Initial Blow-up of Solutions of Semilinear Parabolic Inequalities

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Abstract

We study classical nonnegative solutions $u(x,t)$ of the semilinear parabolic inequalities

$$0 \le u_t - \Delta u \le u^p \qquad \text{in} \qquad \Omega \times (0,1)$$

where p is a positive constant and Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 1$.

We show that a necessary and sufficient condition on p for such solutions u to satisfy a pointwise a priori bound on compact subsets K of Ω as $t \to 0^+$ is $p \le 1 + 2/n$ and in this case the bound on u is

$$\max_{x \in K} u(x,t) = O(t^{-n/2}) \qquad \text{as} \qquad t \to 0^+.$$

If in addition, Ω is smooth, u satisfies the boundary condition $u = 0$ on $\partial\Omega \times (0,1)$, and $p < 1 + 2/n$, then we obtain a bound for u on the entire set Ω as $t \to 0^+$.

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1 Introduction

It is not hard to prove that if u is a nonnegative solution of the heat equation

$$u_t - \Delta u = 0 \qquad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

where T is a positive constant and Ω is an open subset of $\mathbb{R}^n$, $n \ge 1$, then for each compact subset K of Ω, we have

$$\max_{x \in K} u(x,t) = O(t^{-n/2}) \qquad \text{as} \quad t \to 0^+. \tag{1.2}$$

The exponent $-n/2$ in (1.2) is optimal because the Gaussian

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0 \\ 0, & t \le 0 \end{cases} \tag{1.3}$$

is a nonnegative solution of the heat equation in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$ and

$$\Phi(0,t) = (4\pi t)^{-n/2} \qquad \text{for} \quad t > 0. \tag{1.4}$$

It is also not hard to prove that if u is a nonnegative solution of the Dirichlet problem

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0,T) \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \end{aligned} \tag{1.5}$$

where $T > 0$ and Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 1$, then

$$u(x,t) = O\left(\frac{\frac{\rho(x)}{\sqrt{t}} \wedge 1}{\sqrt{t}^{n+1}} \right) \quad \text{in} \quad \Omega \times (0, T/2), \tag{1.6}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and $a \wedge b = \min\{a,b\}$ for $a,b \in \mathbb{R}$.

Note that (1.6) is a pointwise a priori bound for u on the entire set Ω rather than on compact subsets of Ω. As we discuss and state precisely in the third paragraph after Theorem 1.4, the bound (1.6) is optimal for x near the boundary of Ω and t small.

In this paper, we investigate when similar results hold for nonnegative solutions $u(x,t)$ of the inequalities

$$0 \leq u_t - \Delta u \leq u^p + \frac{1}{\sqrt{t}^\alpha} \qquad \text{in} \qquad \Omega \times (0,T), \tag{1.7}$$

where $T > 0$, $p > 0$, and $\alpha \in \mathbb{R}$ are constants and where we sometimes omit either u^p or $1/\sqrt{t}^\alpha$ on the right side of (1.7). Note that nonnegative solutions of the heat equation (1.1) satisfy (1.7).

Our first result deals with nonnegative solutions u of (1.7) when no boundary conditions are imposed on u.

Theorem 1.1. *Suppose $u(x,t)$ is a $C^{2,1}$ nonnegative solution of*

$$0 \leq u_t - \Delta u \leq u^{1+2/n} + \frac{1}{\sqrt{t}^{n+2}} \quad \textit{in} \quad \Omega \times (0,T), \tag{1.8}$$

where $T > 0$ and Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$. Then, for each compact subset K of Ω, u satisfies (1.2).

We proved Theorem 1.1 in [22] with the strong added assumption that

$$\text{for some } x_0 \in \Omega,\ u \text{ is continuous on } (\Omega \times [0,T)) \setminus \{(x_0, 0)\}. \tag{1.9}$$

Theorem 1.1 is optimal in two ways. First, the exponent $-n/2$ on t in (1.2) cannot be improved because, as already pointed out, the Gaussian (1.3) is a C^∞ nonnegative solution of the heat equation in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$ satisfying (1.4).

And second, the exponent $1 + 2/n$ on u in (1.8) cannot be increased by the following theorem in [22].

Theorem 1.2. *Let $p > 1 + 2/n$ and $\psi\colon (0,1) \to (0,\infty)$ be a continuous function. Then there exists a C^∞ nonnegative solution $u(x,t)$ of*

$$0 \leq u_t - \Delta u \leq u^p \quad \textit{in} \quad (\mathbb{R}^n \times \mathbb{R}) \setminus \{(0,0)\}$$

such that

$$u(0,t) \neq O(\psi(t)) \quad \textit{as} \quad t \to 0^+.$$

By Theorems 1.1 and 1.2, a necessary and sufficient condition on a positive constant p for $C^{2,1}$ nonnegative solutions $u(x,t)$ of

$$0 \le u_t - \Delta u \le u^p \quad \text{in} \quad \Omega \times (0,T)$$

to satisfy a pointwise a priori bound on compact subsets K of Ω as $t \to 0$ is $p \le 1 + 2/n$. In this case, the optimal bound is the same as the one for the heat equation (1.1).

M.-F. Bidaut-Véron [3], using methods very different than ours, proved Theorem 1.1 when the differential *inequalities* (1.8) are replaced with the *equation*

$$u_t - \Delta u = u^p \quad \text{in} \quad \Omega \times (0,T) \quad \text{where} \quad 1 < p < n(n+2)/(n-1)^2. \tag{1.10}$$

If in addition, $p > 1 + 2/n$ and K is a compact subset of Ω then she shows nonnegative solutions of (1.10) satisfy

$$u(x,t) \le Ct^{-1/(p-1)} \quad \text{in} \quad K \times (0, T/2)$$

where the constant C does not depend on u.

Our next result deals with nonnegative solutions u of (1.7) when no boundary conditions are imposed on u and when the term u^p is omitted from the right side of (1.7).

Theorem 1.3. *Suppose u is a $C^{2,1}$ nonnegative solution of*

$$0 \le u_t - \Delta u \le \frac{1}{\sqrt{t}^{\alpha+2}} \quad \textit{in} \quad \Omega \times (0,T),$$

where $\alpha \in \mathbb{R}$, $T > 0$, and Ω is an open subset of $\mathbb{R}^n$, $n \ge 1$. Then for each compact subset K of Ω,

$$\max_{x \in K} u(x,t) = \begin{cases} o\left(\frac{1}{\sqrt{t}^{\alpha}}\right) & \textit{if } \alpha > n \\ O\left(\frac{1}{\sqrt{t}^{n}}\right) & \textit{if } \alpha \le n \end{cases}$$

as $t \to 0^+$.

When $\alpha \le n$, Theorem 1.3 follows from Theorem 1.1. We include the case $\alpha \le n$ in Theorem 1.3 for completeness.

The rest of our results deal with nonnegative solutions of (1.7) satisfying a Dirichlet boundary condition. To state our results, we define $d(x,t) := \rho(x) \wedge \sqrt{t}$ to be the parabolic distance from (x,t) to the parabolic boundary of $\Omega \times (0,T)$.

Theorem 1.4. *Suppose $u \in C^{2,1}(\overline{\Omega} \times (0,T))$ is a nonnegative solution of*

$$\begin{aligned} 0 \le u_t - \Delta u \le u^p + \frac{1}{\sqrt{t}^{\alpha}} \qquad & \textit{in} \quad \Omega \times (0,T) \\ u = 0 \qquad & \textit{on} \quad \partial\Omega \times (0,T), \end{aligned} \tag{1.11}$$

where $T > 0$, $p > 0$, and $\alpha \in \mathbb{R}$ are constants and Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \ge 1$. Then

(i) *if $p < 1 + 2/(n+1)$ and $\alpha < n + 3$, then u satisfies (1.6);*

(ii) *if $p = 1 + 2/(n+1)$ and $\alpha \le n + 3$, then*

$$u(x,t) = O(d(x,t)^{-(n+1)}) \quad \textit{in} \quad \Omega \times (0, T/2); \tag{1.12}$$

(iii) *if* $1+2/(n+1) \le p < 1+2/n$ *and* $\alpha \le pq$ *where* $q = 2/(n+2-np)$*, then*

$$u(x,t) = O(d(x,t)^{-(pq-2)}) \quad in \quad \Omega \times (0,T/2). \tag{1.13}$$

Part (ii) of Theorem 1.4 is a special case of part (iii). We state part (ii) separately because it deals with the value of p at which the form of the bound for u changes and because it facilitates our discussion below.

If we define the inner region D_{inn} of $\Omega \times (0,T/2)$ by

$$D_{inn} := \{(x,t) \in \Omega \times (0,T/2) : \rho(x) > \sqrt{t}\}$$

then the bounds (1.6) and (1.12) for u in Theorem 1.4 parts (i) and (ii) are the same in D_{inn} and their common value there is $1/\sqrt{t}^{n+1}$.

The bound (1.6) for u in Theorem 1.4(i) is, like u, zero on $\partial\Omega \times (0,T)$. Furthermore, the bound (1.6) is optimal for x near the boundary of Ω and t small. More precisely, let $x_0 \in \partial\Omega$, $G(x,y,t)$ be the heat kernel of the Dirichlet Laplacian for Ω, and η be the unit inward normal to Ω at x_0. Then using the lower bound for G in [25], it is easy to show that

$$u(x,t) := \lim_{r \to 0^+} \frac{G(x, x_0 + r\eta, t)}{r}$$

is a nonnegative solution of (1.5), and hence of (1.11), such that for some $t_0 > 0$,

$$\frac{u(x,t)}{(\frac{\rho(x)}{\sqrt{t}} \wedge 1)/\sqrt{t}^{n+1}}$$

is bounded between positive constants for all $(x,t) \in \Omega \times (0,t_0)$ satisfying $|x - x_0| < \sqrt{t}$.

On the other hand, since u in Theorem 1.4(ii) is zero on $\partial\Omega \times (0,T)$ and the bound $1/\rho(x)^{n+1}$ for u in Theorem 1.4(ii) in $D_{out} := \Omega \times (0,T/2) \backslash D_{inn}$ is infinite on $\partial\Omega \times (0,T)$, one might conjecture that the bound (1.12) for u could be considerably improved in D_{out}. However, the following theorem casts some doubt on this conjecture. It also shows that the exponent $p = 1 + 2/(n+1)$ on u in Theorem 1.4(ii) is optimal for (1.12) to hold.

