,, & r > 0, \end{cases} \tag{5.4}$$

and $v \geq u$. Thus $V(x) = v(|x|)$ satisfies

$$-\Delta V(x) = \varphi(|x|) f(u(|x|)) \geq \varphi(|x|) f(V(x)) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since the integrals in (5.4) are finite, it is easy to check that

$$\lim_{|x|\to 0} |x|^{N-2} V(x) = a\,, \quad \lim_{|x|\to\infty} V(x) = b.$$

Therefore, $v_{a,b} \equiv V$ satisfies the requirements in Lemma 5.1. □

If K is a finite set of points and V is any solution of $-\Delta V \geq \varphi(|x|)f(V)$ in $\mathbb{R}^N \setminus \{0\}$ then $U(x) := \sum_{y\in K} V(x-y)$ is a solution of (1.1). This concludes our proof.

Under the conditions (1.6)-(1.7), the existence of the minimal solution $\tilde{u}$ of (1.1) is obtained with the same proof as in Theorem 1.1. Note that $\tilde{u}$ is obtained as a pointwise limit of the sequence $\{u_n\}$ where u_n satisfies (4.2) in which $K_1 = \emptyset$ and $K_2 = K$. It remains to prove that $\tilde{u}$ can be continuously extended to a positive continuous function in $\mathbb{R}^N$ if and only if $\int_0^1 r\varphi(r)dr < \infty$.

Assume first that the minimal solution $\tilde{u}$ of (1.1) is bounded. Using a translation argument, one can also assume that $0 \in K$. Fix $r_0 > 0$ such that $\delta_K(x) = |x|$ for all $x \in \overline{B}_{r_0}(0)$. Then averaging (1.1) we obtain

$$-(r^{N-1}\bar{u}'(r))' \geq cr^{N-1}\varphi(r) \quad \text{for all } 0 < r \leq r_0, \tag{5.5}$$

where $c > 0$. Hence $r^{N-1}\bar{u}'(r)$ is decreasing and its limit as $r \searrow 0$ must be zero for otherwise $\bar{u}$−and hence u−would be unbounded for small $r > 0$. Thus integrating (5.5) twice we obtain

$$\infty > \Big(\limsup_{r\searrow 0} \bar{u}(r)\Big) - \bar{u}(r_0) \geq c\int_0^{r_0} t^{1-N} \int_0^t s^{N-1}\varphi(s)dsdt,$$

which by Lemma 2.6(ii) yields $\int_0^1 r\varphi(r)dr < \infty$.

Assume now that $\int_0^1 r\varphi(r)dr < \infty$. The conclusion will follow by Lemma 4.3 once we prove that $\tilde{u}$ is bounded around each point of K. Again by translation and a scaling argument we may assume that $0 \in K$ and $\delta_K(x) = |x|$ for all $x \in B_1(0)$. Set

$$v(x) := M\int_{|x|}^2 t^{1-N} \int_0^t s^{N-1}\varphi(s)dsdt, \quad \text{for all } x \in B_2(0).$$

By Lemma 2.6(i), v is bounded and positive in $B_2(0)$ and

$$-\Delta v(x) = M\varphi(|x|) = M\varphi(\delta_K(x)) \quad \text{in } B_1(0)\setminus\{0\}. \tag{5.6}$$

Therefore, we can take $M > 1$ large enough such that

$$-\Delta v(x) \ge \varphi(\delta_K(x))f(v(x)) \quad \text{in } B_1(0)\setminus\{0\} \quad \text{and} \quad v \ge \tilde u \quad \text{on } \partial B_1(0). \tag{5.7}$$

Let u_n be the solution of (4.2) with $K_1 = \emptyset$ and $K_2 = K$. Recall that $\{u_n\}$ converges pointwise to $\tilde u$. Since $\tilde u \ge u_n$ in $\mathbb{R}^N \setminus K$, from (5.7) we have $v \ge u_n$ on $\partial B_1(0)$. According to the definition of u_n, this inequality also holds true on $\partial B_{1/n}(0)$. Now, by (5.7) and Lemma 2.1 it follows that $v \ge u_n$ in $B_1(0)\setminus B_{1/n}(0)$. Passing to the limit with $n\to\infty$ it follows that $v \ge \tilde u$ in $B_1(0)\setminus\{0\}$ and so, $\tilde u$ is bounded around zero. Proceeding similarly we derive that $\tilde u$ is bounded around every point of K. By Lemma 4.3 we now obtain that $\tilde u$ can be continuously extended on K. This finishes the proof of Theorem 1.4.

□

6. Proof of Theorem 1.6

We shall divide our arguments into three steps.

Step 1: There exists a minimal solution $\xi : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ which in addition satisfies

$$\lim_{|x|\to 0} |x|^{N-2}\xi(x) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \xi(x) = 0. \tag{6.1}$$

Indeed, by Lemma 2.2 there exists a unique function ξ_n such that

$$\begin{cases} -\Delta \xi_n = \varphi(|x|)f(\xi_n),\ \xi_n > 0 & \text{in } B_n(0)\setminus \overline{B}_{1/n}(0),\\ \xi_n = 0 & \text{on } \partial B_n(0)\cup \partial B_{1/n}(0). \end{cases} \tag{6.2}$$

By uniqueness, it also follows that ξ_n is radially symmetric. We next extend $\xi_n = 0$ outside $B_n(0)\setminus \overline{B}_{1/n}(0)$. Now, by Lemma 2.1 we have that $\{\xi_n\}$ is nondecreasing. For any $\varepsilon > 0$, let v_ε be the function constructed in Lemma 5.1 for $a = \varepsilon$ and $b = \varepsilon$. Then, again by Lemma 2.1 we have $\xi_n \le v_\varepsilon$ in $\overline{B}_n(0)\setminus B_{1/n}(0)$.

