Blow-up of Solutions of Nonlinear Parabolic Inequalities

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Abstract

We study nonnegative solutions $u(x,t)$ of the nonlinear parabolic inequalities

$$au^\lambda \le u_t - \Delta u \le u^\lambda$$

in various subset of $\mathbf{R}^n \times \mathbf{R}$, where $\lambda > \frac{n+2}{n}$ and $a \in (0,1)$ are constants. We show that changing the value of a in the *open* interval $(0,1)$ can dramatically affect the blow-up of these solutions.

1 Introduction

In this paper, we study nonnegative solutions $u(x,t)$ of the nonlinear parabolic inequalities

$$au^\lambda \le u_t - \Delta u \le u^\lambda \tag{1.1}$$

in various subsets of $\mathbf{R}^n \times \mathbf{R}$, where $a \in (0,1)$ is a constant and $n \ge 1$ is an integer.

In order to state our results, we define $|(x,t)|$ for $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ by

$$|(x,t)| = \max\{|x|, |t|^{1/2}\} \tag{1.2}$$

where $|x|$ is the usual Euclidean norm of x in $\mathbf{R}^n$, and we define

$$\lambda_B = \begin{cases} \frac{n+2}{n}(\frac{n}{n-1})^2, & \text{if } n \ge 2 \\ \infty, & \text{if } n = 1. \end{cases}$$

Note that $\lambda_B > \frac{n+2}{n}$.

Our result on the blow-up at the origin of nonnegative solutions of (1.1) is

Theorem 1. *Suppose $\lambda > \frac{n+2}{n}$. Then there exists $a = a(n,\lambda) \in (0,1)$ and $C = C(n,\lambda) \in (1,\infty)$ such that for each continuous function*

$$\varphi\colon (0,1) \to (0,\infty) \qquad (\text{resp. } \varphi\colon (-1,0) \to (0,\infty))$$

there exists a C^∞ positive solution $u(x,t)$ of (1.1) in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$ such that

$$u(0,t) \ne O(\varphi(t)) \quad \textit{as} \quad t \to 0^+ \quad (\text{resp. } t \to 0^-)$$

and $|(x,t)|^{\frac{2}{\lambda-1}} u(x,t)$ is bounded between $1/C$ and C in the region

$$(\mathbf{R}^n \times \mathbf{R}) - \{(x,t)\colon |x|^2 \le t \le 1\} \qquad (\text{resp. } (\mathbf{R}^n \times \mathbf{R}) - \{(x,t)\colon -1 \le t \le -|x|^2\}).$$

Theorem 1 is in strong contrast to the following result of Poláčik, Quittner, and Souplet [11, 15].

Theorem 2. *Suppose* $1 < \lambda < \lambda_B$. *Then there exists* $a = a(n,\lambda) \in (0,1)$ *and* $C = C(n,\lambda) \in (1,\infty)$ *such that if* $u(x,t)$ *is a* $C^{2,1}$ *nonnegative solution of* (1.1) *in*

$$B_2(0) \times (0,2) \qquad (\text{resp. } B_2(0) \times (-2,0))$$

then $u(x,t) \le C|t|^{\frac{-1}{\lambda-1}}$ *for*

$$(x,t) \in B_1(0) \times (0,1) \qquad (\text{resp. } (x,t) \in B_1(0) \times (-1,0)).$$

Our result on the blow-up at $t = \pm\infty$ of nonnegative solutions of (1.1) is

Theorem 3. *Suppose* $\lambda > \frac{n+2}{n}$. *Then there exists* $a = a(n,\lambda) \in (0,1)$ *and* $C = C(n,\lambda) \in (1,\infty)$ *such that for each continuous function*

$$\varphi\colon (1,\infty) \to (0,\infty) \qquad (\text{resp. } \varphi\colon (-\infty,-1) \to (0,\infty))$$

there exists a C^∞ *positive solution* $u(x,t)$ *of* (1.1) *in* $\mathbf{R}^n \times \mathbf{R}$ *such that*

$$u(0,t) \ne O(\varphi(t)) \quad \text{as} \quad t \to \infty \quad (\text{resp. } t \to -\infty)$$

and

$$(1 + |(x,t)|)^{\frac{2}{\lambda-1}} u(x,t)$$

is bounded between $1/C$ *and* C *in the region*

$$\{(x,t)\colon t < |x|^2\} \qquad (\text{resp. } \{(x,t)\colon t > -|x|^2\}).$$

Theorem 3 is in strong contrast to the following result of Poláčik, Quittner, and Souplet [11, 15].

Theorem 4. *Suppose* $1 < \lambda < \lambda_B$. *Then there exists* $a = a(n,\lambda) \in (0,1)$ *and* $C = C(n,\lambda) \in (1,\infty)$ *such that if* $u(x,t)$ *is a* $C^{2,1}$ *nonnegative solution of* (1.1) *in*

$$\{(x,t)\colon t > |x|^2\} \qquad (\text{resp. } \{(x,t)\colon t < -|x|^2\})$$

then $u(x,t) \le C|t|^{\frac{-1}{\lambda-1}}$ *for*

$$(x,t) \in B_1(0) \times (2,\infty) \qquad (\text{resp. } (x,t) \in B_1(0) \times (-\infty,-2)).$$

When $\frac{n+2}{n} < \lambda < \lambda_B$, these four theorems show that changing the value of a in the *open* interval $(0,1)$ can dramatically affect the blow-up of positive solutions of (1.1).

Theorem 1 is not true when $\lambda \le \frac{n+2}{n}$. In fact, we prove in [17] that if $u(x,t)$ is a $C^{2,1}$ nonnegative solution of the parabolic inequalities

$$0 \le u_t - \Delta u \le u^{\frac{n+2}{n}} + 1$$

in a punctured neighborhood of the origin in $\mathbf{R}^n \times [0,\infty)$ then

$$u(x,t) = O(t^{-n/2}) \quad \text{as} \quad (x,t) \to (0,0),\ t > 0.$$

If $\lambda > \frac{n+2}{n}$, then by Theorem 1, there exists $a \in (0,1)$ such that (1.1) has $C^{2,1}$ positive solutions in $B_1(0) \times (0,1)$ which are arbitrarily large as (x,t) approaches $(0,0)$ along the positive t-axis. Let $I_1 = I_1(n,\lambda)$ be the set of all such a.