Theorem 1.5. *Suppose* Ω *is a* C^2 *bounded domain in* $\mathbb{R}^n$*,* $n \ge 1$*, and* $p > 1 + 2/(n+1)$*. Then there exists* $\varepsilon = \varepsilon(n,p) > 0$ *such that for each* $x_0 \in \partial\Omega$ *there exists a nonnegative solution* $u \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ *of*

$$\begin{aligned} 0 \le u_t - \Delta u \le u^p \qquad & in \quad \Omega \times (0,\infty) \\ u = 0 \qquad & on \quad \partial\Omega \times (0,\infty) \end{aligned} \tag{1.14}$$

and a sequence $\{(x_j,t_j)\}_{j=1}^{\infty} \subset \Omega \times (0,1)$ *such that as* $j \to \infty$ *we have* $(x_j,t_j) \to (x_0,0)$*,*

$$\frac{\rho(x_j)}{\sqrt{t_j}^{1+\varepsilon}} \to 0, \qquad and \qquad u(x_j,t_j)\rho(x_j)^{n+1+\varepsilon} \to \infty.$$

Thus, the bound (1.12) for u in Theorem 1.4(ii) does not hold for any $p > 1 + 2/(n+1)$ because the bound (1.12) is not large enough in the outer region D_{out}.

Theorem 1.4 deals with problem (1.11) when p satisfies $0 < p < 1 + 2/n$. The rest of our results deal with problem (1.11) when $p \ge 1 + 2/n$.

Theorem 1.6. *Suppose $p > 1+2/n$, Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $\psi : \Omega \times (0,2) \to (0,\infty)$ is a continuous function. Then there exists a nonnegative solution $u \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ of* (1.14) *such that*

$$u(x,t) \neq O(\psi(x,t)) \qquad \textit{in} \quad \Omega \times (0,1).$$

In other words, in contrast to Theorem 1.4, there does not exist a pointwise a priori bound on $\Omega \times (0,1)$ for nonegative solutions of (1.14) when $p > 1+2/n$ and it is natural to ask the

Open Question. If $T > 0$ and Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 1$, then for what $\alpha \in \mathbb{R}$, if any, do nonnegative solutions $u \in C^{2,1}(\overline{\Omega} \times (0,T))$ of

$$\begin{aligned} 0 \leq u_t - \Delta u \leq u^{1+2/n} + \frac{1}{\sqrt{t}^{\alpha}} \qquad & \text{in} \quad \Omega \times (0,T) \\ u = 0 \qquad & \text{on} \quad \partial\Omega \times (0,T) \end{aligned} \tag{1.15}$$

satisfy a pointwise a priori bound on $\Omega \times (0, T/2)$?

By the following theorem, if such a bound does exist, it must be very large.

Theorem 1.7. *Suppose Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 1$, and β is a positive constant. Then there exists a nonnegative solution $u \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ of*

$$\begin{aligned} 0 \leq u_t - \Delta u \leq u^{1+2/n} \qquad & \textit{in} \quad \Omega \times (0,\infty) \\ u = 0 \qquad & \textit{on} \quad \partial\Omega \times (0,\infty) \end{aligned}$$

such that

$$u(x,t) \neq O(d(x,t)^{-\beta}) \quad \textit{in} \quad \Omega \times (0,1).$$

Theorem 1.4 can be strengthened by weakening the boundary condition $u = 0$ and, in parts (ii) and (iii), by replacing the term $1/\sqrt{t}^{\alpha}$ in (1.11) with a larger term which is infinite on $\partial\Omega \times (0,T)$. We state and prove this strengthend version of Theorem 1.4 in Sections 4 and 5.

The proof of Theorems 1.1 and 1.3 (resp. Theorem 1.4) relies heavily on Lemma 2.1 (resp. Lemma 2.2), which we state and prove in Section 2. We are able to prove Theorem 1.1 without condition (1.9) because we do not impose this kind of condition on the function u in Lemma 2.1.

As in [22], a crucial step in the proof of Theorem 1.1 (resp. 1.4) is an adaptation and extension to parabolic inequalities of a method of Brezis [4] concerning elliptic equations and based on Moser's iteration. This method is used to obtain an estimate of the form

$$\|u_j\|_{L^{\frac{n+2}{n}q}(D')} \leq C\|u_j\|_{L^q(D)}$$

where $q > 1$, $D' \subset D$, C is a constant which does not depend on j, and u_j, $j = 1, 2, \ldots$, is obtained from the function u in Theorem 1.1 (resp. 1.4) by appropriately scaling u about (x_j, t_j) where $(x_j, t_j) \in \Omega \times (0,T)$ is a sequence such that $t_j \to 0^+$ and for which the desired bound for u is violated.

Our proofs also rely on upper and lower bounds for the heat kernel of the Dirichlet Laplacian. We use the upper bound in [11] and the lower one in [25].

P. Souplet and P. Quittner communicated to us a proof of Theorem 1.4(i) in the special case that $\alpha = 0$. Their method of proof, which is very different from ours being based on [7, Theorem 4, Remark 3.2(b)] and the comparison principle, does not seem to work for our Theorem 1.4(i) as stated. See also [20, Theorem 26.14(i)].

Poláčik, Quittner, and Souplet [18, Theorem 3.1] obtained estimates of the form (1.12) and (1.13) for solutions of the *equation* (1.10) without imposing boundary conditions on u. Their method of proof, which is very different from ours being based on a parabolic Liouville-type theorem of Bidaut-Véron [3], does not seem to work for the *inequalities* (1.11), even if the term $1/\sqrt{t}^\alpha$ is omitted in (1.11).

The blow-up of solutions of the *equation*

$$u_t - \Delta u = u^p \tag{1.16}$$

has been extensively studied in [1, 2, 3, 5, 6, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 21, 24] and elsewhere. The book [20] is an excellent reference for many of these results. However, other than [22], we know of no previous blow-up results for the *inequalities*

$$0 \le u_t - \Delta u \le u^p.$$

Also, blow-up of solutions of $au^p \le u_t - \Delta u \le u^p$, where $a \in (0,1)$, has been studied in [23].

2 Preliminary lemmas

For the proofs in Section 3 of Theorems 1.1 and 1.3, we will need the following lemma.

Lemma 2.1. *Suppose u is a $C^{2,1}$ nonnegative solution of*

$$Hu \ge 0 \quad in \quad B_4(0) \times (0,3) \subset \mathbb{R}^n \times \mathbb{R}, \qquad n \ge 1, \tag{2.1}$$

where $Hu = u_t - \Delta u$ is the heat operator. Then

$$u, Hu \in L^1(B_2(0) \times (0,2)) \tag{2.2}$$

and there exist a finite positive Borel measure μ on $B_2(0)$ and $h \in C^{2,1}(B_1(0) \times (-1,1))$ satisfying

$$Hh = 0 \quad in \quad B_1(0) \times (-1,1) \tag{2.3}$$

$$h = 0 \quad in \quad B_1(0) \times (-1,0] \tag{2.4}$$

such that

$$u = N + v + h \quad in \quad B_1(0) \times (0,1) \tag{2.5}$$

where

$$N(x,t) := \int_0^2 \int_{|y|<2} \Phi(x-y, t-s) Hu(y,s)\, dy\, ds, \tag{2.6}$$

$$v(x,t) := \int_{|y|<2} \Phi(x-y,t)\, d\mu(y), \tag{2.7}$$

and Φ is the Gaussian (1.3).

Proof. Let $\varphi_1 \in C^2(\overline{B_3(0)})$ and $\lambda > 0$ satisfy

$$\left.\begin{aligned} -\Delta\varphi_1 &= \lambda\varphi_1 \\ \varphi_1 &> 0 \end{aligned}\right\} \quad \text{for } |x| < 3$$
$$\varphi_1 = 0 \qquad\qquad \text{for } |x| = 3.$$

Then for $0 < t \leq 2$, we have by (2.1) that

$$
\begin{aligned}
0 &\leq \int\limits_{|x|<3} [Hu(x,t)]\varphi_1(x)\,dx \\
&= \int\limits_{|x|<3} u_t(x,t)\varphi_1(x)\,dx + \lambda \int\limits_{|x|<3} u(x,t)\varphi_1(x)\,dx + \int\limits_{|x|=3} u(x,t)\frac{\partial \varphi_1(x)}{\partial \eta}\,dS_x \\
&\leq U'(t) + \lambda U(t)
\end{aligned}
$$

where $U(t) = \int\limits_{|x|<3} u(x,t)\varphi_1(x)\,dx$. Thus $(U(t)e^{\lambda t})' \geq 0$ for $0 < t \leq 2$ and consequently for some $U_0 \in [0,\infty)$ we have

$$
U(t) = (U(t)e^{\lambda t})e^{-\lambda t} \to U_0 \quad \text{as} \quad t \to 0^+. \tag{2.8}
$$

Thus $u\varphi_1 \in L^1(B_3(0) \times (0,2))$. Hence, since for $0 < t \leq 2$,

$$
\begin{aligned}
\int_t^2 \int\limits_{|x|<3} Hu(x,\tau)\varphi_1(x)\,dx\,d\tau &= \int\limits_{|x|<3} \left(\int_t^2 u_t(x,\tau)\,d\tau\right)\varphi_1(x)\,dx - \int_t^2 \int\limits_{|x|<3} (\Delta u(x,\tau))\varphi_1(x)\,dx\,d\tau \\
&= \int\limits_{|x|<3} u(x,2)\varphi_1(x)\,dx - \int\limits_{|x|<3} u(x,t)\varphi_1(x)\,dx \\
&\quad + \int_t^2 \int\limits_{|x|=3} u(x,\tau)\frac{\partial \varphi_1(x)}{\partial \eta}\,dS_x\,d\tau \\
&\quad + \lambda \int_t^2 \int\limits_{|x|<3} u(x,\tau)\varphi_1(x)\,dx\,d\tau,
\end{aligned} \tag{2.9}
$$

we see that $(Hu)\varphi_1 \in L^1(B_3(0) \times (0,2))$. So (2.2) holds.

By (2.8),

$$
\int\limits_{|x|\leq 2} u(x,t)\,dx \quad \text{is bounded for} \quad 0 < t \leq 2. \tag{2.10}
$$

Hence there exists a finite positive Borel measure $\hat{\mu}$ on $\overline{B_2(0)}$ and a sequence t_j decreasing to 0 such that for all $g \in C(\overline{B_2(0)})$ we have

$$
\int\limits_{|x|\leq 2} g(x)u(x,t_j)\,dx \longrightarrow \int\limits_{|x|\leq 2} g(x)\,d\hat{\mu} \quad \text{as} \quad j \to \infty.
$$

In particular, for all $\varphi \in C_0^\infty(B_2(0))$ we have

$$
\int\limits_{|x|<2} \varphi(x)u(x,t_j)\,dx \longrightarrow \int\limits_{|x|<2} \varphi(x)\,d\mu \quad \text{as} \quad j \to \infty, \tag{2.11}
$$

where we define μ to be the restriction of $\hat{\mu}$ to $B_2(0)$.

For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, let $v(x,t)$ be defined by (2.7). Then $v \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$, $Hv = 0$ in $\mathbb{R}^n \times (0,\infty)$, and

$$
\int\limits_{\mathbb{R}^n} v(x,t)\,dx = \int\limits_{|y|<2} d\mu(y) < \infty \quad \text{for} \quad t > 0. \tag{2.12}
$$

Thus $v \in L^1(\mathbb{R}^n \times (0,2))$.