Hence, there exists $\xi(x) := \lim_{n\to\infty} \xi_n(x)$, $x \in \mathbb{R}^N\setminus\{0\}$ and $\xi \le v_\varepsilon$. Also ξ is radially symmetric and by standard elliptic arguments it follows that ξ is a solution of (1.11). From $\xi \le v_\varepsilon$ it follows that $\lim_{|x|\to 0}|x|^{N-2}\xi(x) \le \varepsilon$ and $\lim_{|x|\to\infty}\xi(x) \le \varepsilon$. Now, since $\varepsilon > 0$ was arbitrarily chosen, we deduce that ξ satisfies (6.1).

Finally, if u is an arbitrary solution of (1.11), by Lemma 2.1 we deduce $\xi_n \le u$ in $B_n(0)\setminus \overline{B}_{1/n}(0)$ which next produces $\xi \le u$ in $\mathbb{R}^N\setminus\{0\}$. Therefore ξ is the minimal solution of (1.11).

Step 2: Proof of (i).

Fix $a, b \ge 0$. We shall construct a radially symmetric solution of (1.11) that satisfies (1.12) with the aid of the minimal solution ξ constructed at Step 1. By virtue of Lemma 2.2, for any $n \ge 2$ there exists a unique function

$$u_n \in C^2(B_n(0)\setminus \overline{B}_{1/n}(0)) \cap C(\overline{B}_n(0)\setminus B_{1/n}(0))$$

such that

$$\begin{cases} -\Delta u_n = |x|^\alpha u_n^{-p}, \ u_n > 0 & \text{in } B_n(0) \setminus \overline{B}_{1/n}(0), \\ \quad u_n = a|x|^{2-N} + b + \xi(x) & \text{on } \partial B_n(0) \cup \partial B_{1/n}(0). \end{cases} \tag{6.3}$$

Since ξ is radially symmetric, so is u_n. Furthermore, $a|x|^{2-N} + b$ is a sub-solution while $a|x|^{2-N} + b + \xi(x)$ is a super-solution of (6.3). Thus, in view of Lemma 2.1, we obtain

$$a|x|^{2-N} + b \le u_n(x) \le a|x|^{2-N} + b + \xi(x) \quad \text{ in } B_n(0) \setminus B_{1/n}(0). \tag{6.4}$$

As usual we extend $u_n = 0$ outside $B_n(0) \setminus \overline{B}_{1/n}(0)$. By standard elliptic regularity and a diagonal process, up to a subsequence there exists

$$u_{a,b}(x) := \lim_{n\to\infty} u_n(x), \quad x \in \mathbb{R}^N \setminus \{0\}$$

and $u_{a,b}$ is a solution of problem (1.11). Furthermore, from (6.4) we deduce that $u_{a,b}$ satisfies

$$a|x|^{2-N} + b \le u_{a,b}(x) \le a|x|^{2-N} + b + \xi(x) \quad \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{6.5}$$

Now, (6.1) and (6.5) imply (1.12).

Step 3: Proof of (ii).

Let u be a solution of (1.11). By Lemma 2.4 in [10] (see also Brezis and Lions [2]) we have $u \in L^1_{loc}(\mathbb{R}^N)$ so there exists $a \ge 0$ such that

$$\Delta u + \varphi(|x|)u^{-p} + a\delta(0) = 0 \quad \text{ in } \mathcal{D}'(\mathbb{R}^N),$$

where $\delta(0)$ denotes the Dirac mass concentrated at zero. Now, by the representation formula in [3, Theorem 2.4] we have

$$u(x) = a|x|^{2-N} + b + C(N)\int_{\mathbb{R}^N} \frac{\varphi(|y|)f(u(y))}{|x-y|^{N-2}}dy \quad \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Since ξ is also a solution of (1.11) that satisfies (6.1) we have

$$\xi(x) = C(N)\int_{\mathbb{R}^N} \frac{\varphi(|y|)f(\xi(y))}{|x-y|^{N-2}}dy \quad \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Using now the monotonicity of f we deduce

$$a|x|^{2-N} + b \le u \le a|x|^{2-N} + b + \xi \quad \text{ in } \mathbb{R}^N \setminus \{0\}.$$

This implies $\lim_{|x|\to 0} |x|^{N-2}u(x) = a$ and $\lim_{|x|\to\infty} u(x) = b$.

Let now $u_{a,b}$ be the solution of (1.11) that satisfies (1.12). We claim that $u \equiv u_{a,b}$. To this aim, for $\varepsilon > 0$ define

$$u_\varepsilon(x) := u(x) + \varepsilon(|x|^{2-N} + 1), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Then, we can find $R = R(\varepsilon) > 0$ such that $u_\varepsilon \ge u_{a,b}$ if $|x| > R$ or $0 < |x| < 1/R$. By means of Lemma 2.1 the same inequality is true in $B_R(0) \setminus B_{1/R}(0)$, so $u_\varepsilon \ge u_{a,b}$ in $\mathbb{R}^N \setminus \{0\}$. Passing now to the limit with $\varepsilon \to 0$ it follows that $u \ge u_{a,b}$ in $\mathbb{R}^N \setminus \{0\}$. In the same way we obtain the reverse inequality so $u \equiv u_{a,b}$. This finishes the proof of our Theorem 1.6. □

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