If $1 < \lambda < \lambda_B$, then by Theorem 2, there exists $a \in (0,1)$ such that every $C^{2,1}$ positive solution $u(x,t)$ of (1.1) in $B_1(0) \times (0,1)$ satisfies

$$u(0,t) = O(t^{\frac{-1}{\lambda-1}}) \quad \text{as} \quad t \to 0^+.$$

Let $I_2 = I_2(n,\lambda)$ be the set of all such a.

An interesting open question is whether

$$I_1(n,\lambda) \cup I_2(n,\lambda) = (0,1) \quad \text{for all} \quad \lambda \in (\frac{n+2}{n}, \lambda_B) \text{ and } n \geq 1.$$

If not, how do the $C^{2,1}$ positive solutions of (1.1) in $B_1(0) \times (0,1)$ behave as (x,t) approaches the origin along the positive t-axis when $a \in (0,1) - (I_1 \cup I_2)$? A similar question can be asked about Theorems 3 and 4. These questions seem to be very difficult.

The blow-up of solutions of the *equation*

$$u_t - \Delta u = u^\lambda \tag{1.3}$$

has been extensively studied in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 18] and elsewhere. See [13] and [5] for a summary of many of these results. However, other than [15], we know of no previous results for the *inequalities* (1.1). When $\frac{n+2}{n} < \lambda < \lambda_B$, our results show that it is more appropriate to study the *inequalities* (1.1) rather than the *equation* (1.3).

An elliptic analog of the results in this paper can be found in [16].

2 Preliminary results

In this section, we introduce some notation and obtain some results that will be used in Sections 3 and 4 to prove Theorems 1 and 3, respectively.

Lemma 1. *Let f be a C^∞ nonnegative function in an open subset Ω of $\mathbf{R}^n \times \mathbf{R}$ and define*

$$u(x,t) := \iint\limits_{\Omega} \Phi(x-y, t-s) f(y,s)\, dy\, ds \quad \text{for} \quad (x,t) \in \Omega \tag{2.1}$$

where

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & \text{for } t > 0 \\ 0, & \text{for } t \leq 0 \end{cases} \tag{2.2}$$

is the heat kernel. If $u \in L^1_{\mathrm{loc}}(\Omega)$ then u is C^∞ in Ω and $Hu = f$ in Ω where $Hu = u_t - \Delta u$ is the heat operator.

Proof. Let $\psi \in C_0^\infty(\Omega)$. Multiplying (2.1) by $H^*\psi := \psi_t + \Delta\psi$, integrating the resulting equation over Ω, and using Fubini's theorem and the fact that $H\Phi = \delta$, we see that $Hu = f$ in $\mathcal{D}'(\Omega)$. Thus by standard parabolic regularity theory, $u \in C^\infty(\Omega)$. ☐

If $(x,t), (y,s) \in \mathbf{R}^n \times \mathbf{R}$ and $c \in \mathbf{R}$, then it follows from (1.2) that

$$|(x,t) + (y,s)| \leq |(x,t)| + |(y,s)|$$

and $|(cx, c^2t)| = |c|\ |(x,t)|$.

Throughout this section we assume $\lambda > \frac{n+2}{n}$, which implies

$$n > \frac{2}{\lambda - 1} \quad \text{and} \quad 2 < \frac{2\lambda}{\lambda - 1} < n + 2. \tag{2.3}$$

Define $W\colon (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\} \to \mathbf{R}$ by

$$W(y,s) = (|y|^4 + s^2)^{-\frac{1}{2(\lambda-1)}}.$$

Then W is C^∞ on $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$ and

$$W(y,s) \sim |(y,s)|^{-\frac{2}{\lambda-1}} \quad \text{for} \quad 0 < |(y,s)| < \infty. \tag{2.4}$$

(Here and later the notation $X \sim Y$ (resp. $X \lesssim Y$) means $\frac{1}{C} Y \le X \le CY$ (resp. $X \le CY$) for some positive constant C which depends *only* on n and λ.)

Define $W_0\colon (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\} \to \mathbf{R}$ by

$$W_0 = \varphi W \tag{2.5}$$

where $\varphi\colon \mathbf{R}^n \times \mathbf{R} \to [0,1]$ is a C^∞ function satisfying $\varphi(y,s) = 1$ for $|(y,s)| \le 1$ and

$$\varphi(y,s) = 0 \quad \text{for} \quad |(y,s)| \ge \sqrt{\frac{3}{2}}. \tag{2.6}$$

Define $w, w_0\colon (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\} \to \mathbf{R}$ by

$$w(x,t) = \iint\limits_{\mathbf{R}^n \times \mathbf{R}} \Phi(x-y, t-s) W(y,s)^\lambda \, dy \, ds \tag{2.7}$$

and

$$w_0(x,t) = \iint\limits_{\mathbf{R}^n \times \mathbf{R}} \Phi(x-y, t-s) W_0(y,s)^\lambda \, dy \, ds.$$

It follows from (2.3), (2.4), and (2.5) that w and w_0 are locally bounded in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$. Thus by Lemma 1, w and w_0 are C^∞ in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$, $Hw = W^\lambda$ and $Hw_0 = W_0^\lambda$ in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$, and

$$\begin{aligned} 0 \le w(x,t) - w_0(x,t) &\lesssim \int\limits_{|(y,s)| \ge 1} \Phi(x-y, t-s) |(y,s)|^{-\frac{2\lambda}{\lambda-1}} \, dy \, ds \\ &\lesssim 1 \quad \text{for} \quad 0 < |(x,t)| \le 1. \end{aligned} \tag{2.8}$$