For $\varphi \in C_0^\infty(B_2(0))$ and $t > 0$ we have

$$\int\limits_{|x|<2} \varphi(x) v(x,t)\, dx = \int\limits_{|y|<2} \left(\int\limits_{|x|<2} \Phi(x-y,t)\varphi(x)\, dx \right) d\mu(y) \longrightarrow \int\limits_{|y|<2} \varphi(y)\, d\mu(y) \quad \text{as} \quad t \to 0^+,$$

and hence it follows from (2.11) that

$$\int\limits_{|x|<2} \varphi(x)(u(x,t_j) - v(x,t_j))\, dx \to 0 \quad \text{as} \quad j \to \infty. \tag{2.13}$$

Let

$$f := \begin{cases} Hu, & \text{in } B_2(0) \times (0,2) \\ 0, & \text{elsewhere in } \mathbb{R}^n \times \mathbb{R}. \end{cases}$$

Then by (2.2),

$$f \in L^1(\mathbb{R}^n \times \mathbb{R}). \tag{2.14}$$

Let

$$w := \begin{cases} u - v, & \text{in } B_2(0) \times (0,2) \\ 0, & \text{elsewhere in } \mathbb{R}^n \times \mathbb{R}. \end{cases}$$

Then

$$\begin{aligned} &w \in C^{2,1}(B_2(0) \times (0,2)) \cap L^1(\mathbb{R}^n \times \mathbb{R}), \\ &Hw = f \quad \text{in} \quad B_2(0) \times (0,2), \end{aligned} \tag{2.15}$$

and

$$\int\limits_{|x|<2} |w(x,t)|\, dx \quad \text{is bounded for} \quad 0 < t < 2 \tag{2.16}$$

by (2.10) and (2.12). Let $\Omega = B_1(0) \times (-1,1)$ and define $\Lambda \in \mathcal{D}'(\Omega)$ by $\Lambda = -Hw + f$, that is

$$\Lambda\varphi = \int wH^*\varphi + \int f\varphi \quad \text{for} \quad \varphi \in C_0^\infty(\Omega),$$

where $H^*\varphi := \varphi_t + \Delta\varphi$. We now show $\Lambda = 0$. Let $\varphi \in C_0^\infty(\Omega)$, let j be a fixed positive integer, and let $\psi_\varepsilon \colon \mathbb{R} \to [0,1]$, ε small and positive, be a one parameter family of smooth nondecreasing functions such that

$$\psi_\varepsilon(t) = \begin{cases} 1, & t > t_j + \varepsilon \\ 0, & t < t_j - \varepsilon, \end{cases}$$

where t_j is as in (2.11). Then for $0 < \varepsilon < t_j$, we have

$$\begin{aligned} -\int f\varphi\psi_\varepsilon = -\int (Hw)\varphi\psi_\varepsilon &= \int wH^*(\varphi\psi_\varepsilon) \\ &= \int w(\varphi_t\psi_\varepsilon + \varphi\psi_\varepsilon' + \psi_\varepsilon\Delta\varphi) \\ &= \int w\psi_\varepsilon H^*\varphi + \int w\varphi\psi_\varepsilon'. \end{aligned}$$

Letting $\varepsilon \to 0^+$ we get

$$-\int_{t_j}^{1} \int_{|x|<1} f\varphi \, dx \, dt = \int_{t_j}^{1} \int_{|x|<1} wH^*\varphi \, dx \, dt + \int_{|x|<1} w(x,t_j)\varphi(x,t_j) \, dx. \tag{2.17}$$

Also, it follows from (2.16) and (2.13) that

$$\int_{|x|<1} w(x,t_j)\varphi(x,t_j) \, dx = \int_{|x|<1} w(x,t_j)[\varphi(x,t_j) - \varphi(x,0)] \, dx + \int_{|x|<1} w(x,t_j)\varphi(x,0) \, dx$$
$$\to 0 \quad \text{as} \quad j \to \infty.$$

Thus letting $j \to \infty$ in (2.17) and using (2.14) and (2.15) we get $-\int f\varphi = \int wH^*\varphi$. So $\Lambda = 0$.

For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, let $N(x,t)$ be defined by (2.6). Then

$$N(x,t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x-y, t-s) f(y,s) \, dy \, ds$$

and $N \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0)$. By (2.14), we have $N \in L^1(\Omega)$ and $HN = f$ in $\mathcal{D}'(\Omega)$. Thus

$$H(w - N) = -\Lambda + f - f = 0 \quad \text{in} \quad \mathcal{D}'(\Omega)$$

which implies

$$w - N = h \quad \text{in} \quad \mathcal{D}'(\Omega)$$

for some $C^{2,1}$ solution h of (2.3) and (2.4). Hence (2.5) holds. □

For the proof in Sections 4 and 5 of Theorem 1.4, we will need the following lemma.

Lemma 2.2. *Suppose $u \in C^{2,1}(\overline{\Omega} \times (0, 2T))$ is a nonnegative solution of*

$$Hu \geq 0 \qquad \text{in} \qquad \Omega \times (0, 2T),$$

where $Hu = u_t - \Delta u$ is the heat operator, T is a positive constant, and Ω is a bounded C^2 domain in $\mathbb{R}^n$, $n \geq 1$. Then

$$u, \rho Hu \in L^1(\Omega \times (0,T)), \tag{2.18}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega))$. Moreover, there exists $C > 0$ such that

$$\begin{aligned} 0 &\leq u(x,t) - \int_0^t \int_\Omega G(x,y,t-s) Hu(y,s) \, dy \, ds \\ &\leq C \frac{\frac{\rho(x)}{\sqrt{t}} \wedge 1}{t^{\frac{n+1}{2}}} + \sup_{\partial\Omega \times (0,T)} u \qquad \textit{for all } (x,t) \in \Omega \times (0,T), \end{aligned} \tag{2.19}$$

where G is the heat kernel of the Dirichlet Laplacian for Ω.

Proof. For $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$, and $0 < t < T$ we have

$$\begin{aligned} \int_t^T \int_\Omega [Hu(y,\tau)]\varphi(y) \, dy \, d\tau &= \int_\Omega u(y,T)\varphi(y) \, dy - \int_\Omega u(y,t)\varphi(y) \, dy \\ &\quad - \int_t^T \int_\Omega u(y,\tau)\Delta\varphi(y) \, dy \, d\tau + \int_t^T \int_{\partial\Omega} u(y,\tau) \frac{\partial \varphi(y)}{\partial \eta} \, dS_y \, d\tau \end{aligned} \tag{2.20}$$

Let $\varphi_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\lambda > 0$ satisfy

$$\left.\begin{aligned} -\Delta\varphi_1 &= \lambda\varphi_1 \\ 0 < \varphi_1 &< 1 \end{aligned}\right\} \quad \text{in } \Omega \qquad \varphi_1 = 0 \quad \text{on } \partial\Omega.$$

Then for $0 < t < 2T$ we have

$$\begin{aligned} 0 \le \int_\Omega Hu(y,t)\varphi_1(y)\,dy &= U'(t) + \lambda U(t) + \int_{\partial\Omega} u(y,t)\frac{\partial\varphi_1(y)}{\partial\eta}\,dS_y \\ &\le U'(t) + \lambda U(t), \end{aligned}$$

where $U(t) = \int_\Omega u(y,t)\varphi_1(y)\,dy$. Thus $(U(t)e^{\lambda t})' \ge 0$ for $0 < t < 2T$ and hence for some $U_0 \ge 0$ we have

$$U(t) = (U(t)e^{\lambda t})e^{-\lambda t} \to U_0 \quad \text{as} \quad t \to 0^+. \tag{2.21}$$

Consequently $u\varphi_1 \in L^1(\Omega \times (0,T))$. So taking $\varphi = \varphi_1$ in (2.20) we have

$$\varphi_1 Hu \in L^1(\Omega \times (0,T)), \tag{2.22}$$

and taking $\varphi = \varphi_1^2$ in (2.20) we obtain $u|\nabla\varphi_1|^2 \in L^1(\Omega\times(0,T))$. Thus, since $\varphi_1 + |\nabla\varphi_1|^2$ is bounded away from zero on $\overline{\Omega}$, we have $u \in L^1(\Omega \times (0,T))$. Hence, since φ_1/ρ is bounded between positive constants on Ω, it follows from (2.22) that (2.18) holds, and by (2.21) we have

$$\int_\Omega u(y,t)\rho(y)\,dy \quad \text{is bounded for} \quad 0 < t \le T. \tag{2.23}$$

Let $x \in \Omega$ and $0 < \tau < t < T$ be fixed. Then for $\varepsilon > 0$ we have

$$\begin{aligned} &\int_\Omega G(x,y,\varepsilon)u(y,t)\,dy - \int_\tau^t \int_\Omega G(x,y,t+\varepsilon-s)Hu(y,s)\,dy\,ds \\ &\quad = \int_\Omega G(x,y,t+\varepsilon-\tau)u(y,\tau)\,dy - \int_\tau^t \int_{\partial\Omega} u(y,s)\frac{\partial G(x,y,t+\varepsilon-s)}{\partial\eta_y}\,dS_y\,ds \\ &\quad \ge 0. \end{aligned} \tag{2.24}$$

Since $\int_\Omega G(x,y,\zeta)\,dy \le 1$ for $\zeta > 0$, we have

$$\begin{aligned} 0 &\le -\int_\tau^t \int_{\partial\Omega} \frac{\partial G(x,y,t+\varepsilon-s)}{\partial\eta_y}\,dS_y\,ds \\ &= \int_\Omega G(x,y,\varepsilon)\,dy - \int_\Omega G(x,y,t+\varepsilon-\tau)\,dy \le 1 \end{aligned}$$

and

$$\int_\Omega G(x,y,t+\varepsilon-s)Hu(y,s)\,dy \le \max_{\overline{\Omega}\times[\tau,t]} Hu < \infty$$

for $\varepsilon > 0$ and $\tau \le s \le t$. Thus, letting $\varepsilon \to 0^+$ in (2.24) and using the fact that the function $(y,\zeta) \to G(x,y,\zeta)$ is continuous for $(y,\zeta) \in \overline{\Omega} \times (0,\infty)$ we get

$$\begin{aligned} 0 &\le u(x,t) - \int_\tau^t \int_\Omega G(x,y,t-s)Hu(y,s)\,dy\,ds \\ &\le v(x,t,\tau) + \sup_{\partial\Omega\times(0,T)} u \end{aligned} \tag{2.25}$$

where

$$v(x,t,\tau) := \int_\Omega G(x,y,t-\tau)u(y,\tau)\,dy \le C\frac{\frac{\rho(x)}{\sqrt{t-\tau}}\wedge 1}{(t-\tau)^{\frac{n+1}{2}}}\int_\Omega u(y,\tau)\rho(y)\,dy$$

because, as shown by Hui [11, Lemma 1.3], there exists a positive constant $C = C(n,\Omega,T)$ such that if

$$\widehat{G}(r,t) = \frac{C}{t^{n/2}}e^{-r^2/(Ct)} \quad \text{for} \quad r \ge 0 \quad \text{and} \quad t > 0$$

then the heat kernel $G(x,y,t)$ for Ω satisfies

$$G(x,y,t) \le \left(\frac{\rho(x)}{\sqrt{t}}\wedge 1\right)\left(\frac{\rho(y)}{\sqrt{t}}\wedge 1\right)\widehat{G}(|x-y|,t) \quad \text{for} \quad x,y\in\Omega \quad \text{and} \quad 0<t\le T. \tag{2.26}$$

Hence, letting $\tau \to 0^+$ in (2.25) and using (2.23) and the monotone convergence theorem we obtain (2.19). □

For the proofs in Sections 3, 4, and 5 of Theorems 1.1 and 1.4 we will need the following lemma whose proof is an adaptation to parabolic inequalities of a method of Brezis [4] for elliptic equations.