Lemma 2. *The functions $w, W, w_0,$ and W_0 satisfy*

$$\frac{w(x,t)}{W(x,t)} \sim 1 \quad \textit{for} \quad 0 < |(x,t)| < \infty, \tag{2.9}$$

$$\frac{w_0(x,t)}{W_0(x,t)} \sim 1 \quad \textit{for} \quad 0 < |(x,t)| \le 1, \tag{2.10}$$

and

$$\frac{w_0(x,t)}{|(x,t)|^{-n}} \lesssim 1 \quad \textit{and} \quad \frac{W_0(x,t)}{|(x,t)|^{-n}} \lesssim 1 \quad \textit{for} \quad 1 \le |(x,t)| < \infty. \tag{2.11}$$

Proof. Making in (2.7) the change of variables

$$\begin{array}{ll} x = c\xi & t = c^2\tau \\ y = c\eta & s = c^2\zeta \end{array} \quad \text{where } c \text{ is a positive constant} \tag{2.12}$$

we get

$$w(x,t) = c^{-\frac{2}{\lambda-1}} w(\xi,\tau). \tag{2.13}$$

Taking $c = |(x,t)|$ in (2.12) we find

$$|(\xi,\tau)| = \left|\left(\frac{1}{c}x, \frac{1}{c^2}t\right)\right| = \frac{1}{c}|(x,t)| = 1 \tag{2.14}$$

and hence (2.9) follows from (2.13) and (2.4).

It follows from (2.5), (2.4) and (2.8) that for $0 < |(x,t)| \le 1$ we have

$$\begin{aligned} \left|\frac{w_0(x,t)}{W_0(x,t)} - \frac{w(x,t)}{W(x,t)}\right| &= \frac{|w_0(x,t) - w(x,t)|}{W(x,t)} \\ &\lesssim |(x,t)|^{\frac{2}{\lambda-1}} \to 0 \quad \text{as} \quad |(x,t)| \to 0. \end{aligned}$$

Thus (2.10) follows from (2.9) and from the continuity and positivity of w_0 and W_0 on $0 < |(x,t)| \le 1$.

Taking $c = |(x,t)| \ge 4$ in (2.12) we have (2.14) holds, $\frac{2}{|(x,t)|} \le \frac{1}{2}$, and

$$\begin{aligned} \frac{w_0(x,t)}{|(x,t)|^{-n}} &\lesssim |(x,t)|^n \iint\limits_{|(y,s)|\le 2} \Phi(x-y,t-s)|(y,s)|^{-\frac{2\lambda}{\lambda-1}}\,dy\,ds \\ &= |(x,t)|^{n-\frac{2}{\lambda-1}} \iint\limits_{|(\eta,\zeta)|\le 2/|(x,t)|} \Phi(\xi-\eta,\tau-\zeta)|(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta \\ &\lesssim |(x,t)|^{n-\frac{2}{\lambda-1}} \iint\limits_{|(\eta,\zeta)|\le 2/|(x,t)|} |(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta \sim 1. \end{aligned}$$

Thus the first inequality of (2.11) follows from the continuity of $w_0(x,t)$ for $1 \le |(x,t)| \le 4$. The second inequality of (2.11) follows from (2.5) and (2.4). □

For $0 < r \le \frac{1}{2}$, define $W_r \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$W_r(y,s) = (|y|^4 + s^2 + r^4)^{-\frac{1}{2(\lambda-1)}}\varphi(y,s)$$

where φ is the function in (2.5). Then

$$W_r(y,s) \sim r^{-\frac{2}{\lambda-1}}, \quad \text{for} \quad 0 \le |(y,s)| \le r \tag{2.15}$$

$$W_r(y,s) \sim W_0(y,s), \quad \text{for} \quad r \le |(y,s)| < \infty. \tag{2.16}$$

Recall that according to our definition of $X \sim Y$ after equation (2.4), the constants C for the relations (2.15) and (2.16) above and the relations (2.18) and (2.19) below do *not* depend on r.

For $0 < r \le \frac{1}{2}$, define $w_r \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$w_r(x,t) = \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s)W_r(y,s)^\lambda\,dy\,ds. \tag{2.17}$$

It follows from Lemma 1 that w_r is C^∞ in $\mathbf{R}^n \times \mathbf{R}$ and $Hw_r = W_r^\lambda$.

Lemma 3. *For $0 < r \le \frac{1}{2}$ we have*

$$\frac{w_r(x,t)}{W_r(x,t)} \sim 1, \quad \textit{for} \quad 0 \le |(x,t)| \le 1 \tag{2.18}$$

and

$$\frac{w_r(x,t) + W_r(x,t)}{|(x,t)|^{-n}} \lesssim 1, \quad \textit{for} \quad 1 \le |(x,t)| < \infty. \tag{2.19}$$

Proof. It follows from (2.4), (2.5), (2.15), (2.16), and (2.17) that

$$w_r(x,t) \sim I_r(x,t) + J_r(x,t) + K_r(x,t) \quad \text{for} \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}$$

where

$$\begin{aligned} I_r(x,t) &= r^{-\frac{2\lambda}{\lambda-1}} \iint\limits_{|(y,s)|<r} \Phi(x-y,t-s)\,dy\,ds \\ J_r(x,t) &= \iint\limits_{r<|(y,s)|<1} \Phi(x-y,t-s)|(y,s)|^{-\frac{2\lambda}{\lambda-1}}\,dy\,ds \\ K_r(x,t) &= \iint\limits_{1<|(y,s)|<2} \Phi(x-y,t-s)\varphi(y,s)|(y,s)|^{-\frac{2\lambda}{\lambda-1}}\,dy\,ds. \end{aligned} \tag{2.20}$$

For $|(x,t)| \le r \in (0,\frac{1}{2}]$ we have

$$\begin{aligned} \frac{I_r(x,t)}{r^{-\frac{2}{\lambda-1}}} &= r^{-2} \iint\limits_{|(y,s)|<r} \Phi(x-y,t-s)\,dy\,ds \le r^{-2} \int_{-r^2}^{r^2} \left(\int\limits_{y\in\mathbf{R}^n} \Phi(x-y,t-s)dy \right) ds \\ &\le r^{-2} 2r^2 = 2. \end{aligned} \tag{2.21}$$