Lemma 2.3. *Suppose $T > 0$ and $\lambda > 1$ are constants, B is an open ball in $\mathbb{R}^n$, $E = B \times (-T,0)$, and $\varphi \in C_0^\infty(B\times(-T,\infty))$. Then there exists a positive constant C depending only on*

$$n, \lambda, \quad \text{and} \quad \sup_E\left(|\varphi|, |\nabla\varphi|, \left|\frac{\partial\varphi}{\partial t}\right|, |\Delta\varphi|\right) \tag{2.27}$$

such that if Ω is a C^2 bounded domain in $\mathbb{R}^n$, $\Omega\cap B\ne\emptyset$, $D=\Omega\times(-T,0)$, and $u\in C^{2,1}(\overline{D})$ is a nonnegative function satisfying

$$u = 0 \quad \text{on } (\partial\Omega\cap B)\times(-T,0) \tag{2.28}$$

then

$$\left(\iint\limits_{E\cap D}(u^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt\right)^{\frac{n}{n+2}} \le C\left(\iint\limits_{E\cap D}(Hu)^+u^{\lambda-1}\varphi^2\,dx\,dt + \iint\limits_{E\cap D}u^\lambda\,dx\,dt\right). \tag{2.29}$$

We will usually apply Lemma 2.3 when $\Omega = B$. In this case, the condition (2.28) holds vacuously and $E\cap D = E$.

Proof of Lemma 2.3. Let u be as in the lemma. Since

$$\nabla u\cdot\nabla(u^{\lambda-1}\varphi^2) = \frac{4(\lambda-1)}{\lambda^2}|\nabla(u^{\lambda/2}\varphi)|^2 - \frac{\lambda-2}{\lambda^2}\nabla u^\lambda\cdot\nabla\varphi^2 - \frac{4(\lambda-1)}{\lambda^2}u^\lambda|\nabla\varphi|^2 \tag{2.30}$$

we have for $-T<t<0$ that

$$\begin{aligned}\int\limits_{B\cap\Omega}(-\Delta u)u^{\lambda-1}\varphi^2\,dx &= \int\limits_{B\cap\Omega}\nabla u\cdot\nabla(u^{\lambda-1}\varphi^2)\,dx\\ &\ge \frac{4(\lambda-1)}{\lambda^2}\int\limits_{B\cap\Omega}|\nabla(u^{\lambda/2}\varphi)|^2\,dx - C\int\limits_{B\cap\Omega}u^\lambda\,dx\end{aligned} \tag{2.31}$$

where C is a positive constant depending only on the quantities (2.27) whose value may change from line to line. Also, for $x \in B \cap \Omega$ we have

$$\begin{aligned}\int_{-T}^{0} u_t u^{\lambda-1}\varphi^2\,dt &= \frac{1}{\lambda}\int_{-T}^{0}\frac{\partial u^\lambda}{\partial t}\varphi^2\ dt\\ &= \frac{1}{\lambda}\left[u(x,0)^\lambda\varphi(x,0)^2 - \int_{-T}^{0} u^\lambda\frac{\partial\varphi^2}{\partial t}\,dt\right]\\ &\geq -C\int_{-T}^{0} u^\lambda\,dt. \end{aligned}\tag{2.32}$$

Integrating inequality (2.31) with respect to t from $-T$ to 0, integrating inequality (2.32) with respect to x over $B\cap\Omega$, and then adding the two resulting inequalities we get

$$C(I+J) \geq \iint\limits_{E\cap D} |\nabla(u^{\lambda/2}\varphi)|^2\,dx\,dt \tag{2.33}$$

where

$$I = \iint\limits_{E\cap D}(Hu)^+u^{\lambda-1}\varphi^2\,dx\,dt \quad\text{and}\quad J = \iint\limits_{E\cap D} u^\lambda\,dx\,dt.$$

Multiplying (2.33) by

$$M := \max_{-T\leq t\leq 0}\left(\int\limits_{B\cap\Omega} u^\lambda\varphi^2\,dx\right)^{2/n}$$

and using the parabolic Sobolev inequality (see [13, Theorem 6.9]) we obtain

$$C(I+J)M \geq A := \iint\limits_{E\cap D}(u^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt. \tag{2.34}$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}(u^\lambda\varphi^2) &= \lambda u^{\lambda-1}u_t\varphi^2 + 2u^\lambda\,\varphi\varphi_t\\ &= \lambda u^{\lambda-1}\varphi^2(\Delta u + Hu) + 2u^\lambda\varphi\varphi_t\end{aligned}$$

it follows from (2.31) that for $-T < t < 0$ we have

$$\frac{\partial}{\partial t}\int\limits_{B\cap\Omega} u^\lambda\varphi^2\,dx \leq C\int\limits_{B\cap\Omega} u^\lambda\,dx + \lambda\int\limits_{B\cap\Omega} u^{\lambda-1}\varphi^2(Hu)^+\,dx$$

and thus

$$M^{\frac{n}{2}} \leq C(I+J). \tag{2.35}$$

Substituting (2.35) in (2.34) we get

$$A \leq C(I+J)^{\frac{n+2}{n}}$$

which implies (2.29). □

3 Proofs of Theorems 1.1 and 1.3

In this section we prove Theorems 1.1 and 1.3. The following theorem clearly implies Theorem 1.1.

Theorem 3.1. *Suppose u is a $C^{2,1}$ nonnegative solution of*

$$0 \leq u_t - \Delta u \leq b\left(u^{1+2/n} + \frac{1}{\sqrt{t}^{n+2}}\right) \quad \text{in} \quad \Omega \times (0,T), \tag{3.1}$$

where T and b are positive constants and Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$. Then, for each compact subset K of Ω, we have

$$\max_{x \in K} u(x,t) = O(t^{-n/2}) \quad \text{as} \quad t \to 0^+. \tag{3.2}$$

Proof. To prove Theorem 3.1, we claim it suffices to prove Theorem 3.1′ where Theorem 3.1′ is the theorem obtained from Theorem 3.1 by replacing (3.1) with

$$0 \leq u_t - \Delta u \leq \left(u + \frac{b}{\sqrt{t}^n}\right)^{1+2/n} \quad \text{in} \quad B_4(0) \times (0,3) \tag{3.3}$$

and replacing (3.2) with

$$\max_{|x| \leq \frac{1}{2}} u(x,t) = O(t^{-n/2}) \quad \text{as} \quad t \to 0^+. \tag{3.4}$$

To see this, let u be as in Theorem 3.1 and let K be a compact subset of Ω. Since K is compact there exist finite sequences $\{r_j\}_{j=1}^N \subset (0, \sqrt{T}/4)$ and $\{x_j\}_{j=1}^N \subset K$ such that

$$K \subset \bigcup_{j=1}^N B_{r_j/2}(x_j) \subset \bigcup_{j=1}^N B_{4r_j}(x_j) \subset \Omega.$$

Let $v_j(y,s) = r_j^n b^{n/2} u(x,t)$, where $x = x_j + r_j y$ and $t = r_j^2 s$. Then

$$0 \leq Hv_j \leq \left(v_j + \frac{b^{n/2}}{\sqrt{s}^n}\right)^{1+2/n} \quad \text{for} \quad |y| < 4, \quad 0 < s < 16,$$

where $Hv_j := \frac{\partial v_j}{\partial s} - \Delta_y v_j$. Hence by Theorem 3.1′ there exist $s_j \in (0,16)$ and $C_j > 0$ such that

$$\max_{|y| \leq \frac{1}{2}} v_j(y,s) \leq C_j s^{-n/2} \quad \text{for} \quad 0 < s < s_j.$$

That is

$$\max_{|x-x_j| \leq r_j/2} u(x,t) \leq C_j b^{-n/2} t^{-n/2} \quad \text{for} \quad 0 < t < t_j := r_j^2 s_j.$$

So for $0 < t < \min\limits_{1 \leq j \leq N} t_j$ we have

$$\begin{aligned}\max_{x \in K} u(x,t) &\leq \max_{1 \leq j \leq N} \max_{|x-x_j| \leq r_j/2} u(x,t) \\ &\leq (\max_{1 \leq j \leq N} C_j) b^{-n/2} t^{-n/2}.\end{aligned}$$

That is, (3.2) holds.

We now complete the proof of Theorem 3.1 by proving Theorem 3.1′. Suppose u is a $C^{2,1}$ nonnegative solution of (3.3). By Lemma 2.1,

$$u, Hu \in L^1(B_2(0) \times (0,2)) \tag{3.5}$$

and

$$u = N + v + h \quad \text{in} \quad B_1(0) \times (0,1) \tag{3.6}$$

where N, v, and h are as in Lemma 2.1.