Making in (2.20) the change of variables (2.12) with $c = r \in (0,\frac{1}{2}]$, we get

$$\frac{J_r(x,t)}{r^{-\frac{2}{\lambda-1}}} = \widehat{J}_r\left(\frac{x}{r}, \frac{t}{r^2}\right) \tag{2.22}$$

where

$$\widehat{J}_r(\xi,\tau) = \iint\limits_{1\le|(\eta,\zeta)|\le\frac{1}{r}} \Phi(\xi-\eta,\tau-\zeta)|(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta.$$

For $|(x,t)| \le r \in (0,\frac{1}{2}]$ we have $|(\frac{x}{r},\frac{t}{r^2})| \le 1$. Also

$$\begin{aligned} \sup_{|(\xi,\tau)|\le 1} \widehat{J}_r(\xi,\tau) &\le \sup_{|(\xi,\tau)|\le 1} \iint\limits_{1\le|(\eta,\zeta)|\le\infty} \Phi(\xi-\eta,\tau-\zeta)|(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta \\ &= C(n,\lambda) < \infty, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \inf_{|(\xi,\tau)|\le 1} \widehat{J}_r(\xi,\tau) &\ge \inf_{|(\xi,\tau)|\le 1} \iint\limits_{1<|(\eta,\zeta)|<2} \Phi(\xi-\eta,\tau-\zeta)|(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta \\ &\ge C(n,\lambda) > 0. \end{aligned} \tag{2.24}$$

For $r \in (0, \frac{1}{2}]$ and $(x,t) \in \mathbf{R}^n \times \mathbf{R}$, we have

$$\frac{K_r(x,t)}{r^{-\frac{2}{\lambda-1}}} \le \iint\limits_{1<|(y,s)|<2} \Phi(x-y,t-s)\,dy\,ds \le 8. \tag{2.25}$$

Combining (2.21)–(2.25) and using (2.15) we obtain for $r \in (0, \frac{1}{2}]$ that

$$\frac{w_r(x,t)}{W_r(x,t)} \sim \frac{w_r(x,t)}{r^{-\frac{2}{\lambda-1}}} \sim 1 \quad \text{for} \quad |(x,t)| < r.$$

Since for $|(x,t)| \ge r$, $W_r(x,t) \sim W_0(x,t)$ and $w_r(x,t) \le w_0(x,t)$, it follows from Lemma 2 that to complete the proof of Lemma 3 we only need to show

$$\frac{w_r(x,t)}{W_r(x,t)} \gtrsim 1 \quad \text{for} \quad r \le |(x,t)| \le 1. \tag{2.26}$$

To do this, we make in (2.20) the change of variables (2.12) with $c = |(x,t)| \in [r, \frac{1}{2}]$ to get (2.14) and

$$\begin{aligned}\frac{w_r(x,t)}{W_r(x,t)} \gtrsim \frac{J_r(x,t)}{c^{-\frac{2}{\lambda-1}}} &= \iint\limits_{\frac{r}{c}\le|(\eta,\zeta)|\le\frac{1}{c}} \Phi(\xi-\eta,\tau-\zeta)|(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta \\ &\ge \min_{|(\xi,\tau)|=1} \iint\limits_{1\le|(\eta,\zeta)|\le 2} \Phi(\xi-\eta,\tau-\zeta)|(\eta,\zeta)|^{-\frac{2\lambda}{\lambda-1}}\,d\eta\,d\zeta = C(n,\lambda) > 0. \end{aligned}\tag{2.27}$$

Also, for $r \in (0, \frac{1}{2}]$ and $|(x,t)| \in [\frac{1}{2}, 1]$ we have

$$\frac{w_r(x,t)}{W_r(x,t)} \sim w_r(x,t) \gtrsim K_r(x,t) \ge C(n,\lambda) > 0. \tag{2.28}$$

Relation (2.26) follows from (2.27) and (2.28). $\square$

For $0 < r \le \frac{1}{2}$ and $h > 0$, define $W_{h,r} \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$W_{h,r}(y,s) = h^{-\frac{2}{\lambda-1}} W_r\left(\frac{y}{h}, \frac{s}{h^2}\right). \tag{2.29}$$

It follows from (2.5), (2.15), (2.16), and (2.6) that

$$W_{h,r}(y,s) \sim (hr)^{-\frac{2}{\lambda-1}}, \qquad \text{for} \quad 0 \le |(y,s)| \le hr \tag{2.30}$$

$$W_{h,r}(y,s) \sim |(y,s)|^{-\frac{2}{\lambda-1}}, \qquad \text{for} \quad hr \le |(y,s)| \le h \tag{2.31}$$

$$W_{h,r}(y,s) \sim |(y,s)|^{-\frac{2}{\lambda-1}} \varphi\left(\frac{y}{h}, \frac{s}{h^2}\right), \quad \text{for} \quad h \le |(y,s)| \le \sqrt{\frac{3}{2}}\,h \tag{2.32}$$

$$W_{h,r}(y,s) = 0, \qquad \text{for} \quad |(y,s)| \ge \sqrt{\frac{3}{2}}\,h. \tag{2.33}$$

For $0 < r \le \frac{1}{2}$ and $h > 0$, define $w_{h,r} \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$w_{h,r}(x,t) = \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s) W_{h,r}(y,s)^\lambda\,dy\,ds. \tag{2.34}$$

Making in (2.34) the change of variables (2.12) with $c = h$ and using (2.17) and (2.29), we get

$$w_{h,r}(x,t) = h^{-\frac{2}{\lambda-1}} w_r\left(\frac{x}{h}, \frac{t}{h^2}\right). \tag{2.35}$$

The following lemma follows immediately from Lemma 3 and equations (2.29) and (2.35).

Lemma 4. *For $0 < r \le \frac{1}{2}$ and $h > 0$ we have*

$$\begin{aligned} &\frac{w_{h,r}(x,t)}{W_{h,r}(x,t)} \sim 1 && \textit{for} \quad 0 \le |(x,t)| \le h \\ &\frac{w_{h,r}(x,t) + W_{h,r}(x,t)}{h^{n-\frac{2}{\lambda-1}}|(x,t)|^{-n}} \lesssim 1 && \textit{for} \quad h \le |(x,t)| < \infty. \end{aligned}$$

Define $\widehat{W}, \widehat{w}\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$\begin{aligned} \widehat{W}(y,s) &= (|y|^4 + s^2 + 1)^{-\frac{1}{2(\lambda-1)}} \\ \widehat{w}(x,t) &= \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s)\widehat{W}(y,s)^\lambda \, dy\, ds. \end{aligned} \tag{2.36}$$

Then

$$\widehat{W}(y,s) \sim (1 + |(y,s)|)^{-\frac{2}{\lambda-1}} \quad \text{for} \quad (y,s) \in \mathbf{R}^n \times \mathbf{R} \tag{2.37}$$

which implies

$$\widehat{W}(y,s) \sim 1 \quad \text{for} \quad 0 \le |(y,s)| \le 1$$

and

$$\widehat{W}(y,s) \sim |(y,s)|^{-\frac{2}{\lambda-1}} \quad \text{for} \quad |(y,s)| \ge 1 \tag{2.38}$$

and it follows from (2.3) and Lemma 1 that $\widehat{w}$ is C^∞ in $\mathbf{R}^n \times \mathbf{R}$ and $H\widehat{w} = \widehat{W}^\lambda$.

For $|(x,t)| \ge 1$, we obtain from (2.4) and (2.9) that

$$|(x,t)|^{\frac{2}{\lambda-1}} \widehat{w}(x,t) \le |(x,t)|^{\frac{2}{\lambda-1}} w(x,t) \sim \frac{w(x,t)}{W(x,t)} \sim 1$$

and making the change of variables (2.12) with $c = |(x,t)| \ge 1$ and using (2.14) and (2.38) we get

$$\begin{aligned} |(x,t)|^{\frac{2}{\lambda-1}} \widehat{w}(x,t) &\gtrsim |(x,t)|^{\frac{2}{\lambda-1}} \iint\limits_{|(y,s)|\ge 2} \Phi(x-y,t-s) \frac{1}{|(y,s)|^{\frac{2\lambda}{\lambda-1}}} \, dy\, ds \\ &= \iint\limits_{|(\eta,\zeta)|\ge \frac{2}{|(x,t)|}} \Phi(\xi-\eta,\tau-\zeta) \frac{1}{|(\eta,\zeta)|^{\frac{2\lambda}{\lambda-1}}} \, d\eta\, d\zeta \\ &\ge \min_{|(\xi,\tau)|=1} \iint\limits_{2\le|(\eta,\zeta)|\le 3} \Phi(\xi-\eta,\tau-\zeta) \frac{1}{|(\eta,\zeta)|^{\frac{2\lambda}{\lambda-1}}} \, d\eta\, d\zeta \\ &= C(n,\lambda) > 0. \end{aligned}$$

So $\widehat{w}(x,t) \sim |(x,t)|^{-\frac{2}{\lambda-1}}$ for $|(x,t)| \geq 1$, and thus by (2.38),

$$\left(\frac{\widehat{w}^\lambda}{H\widehat{w}}\right)^{1/\lambda} = \frac{\widehat{w}}{\widehat{W}} \sim 1 \quad \text{in} \quad \mathbf{R}^n \times \mathbf{R}. \tag{2.39}$$

For $0 < r \leq \frac{1}{2}$ and $h > 0$, define $V^+_{h,r}, V^-_{h,r} \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$V^+_{h,r}(x,t) = W_{h,r}((x,t) - (0,2h^2))$$
$$V^-_{h,r}(x,t) = W_{h,r}((x,t) + (0,2h^2)).$$

We abbreviate these last two equations by writing

$$V^\pm_{h,r}(x,t) = W_{h,r}((x,t) \mp (0,2h^2))$$

and in what follows we abbreviate other pairs of equations in a similar way.

For $0 < r \leq \frac{1}{2}$ and $h > 0$, define $v^\pm_{h,r} \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by

$$v^\pm_{h,r}(x,t) = \iint\limits_{\mathbf{R}^n \times \mathbf{R}} \Phi(x-y, t-s) V^\pm_{h,r}(y,s)^\lambda \, dy \, ds. \tag{2.40}$$

Then

$$Hv^\pm_{h,r} = (V^\pm_{h,r})^\lambda \tag{2.41}$$

and by (2.34)

$$v^\pm_{h,r}(x,t) = w_{h,r}((x,t) \mp (0,2h^2)).$$

Thus the following lemma follows directly from Lemma 4.

Lemma 5. *For $0 < r \leq \frac{1}{2}$ and $h > 0$ we have*

$$\frac{v^\pm_{h,r}(x,t)}{V^\pm_{h,r}(x,t)} \sim 1, \qquad \textit{for} \quad |(x,t) \mp (0,2h^2)| \leq h$$
$$\frac{v^\pm_{h,r}(x,t) + V^\pm_{h,r}(x,t)}{h^{n-\frac{2}{\lambda-1}} |(x,t) \mp (0,2h^2)|^{-n}} \lesssim 1, \quad \textit{for} \quad |(x,t) \mp (0,2h^2)| \geq h.$$

Lemma 5 and equations (2.30), (2.31), (2.4), and (2.9) imply

$$v^\pm_{h,r} \sim V^\pm_{h,r} \gtrsim h^{-\frac{2}{\lambda-1}} \sim |(x,t)|^{-\frac{2}{\lambda-1}} \sim W \sim w \quad \text{for} \quad |(x,t) \mp (0,2h^2)| \leq h. \tag{2.42}$$