Suppose for contradiction that (3.4) does not hold. Then there exists a sequence $\{(x_j, t_j)\} \subset \overline{B_{1/2}(0)} \times (0, 1/4)$ such that for some $x_0 \in \overline{B_{1/2}(0)}$ we have $(x_j, t_j) \to (x_0, 0)$ as $j \to \infty$ and

$$\lim_{j\to\infty} t_j^{n/2} u(x_j, t_j) = \infty. \tag{3.7}$$

Clearly

$$(4\pi t)^{n/2} v(x,t) \le \int\limits_{|y|<2} d\mu(y) < \infty \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{3.8}$$

For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, let

$$E_r(x,t) := \{(y,s) \in \mathbb{R}^n \times \mathbb{R} \colon \ |y - x| < \sqrt{r} \quad \text{and} \quad t - r < s < t\}. \tag{3.9}$$

In what follows, the variables (x,t) and (ξ, τ) are related by

$$x = x_j + \sqrt{t_j}\, \xi \quad \text{and} \quad t = t_j + t_j \tau \tag{3.10}$$

and the variables (y, s) and (η, ζ) are related by

$$y = x_j + \sqrt{t_j}\, \eta \quad \text{and} \quad s = t_j + t_j \zeta. \tag{3.11}$$

For each positive integer j, define

$$f_j(\eta, \zeta) = \sqrt{t_j}^{n+2} Hu(y,s) \quad \text{for} \quad (y,s) \in E_{t_j}(x_j, t_j) \tag{3.12}$$

and define

$$u_j(\xi, \tau) = \sqrt{t_j}^{n} \iint\limits_{E_{t_j}(x_j,t_j)} \Phi(x - y, t - s) Hu(y,s)\, dy\, ds \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{3.13}$$

By (3.5) we have

$$\iint\limits_{E_{t_j}(x_j,t_j)} Hu(y,s)\, dy\, ds \to 0 \quad \text{as} \quad j \to \infty \tag{3.14}$$

and thus making the change of variables (3.11) in (3.14) and using (3.12) we get

$$\iint\limits_{E_1(0,0)} f_j(\eta,\zeta)\, d\eta\, d\zeta \to 0 \quad \text{as} \quad j \to \infty. \tag{3.15}$$

Since

$$\Phi(x-y,t-s)=\frac{1}{\sqrt{t_j}^n}\Phi(\xi-\eta,\tau-\zeta)$$

it follows from (3.13) and (3.12) that

$$u_j(\xi,\tau)=\iint\limits_{E_1(0,0)}\Phi(\xi-\eta,\tau-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta. \tag{3.16}$$

It is easy to check that for $1<q<\frac{n+2}{n}$ and $(\xi,\tau)\in\mathbb{R}^n\times(-1,0]$ we have

$$\left(\iint\limits_{\mathbb{R}^n\times(-1,0)}\Phi(\xi-\eta,\tau-\zeta)^q\,d\eta\,d\zeta\right)^{1/q}<C(n,q)<\infty. \tag{3.17}$$

Thus for $1<q<\frac{n+2}{n}$ we have by (3.16) and standard L^p estimates for the convolution of two functions that

$$\|u_j\|_{L^q(E_1(0,0))}\le C(n,q)\|f_j\|_{L^1(E_1(0,0))}\to 0\quad\text{as}\quad j\to\infty \tag{3.18}$$

by (3.15). If

$$(x,t)\in\overline{E_{t_j/4}(x_j,t_j)}\quad\text{and}\quad(y,s)\in\mathbb{R}^n\times(0,\infty)\setminus E_{t_j}(x_j,t_j) \tag{3.19}$$

then

$$\Phi(x-y,t-s)\le\max_{0\le\tau<\infty}\Phi\left(\frac{\sqrt{t_j}}{2},\tau\right)\le\frac{C(n)}{\sqrt{t_j}^n}.$$

Thus for $(x,t)\in\overline{E_{t_j/4}(x_j,t_j)}$ we have

$$\iint\limits_{B_2(0)\times(0,2)\setminus E_{t_j}(x_j,t_j)}\Phi(x-y,t-s)Hu(y,s)\,dy\,ds\le\frac{C(n)}{\sqrt{t_j}^n}\iint\limits_{B_2(0)\times(0,2)}Hu(y,s)\,dy\,ds.$$

It follows therefore from (3.6), (3.8), (3.5), and (3.13) that

$$u(x,t)\le\frac{u_j(\xi,\tau)+C}{\sqrt{t_j}^n}\quad\text{for}\quad(x,t)\in\overline{E_{t_j/4}(x_j,t_j)} \tag{3.20}$$

where C is a positive constant which does not depend on j or (x,t).

Substituting $(x,t)=(x_j,t_j)$ in (3.20) and using (3.7) we obtain

$$u_j(0,0)\to\infty\quad\text{as}\quad j\to\infty. \tag{3.21}$$

For $(\xi,\tau)\in E_1(0,0)$ we have by (3.13) that

$$Hu_j(\xi,\tau)=\sqrt{t_j}^{n+2}Hu(x,t).$$

Hence for $(\xi,\tau)\in E_1(0,0)$ we have by (3.12) that

$$Hu_j(\xi,\tau)=f_j(\xi,\tau) \tag{3.22}$$

and for $(\xi,\tau)\in E_{1/4}(0,0)$ we have by (3.3) and (3.20) that

$$\begin{aligned} Hu_j(\xi,\tau) &\le \sqrt{t_j}^{\,n+2}\left(u(x,t)+\sqrt{\frac{4}{3}}^{\,n} b\frac{1}{\sqrt{t_j}^{\,n}}\right)^{\frac{n+2}{n}} \\ &\le \sqrt{t_j}^{\,n+2}\left(\frac{u_j(\xi,\tau)+C}{\sqrt{t_j}^{\,n}}\right)^{\frac{n+2}{n}} \\ &= (u_j(\xi,\tau)+C)^{\frac{n+2}{n}} \\ &=: v_j(\xi,\tau)^{\frac{n+2}{n}} \end{aligned} \tag{3.23}$$

where the last equation is our definition of v_j. Thus

$$v_j(\xi,\tau)=u_j(\xi,\tau)+C \quad \text{for} \quad (\xi,\tau)\in E_{1/4}(0,0) \tag{3.24}$$

where C is a positive constant which does not depend on (ξ,τ) or j. Hence in $E_{1/4}(0,0)$ we have $Hu_j=Hv_j$ and

$$\left(\frac{Hv_j}{v_j}\right)^{\frac{n+2}{2}} = Hu_j\left(\frac{Hu_j}{v_j^{\frac{n+2}{n}}}\right)^{n/2} \le Hu_j = f_j$$

by (3.23) and (3.22). Thus

$$\iint\limits_{E_{1/4}(0,0)} \left(\frac{Hv_j}{v_j}\right)^{\frac{n+2}{2}} d\eta\, d\zeta \to 0 \quad \text{as} \quad j\to\infty \tag{3.25}$$

by (3.15).

Let $0<R<1/8$ and $\lambda>1$ be constants and let $\varphi\in C_0^\infty(B_{\sqrt{2R}}(0)\times(-2R,\infty))$ satisfy $\varphi\equiv 1$ on $E_R(0,0)$ and $\varphi\ge 0$ on $\mathbb{R}^n\times\mathbb{R}$. Then

$$\begin{aligned} \iint\limits_{E_{2R}(0,0)} (Hv_j)v_j^{\lambda-1}\varphi^2\, d\xi\, d\tau &= \iint\limits_{E_{2R}(0,0)} \frac{Hv_j}{v_j} v_j^\lambda \varphi^2\, d\xi\, d\tau \\ &\le \left(\iint\limits_{E_{2R}(0,0)} \left(\frac{Hv_j}{v_j}\right)^{\frac{n+2}{2}} d\xi\, d\tau\right)^{\frac{2}{n+2}} \left(\iint\limits_{E_{2R}(0,0)} (v_j^\lambda\varphi^2)^{\frac{n+2}{n}} d\xi\, d\tau\right)^{\frac{n}{n+2}}. \end{aligned}$$

Hence, using (3.25) and applying Lemma 2.3 with $T=2R$, $B=\Omega=B_{\sqrt{2R}}(0)$, $E=E_{2R}(0,0)$, and $u=v_j$ we have

$$\iint\limits_{E_{2R}(0,0)} (v_j^\lambda\varphi^2)^{\frac{n+2}{n}} d\xi\, d\tau \le C\left(\iint\limits_{E_{2R}(0,0)} v_j^\lambda\, d\xi\, d\tau\right)^{\frac{n+2}{n}}$$

where C does not depend on j. Therefore

$$\iint\limits_{E_R(0,0)} v_j^{\lambda\frac{n+2}{n}} d\xi\, d\tau \le C\left(\iint\limits_{E_{2R}(0,0)} v_j^\lambda\, d\xi\, d\tau\right)^{\frac{n+2}{n}}. \tag{3.26}$$

Starting with (3.18) with $q = \frac{n+1}{n}$ and applying (3.26) a finite number of times we find for each $p > 1$ there exists $\varepsilon > 0$ such that the sequence v_j is bounded in $L^p(E_\varepsilon(0,0))$ and thus the same is true for the sequence f_j by (3.23) and (3.22). Thus by (3.17) and Hölder's inequality we have

$$\limsup_{j\to\infty} \iint_{E_\varepsilon(0,0)} \Phi(-\eta,-\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta < \infty \tag{3.27}$$

for some $\varepsilon > 0$. Also by (3.15)

$$\lim_{j\to\infty} \iint_{E_1(0,0)\setminus E_\varepsilon(0,0)} \Phi(-\eta,-\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta = 0. \tag{3.28}$$

Adding (3.27) and (3.28), and using (3.16), we contradict (3.21) and thereby complete the proof of Theorem 3.1. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. By Theorem 1.1, we can assume $\alpha > n$. By using a procedure very similar to the one used in the first paragraph of the proof of Theorem 3.1, we can assume $\Omega \times (0,T) = B_4(0) \times (0,3)$ and $K = \overline{B_{1/2}(0)}$.

By Lemma 2.1,

$$u, Hu \in L^1(B_2(0) \times (0,2)) \tag{3.29}$$

and

$$u = N + v + h \qquad \text{in} \quad B_1(0) \times (0,1), \tag{3.30}$$

where N, v, and h are as in Lemma 2.1.

Let $(x,t) \in \overline{B_{1/2}(0)} \times (0,1/4]$. Then $\overline{E_{t/4}(x,t)} \subset B_1(0) \times (0,1/4]$, where $E_r(x,t)$ is defined by (3.9). Clearly

$$(4\pi t)^{n/2} v(x,t) \le \int_{|y|<2} d\mu(y) < \infty. \tag{3.31}$$

It is easily verified that for $(y,s) \in B_2(0) \times (0,t) \setminus E_{t/m^2}(x,t)$, where $m \ge 2$, we have

$$\Phi(x-y,t-s) \le m^n C(n)/\sqrt{t}^n.$$

Thus, for $m \ge 2$, we have

$$\iint_{B_2(0)\times(0,2)\setminus E_{t/m^2}(x,t)} \Phi(x-y,t-s) Hu(y,s)\, dy\, ds \le \frac{m^n C(n)}{\sqrt{t}^n} \iint_{B_2(0)\times(0,2)} Hu(y,s)\, dy\, ds.$$

Also for $m \ge 2$,

$$\begin{aligned} \iint_{E_{t/m^2}(x,t)} \Phi(x-y,t-s) Hu(y,s)\, dy\, ds &\le \frac{1}{\sqrt{t(1-1/m^2)}^{\alpha+2}} \iint_{E_{t/m^2}(x,t)} \Phi(x-y,t-s)\, dy\, ds \\ &\le \frac{1}{\sqrt{t(1-1/m^2)}^{\alpha+2}} \frac{t}{m^2} = \frac{1}{m^2\sqrt{1-1/m^2}^{\alpha+2}} \frac{1}{\sqrt{t}^\alpha}. \end{aligned}$$

Thus, given $\varepsilon > 0$ and choosing $m = m(\alpha,\varepsilon) > 2$ such that $1/(m^2\sqrt{1-1/m^2}^{\alpha+2}) < \varepsilon$ it follows from (3.30), (3.29), and (3.31) that

$$\begin{aligned} u(x,t) &\le \frac{\varepsilon}{\sqrt{t}^\alpha} + \frac{m^n C(n) \iint\limits_{B_2(0)\times(0,2)} Hu(y,s)\,dy\,ds + \frac{1}{(4\pi)^{n/2}} \int\limits_{|y|<2} d\mu(y)}{\sqrt{t}^n} + h(x,t) \\ &\le \frac{\varepsilon}{\sqrt{t}^\alpha} + \frac{C}{\sqrt{t}^n} \qquad \text{for} \quad (x,t) \in \overline{B_{1/2}(0)} \times (0,1/4], \end{aligned}$$

where C is a positive constant which does not depend on (x,t). This establishes Theorem 1.3 when $\alpha > n$. □

4 Proof of Theorem 1.4(i)

In this section we prove the following theorem which clearly implies Theorem 1.4(i).