Since for $|(x,t) \mp (0,2h^2)| \geq h$,

$$\frac{h^{n-\frac{2}{\lambda-1}} |(x,t) \mp (0,2h^2)|^{-n}}{|(x,t)|^{-\frac{2}{\lambda-1}}} \lesssim \min\left\{ \left(\frac{h}{|(x,t)|}\right)^{n-\frac{2}{\lambda-1}}, \left(\frac{|(x,t)|}{h}\right)^{\frac{2}{\lambda-1}} \right\},$$

it follows from Lemma 5 that

$$\frac{v^\pm_{h,r}(x,t) + V^\pm_{h,r}(x,t)}{|(x,t)|^{-\frac{2}{\lambda-1}}} \lesssim \min\left\{ \left(\frac{h}{|(x,t)|}\right)^{n-\frac{2}{\lambda-1}}, \left(\frac{|(x,t)|}{h}\right)^{\frac{2}{\lambda-1}} \right\} \tag{2.43}$$

for $|(x,t) \mp (0,2h^2)| \geq h$.

Let $h_j = 3^j$ for $j \in \mathbf{Z}$. Let $\varphi\colon (-\infty,0)\cup(0,\infty) \to (0,\infty)$ be a continuous function. (There is no loss of generality in assuming the functions φ in Theorems 1 and 3 are all positive and continuous on the larger set $(-\infty,0)\cup(0,\infty)$.) Choose $r_j \in (0,\frac{1}{2}]$ such that

$$\frac{(h_j r_j)^{-\frac{2}{\lambda-1}}}{\varphi(\pm 2h_j^2)} \to \infty \quad \text{as} \quad |j| \to \infty. \tag{2.44}$$

Let $V_j^\pm = V_{h_j,r_j}^\pm$ and $v_j^\pm = v_{h_j,r_j}^\pm$. Since by (2.33) the support of $V_j^\pm$ is contained in $R_{\sqrt{\frac{3}{2}}h_j}(0,\pm 2h_j^2)$ where

$$R_h(x,y) := \{(y,s) \in \mathbf{R}^n \times \mathbf{R}\colon |(y,s)-(x,t)| \le h\},$$

we see that the functions $V_j^\pm$, $j \in \mathbf{Z}$, have disjoint supports. By Lemma 5 and equation (2.30),

$$v_j^\pm(0,\pm 2h_j^2) \sim V_j^\pm(0,\pm 2h_j^2) = W_{h_j,r_j}(0,0) \sim (h_j,r_j)^{-\frac{2}{\lambda-1}} \tag{2.45}$$

for $j \in \mathbf{Z}$.

If A is any subset of $\mathbf{Z}$ and $R_j^\pm = R_{h_j}(0,\pm 2h_j^2)$ it follows from (2.43) that for $(x,t) \notin \bigcup_{j\in A} R_j^\pm$ and $(x,t) \neq (0,0)$ we have

$$\begin{aligned}\frac{\sum_{j\in A} v_j^\pm(x,t) + V_j^\pm(x,t)}{|(x,t)|^{-\frac{2}{\lambda-1}}} &\lesssim \sum_{h_j \le |(x,t)|} \left(\frac{h_j}{|(x,t)|}\right)^{n-\frac{2}{\lambda-1}} + \sum_{h_j \ge |(x,t)|} \left(\frac{|(x,t)|}{h_j}\right)^{\frac{2}{\lambda-1}} \\ &\lesssim 1 \end{aligned}\tag{2.46}$$

and thus, since the functions $V_j^\pm$ have disjoint supports,

$$\frac{\sum_{j\in A} V_j^\pm(x,t)^\lambda}{|(x,t)|^{-\frac{2\lambda}{\lambda-1}}} = \left(\frac{\sum_{j\in A} V_j^\pm(x,t)}{|(x,t)|^{-\frac{2}{\lambda-1}}}\right)^\lambda \lesssim 1 \tag{2.47}$$

for $(x,t) \notin \bigcup_{j\in A} R_j^\pm$ and $(x,t) \neq (0,0)$.

3 Proof of Theorem 1

In this section, we use the notation and results in Section 2 to prove Theorem 1.

Proof of Theorem 1. Since the functions $V_j^\pm$, $j \in \mathbf{Z}$, are C^∞ and have disjoint support, $\sum_{j\le -1} (V_j^\pm)^\lambda$ converges in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$ to a C^∞ function. It follows from the monotone convergence theorem and (2.40) that

$$\begin{aligned} v^\pm(x,t) &:= \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s)\left(\sum_{j\le -1} V_j^\pm(y,s)^\lambda\right) dy\,ds \\ &= \sum_{j\le -1} v_j^\pm(x,t) \\ &\lesssim \begin{cases} |(x,t)|^{-\frac{2}{\lambda-1}}, & \text{if } (x,t) \notin \bigcup_{j\le -1} R_j^\pm \\ v_{j_0}^\pm(x,t) + |(x,t)|^{-\frac{2}{\lambda-1}}, & \text{if } (x,t) \in R_{j_0}^\pm \text{ for some } j_0 \le -1 \end{cases} \end{aligned}\tag{3.1}$$

by (2.46). Thus $v^{\pm}$ is bounded on compact subsets of $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$, and so by Lemma 1, $v^{\pm}$ is C^∞ on $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$ and

$$Hv^{\pm} = \sum_{j \le -1} (V_j^{\pm})^\lambda = \sum_{j \le -1} Hv_j^{\pm}$$

by (2.41).

Define $u^{\pm} \colon (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\} \to \mathbf{R}$ by $u^{\pm} = w + v^{\pm}$ where w is given by (2.7). Then $u^{\pm}$ is C^∞ and

$$Hu^{\pm} = W^\lambda + \sum_{j \le -1} (V_j^{\pm})^\lambda. \tag{3.2}$$

We now show

$$Hu^{\pm} \sim (u^{\pm})^\lambda \quad \text{in} \quad (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}, \tag{3.3}$$

which after scaling $u^{\pm}$ if necessary, implies $u^{\pm}$ satisfies (1.1) in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$.