Theorem 4.1. *Suppose $u \in C^{2,1}(\overline{\Omega} \times (0,2T))$ is a nonnegative solution of*

$$\begin{cases} 0 \le u_t - \Delta u \le b\left(u^p + \frac{1}{\sqrt{t}^\alpha}\right) & \text{in } \Omega \times (0,2T) \\ u \le b & \text{on } \partial\Omega \times (0,2T) \end{cases} \tag{4.1}$$

where T and b are positive constants, $0 < p < 1 + 2/(n+1)$, $\alpha < n+3$, and Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \ge 1$. Then there exists a positive constant C such that

$$u(x,t) \le C \frac{\frac{\rho(x)}{\sqrt{t}} \wedge 1}{\sqrt{t}^{n+1}} + \sup_{\partial\Omega\times(0,T)} u \qquad \text{for all } (x,t) \in \Omega \times (0,T), \tag{4.2}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$.

Proof. If $p' > \max\{p, 1 + 1/(n+1), \alpha/(n+1)\}$ and $p' < 1 + 2/(n+1)$ then it is easy to check that for $0 < t < 2T$ and $u \ge 0$ we have

$$u^p + \frac{1}{\sqrt{t}^\alpha} \le C\left(u^{p'} + \frac{1}{\sqrt{t}^{p'(n+1)}}\right)$$

for some constant $C = C(n,T,\alpha) > 0$. Thus we can assume

$$p > 1 + \frac{1}{n+1} \qquad \text{and} \qquad \alpha = p(n+1). \tag{4.3}$$

Suppose for contradiction that (4.2) does not hold. Then there exists a sequence $\{(x_j,t_j)\} \subset \Omega \times (0,T)$ such that $t_j \to 0$ as $j \to \infty$ and

$$\frac{u(x_j,t_j) - \sup_{\partial\Omega\times(0,T)} u}{\left(\frac{\rho(x_j)}{\sqrt{t_j}} \wedge 1\right) / \sqrt{t_j}^{n+1}} \to \infty \qquad \text{as} \quad j \to \infty. \tag{4.4}$$

In what follows the variables (x,t) and (ξ,τ) are related by

$$x = x_j + \sqrt{t_j}\xi \quad \text{and} \quad t = t_j + t_j\tau \tag{4.5}$$

and the variables (y,s) and (η,ζ) are related by

$$y = x_j + \sqrt{t_j}\eta \quad \text{and} \quad s = t_j + t_j\zeta. \tag{4.6}$$

For each positive integer j, define

$$\rho_j(\eta) = \frac{\rho(y)}{\sqrt{t_j}} \quad \text{and} \quad f_j(\eta,\zeta) = \sqrt{t_j}^{\,n+3} Hu(y,s) \quad \text{for } (y,s) \in \overline{\Omega} \times (0,2T) \tag{4.7}$$

and define

$$u_j(\xi,\tau) = \sqrt{t_j}^{\,n+1} \iint\limits_{E_{t_j}(x_j,t_j)\cap(\Omega\times(0,T))} G(x,y,t-s)Hu(y,s)\,dy\,ds \quad \text{for } (x,t) \in \overline{\Omega} \times (0,2T) \tag{4.8}$$

where we define $G(x,y,\tau)=0$ if $\tau \le 0$ and where Hu and G are as in Lemma 2.2 and $E_r(x,t)$ is given by (3.9).

By (2.18) we have

$$\iint\limits_{E_{t_j}(x_j,t_j)\cap(\Omega\times(0,T))} \rho(y)Hu(y,s)\,dy\,ds \to 0 \quad \text{as} \quad j\to\infty, \tag{4.9}$$

and thus making the change of variables (4.6) in (4.9) we get

$$\iint\limits_{E_1(0,0)\cap D_j} f_j(\eta,\zeta)\rho_j(\eta)\,d\eta\,d\zeta \to 0 \quad \text{as} \quad j\to\infty, \tag{4.10}$$

where $D_j = \Omega_j \times (-1,0)$ and $\Omega_j = \{\eta : y \in \Omega\}$.

Since, by (2.26) and (4.7),

$$\begin{aligned} G(x,y,t-s) &\le \left(\frac{\rho(x)}{\sqrt{t-s}} \wedge 1\right)\left(\frac{\rho(y)}{\sqrt{t-s}} \wedge 1\right)\widehat{G}(|x-y|,t-s) \\ &= \left(\frac{\rho_j(\xi)}{\sqrt{\tau-\zeta}} \wedge 1\right)\left(\frac{\rho_j(\eta)}{\sqrt{\tau-\zeta}} \wedge 1\right)\frac{1}{\sqrt{t_j}^{\,n}}\widehat{G}(|\xi-\eta|,\tau-\zeta), \end{aligned}$$

it follows from (4.8) and (4.7) that for $(\xi,\tau) \in \Omega_j \times (-1,0]$ we have

$$u_j(\xi,\tau) \le \iint\limits_{E_1(0,0)\cap D_j} \left(\frac{\rho_j(\xi)}{\sqrt{\tau-\zeta}} \wedge 1\right)\left(\frac{\rho_j(\eta)}{\sqrt{\tau-\zeta}} \wedge 1\right)\widehat{G}(|\xi-\eta|,\tau-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta \tag{4.11}$$

where we define $\widehat{G}(r,\tau)=0$ if $\tau \le 0$. It is easy to check that for $1<q<\frac{n+2}{n+1}$ and $(\xi,\tau) \in \mathbb{R}^n \times (-1,0]$ we have

$$\left(\iint\limits_{\mathbb{R}^n\times(-1,0)} \left(\frac{1}{\sqrt{\tau-\zeta}}\widehat{G}(|\xi-\eta|,\tau-\zeta)\right)^q d\eta\,d\zeta \right)^{\frac{1}{q}} < C(n,q,\Omega,T) < \infty. \tag{4.12}$$

Thus, for $1<q<\frac{n+2}{n+1}$, we have by (4.11) and standard L^p estimates for the convolution of two functions that

$$\|u_j\|_{L^q(E_1(0,0)\cap D_j)} \le C(n,q,\Omega,T)\|f_j\rho_j\|_{L^1(E_1(0,0)\cap D_j)} \to 0 \quad \text{as} \quad j\to\infty \tag{4.13}$$

by (4.10).

If

$$(x,t)\in\overline{E_{t_j/4}(x_j,t_j)}\cap(\Omega\times(0,T))\quad\text{and}\quad(y,s)\in\Omega\times(0,t)\setminus E_{t_j}(x_j,t_j) \tag{4.14}$$

then

$$|x-y|\geq\sqrt{t_j}/2 \tag{4.15}$$

and hence by (2.26) we have

$$\begin{aligned}G(x,y,t-s)&\leq\left(\frac{\rho(x)}{\sqrt{t-s}}\wedge 1\right)\frac{\rho(y)}{\sqrt{t-s}}\widehat{G}\left(\frac{\sqrt{t_j}}{2},t-s\right)\\&\leq\rho(y)\max_{0<\tau<\infty}\left(\frac{\rho(x)}{\sqrt{\tau}}\wedge 1\right)\frac{1}{\sqrt{\tau}}\widehat{G}\left(\frac{\sqrt{t_j}}{2},\tau\right)\leq\frac{C(n,\Omega,T)\rho(y)}{\sqrt{t_j}^{n+1}}\left(\frac{\rho(x)}{\sqrt{t_j}}\wedge 1\right).\end{aligned}$$

Thus for $(x,t)\in\overline{E_{t_j/4}(x_j,t_j)}\cap(\Omega\times(0,T))$ we have

$$\iint\limits_{\Omega\times(0,t)\setminus E_{t_j}(x_j,t_j)}G(x,y,t-s)Hu(y,s)\,dy\,ds\leq\frac{C(n,\Omega,T)}{\sqrt{t_j}^{n+1}}\left(\frac{\rho(x)}{\sqrt{t_j}}\wedge 1\right)\int\limits_{\Omega\times(0,T)}\rho(y)Hu(y,s)\,dy\,ds.$$

It follows therefore from Lemma 2.2 and (4.8) that

$$u(x,t)\leq\frac{u_j(\xi,\tau)+C\left(\frac{\rho(x)}{\sqrt{t_j}}\wedge 1\right)}{\sqrt{t_j}^{n+1}}+\sup_{\partial\Omega\times(0,T)}u\qquad\text{for }(x,t)\in\overline{E_{t_j/4}(x_j,t_j)}\cap(\Omega\times(0,T)) \tag{4.16}$$

where C is a positive constant which does not depend on j or (x,t).

Substituting $(x,t)=(x_j,t_j)$ in (4.16) and using (4.4) we obtain

$$\frac{u_j(0,0)}{\rho_j(0)\wedge 1}\geq\frac{u(x_j,t_j)-\sup_{\partial\Omega\times(0,T)}u}{\left(\frac{\rho(x_j)}{\sqrt{t_j}}\wedge 1\right)/\sqrt{t_j}^{n+1}}-C\to\infty\quad\text{as}\quad j\to\infty. \tag{4.17}$$

For $(\xi,\tau)\in E_1(0,0)\cap D_j$ we have by (4.8) that

$$(Hu_j)(\xi,\tau)=\sqrt{t_j}^{n+3}(Hu)(x,t). \tag{4.18}$$

Hence for $(\xi,\tau)\in E_1(0,0)\cap D_j$ we have by (4.7) that

$$(Hu_j)(\xi,\tau)=f_j(\xi,\tau) \tag{4.19}$$

and for $(\xi,\tau)\in E_{1/4}(0,0)\cap D_j$ we have by (4.1), (4.3), and (4.16) that

$$\begin{aligned}Hu_j(\xi,\tau)&\leq\sqrt{t_j}^{n+3}b\left(u(x,t)+\sqrt{\frac{4}{3}}^{n+1}\frac{1}{\sqrt{t_j}^{n+1}}\right)^p\\&\leq\sqrt{t_j}^{n+3}b\left(\frac{u_j(\xi,\tau)+C}{\sqrt{t_j}^{n+1}}\right)^p\\&=\sqrt{t_j}^{a}b(u_j(\xi,\tau)+C)^p\quad\text{where }a=(n+1)\left(\frac{n+3}{n+1}-p\right)>0\\&=:\sqrt{t_j}^{a}bv_j(\xi,\tau)^p,\end{aligned} \tag{4.20}$$

where the last equation is our definition of v_j. Thus

$$v_j(\xi,\tau) = u_j(\xi,\tau) + C \tag{4.21}$$

where C is a positive constant which does not depend on (ξ,τ) or j. Hence in $E_{1/4}(0,0)\cap D_j$ we have

$$\left(\frac{Hu_j}{v_j}\right)^{\frac{n+2}{2}} \le (\sqrt{t_j}^a b v_j^{p-1})^{\frac{n+2}{2}} \le \sqrt{t_j}^{a(n+2)/2} b^{\frac{n+2}{2}} v_j^q,$$

where $q=(p-1)\frac{n+2}{2} < \frac{2}{n+1}\frac{n+2}{2} = \frac{n+2}{n+1}$. Thus

$$\iint\limits_{E_{1/4}(0,0)\cap D_j} \left(\frac{Hu_j}{v_j}\right)^{\frac{n+2}{2}} d\eta\, d\zeta \le \sqrt{t_j}^{a(n+2)/2} b^{\frac{n+2}{2}} \|v_j\|^q_{L^q(E_1(0,0)\cap D_j)} \to 0 \quad \text{as } j\to\infty \tag{4.22}$$

by (4.13) and (4.21).

Let $0<R<1/8$ and $\lambda>1$ be constants and let $\varphi\in C_0^\infty(B_{\sqrt{2R}}(0,0)\times(-2R,\infty))$ satisfy $\varphi\equiv 1$ on $E_R(0,0)$ and $\varphi\ge 0$ on $\mathbb{R}^n\times\mathbb{R}$. Then using (4.21) we have

$$v_j^\lambda\varphi^2 = (u_j+C)^\lambda\varphi^2 \le 2^\lambda(u_j^\lambda\varphi^2 + C^\lambda\varphi^2) \quad \text{in } E_{1/4}(0,0)\cap D_j$$

and hence

$$\begin{aligned}
&\iint\limits_{E_{2R}(0,0)\cap D_j} (Hu_j)u_j^{\lambda-1}\varphi^2\, d\xi\, d\tau \le \iint\limits_{E_{2R}(0,0)\cap D_j} (Hu_j)v_j^{\lambda-1}\varphi^2\, d\xi\, d\tau \\
&= \iint\limits_{E_{2R}(0,0)\cap D_j} \frac{Hu_j}{v_j} v_j^\lambda\varphi^2\, d\xi\, d\tau \\
&\le \left(\iint\limits_{E_{2R}(0,0)\cap D_j} \left(\frac{Hu_j}{v_j}\right)^{\frac{n+2}{2}} d\xi\, d\tau\right)^{\frac{2}{n+2}} \left(\iint\limits_{E_{2R}(0,0)\cap D_j} (v_j^\lambda\varphi^2)^{\frac{n+2}{n}} d\xi\, d\tau\right)^{\frac{n}{n+2}} \\
&\le C\left(\iint\limits_{E_{2R}(0,0)\cap D_j} \left(\frac{Hu_j}{v_j}\right)^{\frac{n+2}{2}} d\xi\, d\tau\right)^{\frac{2}{n+2}} \left[\left(\iint\limits_{E_{2R}(0,0)\cap D_j} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}} d\xi\, d\tau\right)^{\frac{n}{n+2}} + 1\right]
\end{aligned} \tag{4.23}$$

where C is a positive constant which does not depend on j and whose value may change from line to line. Thus using (4.22) and applying Lemma 2.3 with $T=2R$, $B=B_{\sqrt{2R}}(0)$, $E=E_{2R}(0,0)$, $\Omega=\Omega_j$, and $u=u_j$, we have

$$\left(\iint\limits_{E_{2R}(0,0)\cap D_j} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}} d\xi\, d\tau\right)^{\frac{n}{n+2}} \le C\left(\iint\limits_{E_{2R}(0,0)\cap D_j} u_j^\lambda\, d\xi\, d\tau + 1\right).$$

Consequently,

$$\iint\limits_{E_R(0,0)\cap D_j} u_j^{\lambda\frac{n+2}{n}}\, d\xi\, d\tau \le C\left(\iint\limits_{E_{2R}(0,0)\cap D_j} u_j^\lambda\, d\xi\, d\tau + 1\right)^{\frac{n+2}{n}}. \tag{4.24}$$

By (4.13),

$$\lim_{j\to\infty} \iint\limits_{E_{1/4}(0,0)\cap D_j} u_j^{\frac{n+3}{n+2}}\, d\xi\, d\tau = 0. \tag{4.25}$$

Starting with (4.25) and using (4.24) a finite number of times we find that for each $p > 1$ there exists $\varepsilon > 0$ such that the sequence u_j is bounded in $L^p(E_\varepsilon(0,0) \cap D_j)$ and thus the same is true for the sequences v_j, Hu_j, and f_j by (4.21), (4.20), and (4.19).

Thus by (4.11), there exists $\varepsilon > 0$ such that

$$\begin{aligned}
\limsup_{j\to\infty} \frac{u_j(0,0)}{\rho_j(0)} &\le \limsup_{j\to\infty} \iint\limits_{E_1(0,0)\cap D_j} \frac{1}{\sqrt{-\zeta}}\left(\frac{\rho_j(\eta)}{\sqrt{-\zeta}} \wedge 1\right) \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta \\
&\le \limsup_{j\to\infty} \Bigg(\iint\limits_{E_\varepsilon(0,0)\cap D_j} \frac{1}{\sqrt{-\zeta}} \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta \\
&\quad + \iint\limits_{(E_1(0,0)\setminus E_\varepsilon(0,0))\cap D_j} \frac{1}{-\zeta} \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\rho_j(\eta)\, d\eta\, d\zeta \Bigg) < \infty
\end{aligned}$$

where we have estimated the first integral using (4.12) and Hölder's inequality and the second integral using (4.10). Similarly by (4.11),

$$\begin{aligned}
\limsup_{j\to\infty} u_j(0,0) &\le \limsup_{j\to\infty} \iint\limits_{E_1(0,0)\cap D_j} \left(\frac{\rho_j(\eta)}{\sqrt{-\zeta}} \wedge 1\right) \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta \\
&\le \limsup_{j\to\infty} \Bigg(\iint\limits_{E_\varepsilon(0,0)\cap D_j} \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta \\
&\quad + \iint\limits_{(E_1(0,0)\setminus E_\varepsilon(0,0))\cap D_j} \frac{1}{\sqrt{-\zeta}} \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\rho_j(\eta)\, d\eta\, d\zeta \Bigg) < \infty.
\end{aligned}$$

Hence

$$\limsup_{j\to\infty} \frac{u_j(0,0)}{\rho_j(0)\wedge 1} < \infty$$

which contradicts (4.17) and completes the proof of Theorem 4.1. $\square$

5 Proof of Theorem 1.4(ii) and (iii)

In this section we prove the following theorem which clearly implies Theorem 1.4(ii) and (iii).

Theorem 5.1. *Suppose $u \in C^{2,1}(\overline{\Omega} \times (0,2T))$ is a nonnegative solution of*

$$\begin{cases} 0 \le u_t - \Delta u \le b\left(u^p + \dfrac{1}{d(x,t)^{qp}}\right) & \text{in } \Omega \times (0,2T) \\ u \le b, & \text{on } \partial\Omega \times (0,2T) \end{cases} \tag{5.1}$$

where $T > 0$, $b > 0$,

$$1+\frac{2}{n+1}\le p<1+\frac{2}{n}, \qquad \text{and} \qquad q=\frac{2}{n+2-np} \tag{5.2}$$

are constants, Ω is a C^2 bounded domain in $\mathbb{R}^n$, $n \ge 1$, and $d(x,t) = \rho(x) \wedge \sqrt{t}$ is the parabolic distance from (x,t) to the parabolic boundary of $\Omega \times (0,2T)$. Then

$$d(x,t)^{pq-2}u(x,t) \quad \text{is bounded in} \quad \Omega\times(0,T). \tag{5.3}$$

Proof. First we note for later that (5.2) implies

$$q \ge n+1 \qquad \text{and} \qquad pq-q-2 = 2(q-1-n)/n \ge 0. \tag{5.4}$$

Suppose for contradiction that (5.3) does not hold. Then there exists a sequence $\{(x_j,t_j)\} \subset \Omega \times (0,T)$ such that $t_j \to 0$ as $j \to \infty$ and

$$\lim_{j\to\infty} d_j^{pq-2}u(x_j,t_j) = \infty \tag{5.5}$$

where $d_j = d(x_j,t_j)/2$.

If $E_r(x,t)$ is defined by (3.9) then for $(x,t) \in E_{d_j^2}(x_j,t_j)$ we have

$$d_j \le \frac{\rho(x_j)}{2} < \rho(x) < \frac{3\rho(x_j)}{2} \quad \text{and} \quad 3d_j^2 \le \frac{3t_j}{4} < t < t_j \tag{5.6}$$

and thus $d_j \le d(x,t)$ for $(x,t) \in E_{d_j^2}(x_j,t_j)$. Also, if

$$(x,t)\in\overline{E_{d_j^2/4}(x_j,t_j)} \quad \text{and} \quad (y,s)\in\Omega\times(0,t)\setminus E_{d_j^2}(x_j,t_j) \tag{5.7}$$

then either

$$|x-y| \ge d_j/2 \tag{5.8}$$

or

$$(t-s) \ge \frac{3}{4}d_j^2. \tag{5.9}$$

If (5.7) and (5.8) hold and G is as in Lemma 2.2 then by (2.26)

$$\begin{aligned} G(x,y,t-s) &\le \frac{\rho(y)}{\sqrt{t-s}}\widehat{G}\left(\frac{d_j}{2},t-s\right) \\ &\le \rho(y)\max_{0<\tau<\infty}\frac{1}{\sqrt{\tau}}\widehat{G}\left(\frac{d_j}{2},\tau\right) = \frac{C\rho(y)}{d_j^{n+1}} \end{aligned}$$

where $C = C(n,T,\Omega) > 0$. If (5.7) and (5.9) hold then by (2.26)

$$\begin{aligned} G(x,y,t-s) &\le \frac{\rho(y)}{\sqrt{t-s}}\widehat{G}(0,t-s) = \rho(y)\frac{C}{(t-s)^{\frac{n+1}{2}}} \\ &\le \frac{C\rho(y)}{d_j^{n+1}} \end{aligned}$$

where $C = C(n, T, \Omega) > 0$. Thus for $(x,t) \in \overline{E_{d_j^2/4}(x_j, t_j)}$ we have

$$\iint\limits_{\Omega\times(0,t)\setminus E_{d_j^2}(x_j,t_j)} G(x,y,t-s)Hu(y,s)\,dy\,ds \le \frac{C}{d_j^{n+1}} \int\limits_{\Omega\times(0,T)} \rho(y)Hu(y,s)\,dy\,ds$$

where Hu is as in Lemma 2.2. It follows therefore from (5.6) and Lemma 2.2 that for $(x,t) \in \overline{E_{d_j^2/4}(x_j, t_j)}$ we have

$$u(x,t) \le \frac{C}{d_j^{n+1}} + \iint\limits_{E_{d_j^2}(x_j,t_j)} G(x,y,t-s)Hu(y,s)\,dy\,ds \tag{5.10}$$

where we define $G(x,y,\tau) = 0$ for $\tau \le 0$ and where C is a positive constant which does not depend on j or (x,t).