If $(x,t) \notin \bigcup_{j \le -1} R_j$ and $(x,t) \neq (0,0)$ then by (3.2), (2.4), (2.9), (2.47), and (2.46),

$$\begin{aligned} Hu^{\pm} &= W^\lambda \left[1 + \sum_{j \le -1} \left(\frac{V_j^{\pm}}{W} \right)^\lambda \right] \\ &\sim w^\lambda \left[1 + \sum_{j \le -1} \frac{(V_j^{\pm})^\lambda}{|(x,t)|^{-\frac{2\lambda}{\lambda-1}}} \right] \\ &\sim w^\lambda \sim w^\lambda \left(1 + \left(\frac{v^{\pm}}{w} \right)^\lambda \right) = (u^{\pm})^\lambda. \end{aligned}$$

If $(x,t) \in R_{j_0}$ for some $j_0 \le -1$ then by (3.2), (2.42), (2.46), and Lemma 5,

$$\begin{aligned} Hu^{\pm} &= (V_{j_0}^{\pm})^\lambda + W^\lambda \\ &= (V_{j_0}^{\pm})^\lambda \left[1 + \left(\frac{W}{V_{j_0}^{\pm}} \right)^\lambda \right] \\ &\sim (V_{j_0}^{\pm})^\lambda \sim (v_{j_0}^{\pm})^\lambda \\ &\sim (v_{j_0}^{\pm})^\lambda \left[1 + \sum_{\substack{j \le -1 \\ j \neq j_0}} \frac{v_j^{\pm}}{v_{j_0}^{\pm}} + \frac{w}{v_{j_0}^{\pm}} \right]^\lambda \\ &= (u^{\pm})^\lambda \end{aligned}$$

which proves (3.3).

It follows from (2.44) and (2.45) that $u^{\pm}(0,t) \neq O(\varphi(t))$ as $t \to 0^{\pm}$.

By (2.4), (2.9), and (3.1) we see that $|(x,t)|^{\frac{2}{\lambda-1}} u^{\pm}(x,t) \sim 1$ in the regions stated in Theorem 1.

Taking $u = u^+$ (resp. $u = u^-$), we obtain Theorem 1. ◻

4 Proof of Theorem 3

In this section, we use the notation and results in Section 2 to prove Theorem 3.

Proof of Theorem 3. Since the functions $V_j^\pm$, $j \in \mathbf{Z}$, are C^∞ and have disjoint support, $\sum_{j\geq 1} (V_j^\pm)^\lambda$ converges on $\mathbf{R}^n \times \mathbf{R}$ to a C^∞ function.

Let B be a subset of $\mathbf{N}$. If $|(x,t)| < 1$ then $(x,t) \notin \bigcup_{j\in B} R_j^\pm$ and it therefore follows from (2.43) that

$$\sum_{j\in B} v_j^\pm(x,t) + V_j^\pm(x,t) \lesssim \sum_{j\geq 1} h_j^{-\frac{2}{\lambda-1}} \sim 1.$$

Thus, by (2.46), we have for $(x,t) \notin \bigcup_{j\in B} R_j^\pm$ that

$$\sum_{j\in B} v_j^\pm(x,t) + V_j^\pm(x,t) \lesssim (1 + |(x,t)|)^{-\frac{2}{\lambda-1}}. \tag{4.1}$$

Hence, since the functions $V_j^\pm$ have disjoint support,

$$\sum_{j\in B} V_j^\pm(x,t)^\lambda \lesssim (1 + |(x,t)|)^{-\frac{2\lambda}{\lambda-1}} \tag{4.2}$$

for $(x,t) \notin \bigcup_{j\in B} R_j^\pm$.

It follows from the monotone convergence theorem and (2.40) that

$$\begin{aligned} v^\pm(x,t) &:= \iint_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s) \sum_{j\geq 1} V_j^\pm(y,s)^\lambda \, dy \, ds \\ &= \sum_{j\geq 1} v_j^\pm(x,t) \\ &\lesssim \begin{cases} (1 + |(x,t)|)^{-\frac{2}{\lambda-1}}, & \text{if } (x,t) \notin \bigcup_{j\geq 1} R_j^\pm \\ v_{j_0}(x,t) + (1 + |(x,t)|)^{-\frac{2}{\lambda-1}}, & \text{if } (x,t) \in R_{j_0}^\pm \text{ for some } j_0 \geq 1 \end{cases} \end{aligned} \tag{4.3}$$

by (4.1). Thus $v^\pm$ is bounded on compact subsets of $\mathbf{R}^n \times \mathbf{R}$ and so by Lemma 1, $v^\pm$ is C^∞ in $\mathbf{R}^n \times \mathbf{R}$ and

$$Hv^\pm = \sum_{j\geq 1} (V_j^\pm)^\lambda = \sum_{j\geq 1} Hv_j^\pm$$

by (2.41).

Define $u^\pm \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by $u^\pm = \widehat{w} + v^\pm$ where $\widehat{w}$ is given by (2.36). Then $u^\pm$ is C^∞ and

$$Hu^\pm = \widehat{W}^\lambda + \sum_{j\geq 1} (V_j^\pm)^\lambda. \tag{4.4}$$

We now show

$$Hu^\pm \sim (u^\pm)^\lambda \quad \text{in} \quad \mathbf{R}^n \times \mathbf{R}, \tag{4.5}$$

which after scaling $u^\pm$ if necessary, implies $u^\pm$ satisfies (1.1) in $\mathbf{R}^n \times \mathbf{R}$.

If $(x,t) \notin R_j^{\pm}$ then by (4.4), (4.2), (2.37), (2.39), and (4.1), we have

$$\begin{aligned} Hu^{\pm} &= \widehat{W}^{\lambda}\left[1+\sum_{j\geq 1}\left(\frac{V_j^{\pm}}{\widehat{W}}\right)^{\lambda}\right] \\ &\sim \widehat{w}^{\lambda} \sim \widehat{w}^{\lambda}\left[1+\left(\frac{v^{\pm}}{\widehat{w}}\right)\right]^{\lambda} = (u^{\pm})^{\lambda}. \end{aligned}$$

If $(x,t) \in R_{j_0}^{\pm}$ for some $j_0 \geq 1$ then $|(x,t)| \geq 1$ and so (2.42), (2.37), and (2.39) imply

$$v_{j_0}^{\pm} \sim V_{j_0}^{\pm} \geq h_{j_0}^{-\frac{2}{\lambda-1}} \sim |(x,t)|^{-\frac{2}{\lambda-1}} \sim (1+|(x,t)|)^{-\frac{2}{\lambda-1}} \sim \widehat{W} \sim \widehat{w}.$$

Hence, if $(x,t) \in R_{j_0}^{\pm}$ for some $j_0 \geq 1$ then by (4.4), (4.1) and Lemma 5, we have

$$\begin{aligned} Hu^{\pm} &= (V_{j_0}^{\pm})^{\lambda} + \widehat{W}^{\lambda} = (V_{j_0}^{\pm})^{\lambda}\left[1+\left(\frac{\widehat{W}}{V_{j_0}^{\pm}}\right)^{\lambda}\right] \\ &\sim (V_{j_0}^{\pm})^{\lambda} \sim (v_{j_0}^{\pm})^{\lambda} \sim (v_{j_0}^{\pm})^{\lambda}\left[1+\sum_{\substack{j\geq 1\\ j\neq j_0}}\frac{v_j^{\pm}}{v_{j_0}^{\pm}}+\frac{\widehat{w}}{v_{j_0}^{\pm}}\right]^{\lambda} \\ &= (u^{\pm})^{\lambda} \end{aligned}$$

which proves (4.5).

It follows from (2.44) and (2.45) that $u^{\pm}(0,t) \neq O(\varphi(t))$ as $t \to \pm\infty$. By (2.37), (2.39), and (4.3) we see that $(1+|(x,t)|)^{\frac{2}{\lambda-1}}u^{\pm}(x,t) \sim 1$ in the regions stated in Theorem 3.

Taking $u = u^{+}$ (resp. $u = u^{-}$), we obtain Theorem 3. □

5 Proofs of Theorems 2 and 4

Souplet [15] showed that the proof of Theorem 3.1 in [11] can be very slightly modified to prove the following theorem.

Theorem 5. *Suppose* $1 < \lambda < \lambda_B$ *and* D *is a proper open subset of* $\mathbf{R}^n \times \mathbf{R}$*. Then there exists* $a = a(n,\lambda) \in (0,1)$ *and* $C = C(n,\lambda) \in (1,\infty)$ *such that if* $u(x,t)$ *is a* $C^{2,1}$ *nonnegative solution of* (1.1) *in* D *then*

$$u(x,t) \leq C\left(\inf_{(y,s)\in\partial D}|(y,s)-(x,t)|\right)^{\frac{-2}{\lambda-1}} \quad \text{for all } (x,t) \in D.$$

Theorems 2 and 4 are immediate consequences of Theorem 5.

The proofs of Theorem 5 and [11, Theorem 3.1] rely heavily on the following Liouville-type result of Bidaut-Véron [3].

Theorem 6. *Suppose* $1 < \lambda < \lambda_B$*. Then the only* $C^{2,1}$ *nonnegative solution* $u(x,t)$ *of*

$$u_t - \Delta u = u^{\lambda} \quad \text{in} \quad \mathbf{R}^n \times \mathbf{R}$$

is $u \equiv 0$.

References

[1] D. Andreucci and E. DiBenedetto, On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **18**, 363–441 (1991).

[2] D. Andreucci, M. A. Herrero, and J. J. L. Velázquez, Liouville theorems and blow up behaviour in semilinear reaction diffusion systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14**, 1–53 (1997).

[3] M.-F. Bidaut-Véron, Initial blow-up for the solutions of a semilinear parabolic equation with source term. Équations aux dérivées partielles et applications, 189–198, Gauthier-Villars, Ed. Sci. Méd. Elsevier, Paris, 1998.

[4] Y. Giga and R. V. Kohn, Characterizing blowup using similarity variables, *Indiana Univ. Math. J.* **36** (1987), 1–40.

[5] Y. Giga, S. Matsui, and S. Sasayama, Blow up rate for semilinear heat equations with subcritical nonlinearity, *Indiana Univ. Math. J.* **53** (2004), 483–514.

[6] M. A. Herrero and J. J. L. Velázquez, Explosion de solutions d'équations paraboliques semilinéaires supercritiques, *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994), 141–145.

[7] O. Kavian, Remarks on the large time behaviour of a nonlinear diffusion equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1987), 423–452.

[8] H. Matano and F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation, *Comm. Pure Appl. Math.* **57** (2004), 1494–1541.

[9] F. Merle and H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations, *Comm. Pure Appl. Math.* **51** (1998), 139–196.

[10] N. Mizoguchi, Type-II blowup for a semilinear heat equation, *Adv. Differential Equations* **9** (2004), 1279–1316.

[11] P. Poláčik, P. Quittner, and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: Parabolic equations, *Indiana Univ. Math. J.* **56** (2007), 879-908.

[12] P. Poláčik and E. Yanagida, On bounded and unbounded global solutions of a supercritical semilinear heat equation, *Math. Ann.* **327** (2003), 745–771.

[13] P. Quittner and P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser, Basel 2007.

[14] P. Quittner, P. Souplet, and M. Winkler, Initial blow-up rates and universal bounds for nonlinear heat equations, *J. Differential Equations* **196**, 316-339 (2004).

[15] P. Souplet, personal communication.

[16] S. Taliaferro, Local behavior and global existence of positive solutions of $au^\lambda \le -\Delta u \le u^\lambda$, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **19** (2002), 889–901.

[17] S. Taliaferro, Isolated singularities of nonlinear parabolic inequalities, *Math. Ann.* 338 (2007), 555–586.

[18] L. Véron, Singularities of solutions of second order quasilinear equations. Pitman Research Notes in Mathematics Series, 353. Longman, Harlow, 1996.