Substituting $(x,t) = (x_j, t_j)$ in (5.10) and using (5.5) and (5.4) we obtain

$$d_j^{pq-2} \iint\limits_{E_{d_j^2}(x_j,t_j)} G(x_j,y,t_j-s)Hu(y,s)\,dy\,ds \to \infty \quad \text{as} \quad j\to\infty. \tag{5.11}$$

Also, by (2.18) we have

$$\iint\limits_{E_{d_j^2}(x_j,t_j)} \rho(y)Hu(y,s)\,dy\,ds \to 0 \quad \text{as} \quad j\to\infty.$$

Hence, it follows from (5.6) that

$$\iint\limits_{E_{d_j^2}(x_j,t_j)} d_j Hu(y,s)\,dy\,ds \to 0 \quad \text{as} \quad j\to\infty. \tag{5.12}$$

In what follows the variables (x,t) and (ξ,τ) are related by

$$x = x_j + d_j\xi \quad \text{and} \quad t = t_j + d_j^2\tau$$

and the variables (y,s) and (η,ζ) are related by

$$y = x_j + d_j\eta \quad \text{and} \quad s = t_j + d_j^2\zeta. \tag{5.13}$$

For each positive integer j, define

$$f_j(\eta,\zeta) := d_j^{q+2} Hu(y,s) \quad \text{for} \quad (y,s)\in\Omega\times(0,2T) \tag{5.14}$$

and

$$u_j(\xi,\tau) := d_j^q \iint\limits_{E_{d_j^2}(x_j,t_j)} G(x,y,t-s)Hu(y,s)\,dy\,ds \quad \text{for} \quad (x,t)\in\Omega\times\mathbb{R}. \tag{5.15}$$

Then

$$Hu_j(\xi,\tau) = d_j^{q+2}Hu(x,t) = f_j(\xi,\tau) \quad \text{for} \quad (\xi,\tau)\in E_1(0,0) \tag{5.16}$$

and making the change of variables (5.13) in (5.12) and (5.15) we get

$$\frac{1}{d_j^{q-(n+1)}} \iint\limits_{E_1(0,0)} f_j(\eta,\zeta)\,d\eta\,d\zeta \to 0 \quad \text{as} \quad j \to \infty \tag{5.17}$$

and

$$u_j(\xi,\tau) \le \iint\limits_{E_1(0,0)} \widehat{G}(|\xi-\eta|,\tau-\zeta) f_j(\eta,\zeta)\,d\eta\,d\zeta \quad \text{for} \quad (\xi,\tau) \in \overline{E_1(0,0)} \tag{5.18}$$

where we have used

$$G(x,y,t-s) \le \widehat{G}(|x-y|,t-s) \qquad \text{for} \quad (x,t),(y,s) \in \Omega \times (0,T) \tag{5.19}$$

which follows from (2.26) and where we define $\widehat{G}(r,\tau) = 0$ if $\tau \le 0$.

It is easy to check that for $1 < r < 1 + 2/n$ and $(\xi,\tau) \in \mathbb{R}^n \times [-1,0]$ we have

$$\left(\iint\limits_{\mathbb{R}^n \times (-1,0)} \left(\widehat{G}(|\xi-\eta|,\tau-\zeta) \right)^r d\eta\,d\zeta \right)^{1/r} < C(n,T,\Omega,r) < \infty. \tag{5.20}$$

Thus, applying standard L^p estimates for the convolution of two functions to the right side of (5.18), we have for $1 < r < 1 + 2/n$ that

$$\|u_j\|_{L^r(E_1(0,0))} \le C(n,T,\Omega,r) \|f_j\|_{L^1(E_1(0,0))} \to 0 \quad \text{as} \quad j \to \infty \tag{5.21}$$

by (5.17) and (5.4).

Also, by (5.11), (5.14), and (5.19),

$$d_j^{pq-q-2} \iint\limits_{E_1(0,0)} \widehat{G}(|-\eta|,-\zeta) f_j(\eta,\zeta)\,d\eta\,d\zeta \to \infty \quad \text{as} \quad j \to \infty, \tag{5.22}$$

and for $(x,t) \in E_{d_j^2/4}(x_j,t_j)$ it follows from (5.16), (5.1), (5.6), (5.10), (5.15), and (5.4) that

$$\begin{aligned} Hu_j(\xi,\tau) &= d_j^{q+2} Hu(x,t) \\ &\le d_j^{q+2} b(u(x,t) + d_j^{-q})^p \\ &\le d_j^{q+2} b \left(\frac{u_j(\xi,\tau) + C}{d_j^q} \right)^p \\ &= b d_j^{-(pq-q-2)} (u_j(\xi,\tau) + C)^p \\ &=: b d_j^{-(pq-q-2)} v_j(\xi,\tau)^p \end{aligned} \tag{5.23}$$

where the last equation is our definition of v_j. Thus

$$v_j(\xi,\tau) = u_j(\xi,\tau) + C \qquad \text{for} \qquad (\xi,\tau) \in E_{1/4}(0,0)$$

where $C > 1$ is a constant which does not depend on (ξ,τ) or j. Hence in $E_{1/4}(0,0)$ we have $Hu_j = Hv_j$ and

$$\begin{aligned} \left(\frac{Hv_j}{v_j} \right)^{1+n/2} &= (Hu_j) \left(\frac{Hu_j}{v_j^{1+2/n}} \right)^{n/2} \le (Hu_j)(b d_j^{-(pq-q-2)})^{n/2} \\ &= b^{n/2} d_j^{-(q-n-1)} f_j \end{aligned}$$

by (5.23), (5.2), (5.16), and (5.4). Thus (3.25) holds by (5.17).

Exactly as in the second to last paragraph of the proof of Theorem 3.1, we have for $0 < R < 1/8$ and $\lambda > 1$ that the functions v_j satisfy (3.26) where C does not depend on j.

Starting with (5.21) with $r = 1 + 1/n$ and applying (3.26) a finite number of times we find for each $r > 1$ there exists $\varepsilon > 0$ such that the sequence v_j is bounded in $L^r(E_\varepsilon(0,0))$ and thus the same is true for the sequence $d_j^{pq-q-2} f_j$ by (5.23) and (5.16). Thus by (5.20) and Hölder's inequality we have

$$\limsup_{j\to\infty} d_j^{pq-q-2} \iint\limits_{E_\varepsilon(0,0)} \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta < \infty \tag{5.24}$$

for some $\varepsilon > 0$. Also by (5.17) and (5.4),

$$\lim_{j\to\infty} d_j^{pq-q-2} \iint\limits_{E_1(0,0)\setminus E_\varepsilon(0,0)} \widehat{G}(|-\eta|, -\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta = 0. \tag{5.25}$$

Adding (5.24) and (5.25), we contradict (5.22). □

6 Proof of Theorems 1.5, 1.6, and 1.7

Theorem 6.1 below clearly implies Theorems 1.5, 1.6, and 1.7 in the introduction.

In the following theorem, $g\colon (0,\infty) \to (0,\infty)$ is a continuous function such that as $\rho \to 0^+$ we have

$$\frac{g(\rho)}{\rho} \to \infty \tag{6.1}$$

perhaps very slowly.

Theorem 6.1. *Suppose Ω is a C^2 bounded domain in $\mathbb{R}^n$, $\psi\colon \Omega \times (0,1) \to (0,\infty)$ is a continuous function, and a and p are constants satisfying*

$$a = p - \left(1 + \frac{2}{n+1}\right) > 0. \tag{6.2}$$

Then for each $x_0 \in \partial\Omega$ there exists a nonnegative solution $u \in C^{2,1}(\overline{\Omega} \times (0,\infty))$ of

$$\begin{aligned} 0 \le u_t - \Delta u \le u^p \quad &\text{in } \Omega \times (0,\infty) \\ u = 0 \quad &\text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{6.3}$$

and a sequence $\{(x_j,t_j)\} \subset \Omega \times (0,1)$ satisfying

$$\rho(x_j) = \sqrt{t_j}^{\,1+\frac{(n+1)a}{p+1}} \tag{6.4}$$

such that as $j \to \infty$ we have $(x_j,t_j) \to (x_0,0)$ and

$$u(x_j,t_j) \ne \begin{cases} O\left(g(\rho(x_j))^{-\frac{n+1}{1-n(n+1)a/2}}\right) & \text{if } p < 1 + \dfrac{2}{n} \\ O\left(e^{g(\rho(x_j))^{-1}}\right) & \text{if } p = 1 + \dfrac{2}{n} \\ O(\psi(x_j,t_j)) & \text{if } p > 1 + \dfrac{2}{n}. \end{cases} \tag{6.5}$$

Since (6.2) and (6.4) imply $\rho(x_j) = d(x_j, t_j)$ we see that we can replace $\rho(x_j)$ with $d(x_j, t_j)$ in (6.5).

Proof of Theorem 6.1. By [25, Theorem 1] there exist positive constants T and α depending only on n and Ω such that the heat kernel G of the Dirichlet–Laplacian for Ω satisfies

$$G(x,y,\tau) \geq \left(\frac{\rho(x)}{\sqrt{\tau}} \wedge 1\right)\left(\frac{\rho(y)}{\sqrt{\tau}} \wedge 1\right)\frac{1}{\alpha\tau^{n/2}}e^{-\frac{\alpha|x-y|^2}{\tau}} \tag{6.6}$$

for all $x, y \in \Omega$ and $0 < \tau \leq T$.

We define positive constants β, γ, and δ by

$$\beta = \frac{\omega_n}{2\alpha}e^{-\alpha} \wedge 1, \quad \gamma = (p-1)\beta^p, \quad \gamma\delta^{p-1} = 5, \tag{6.7}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Thus β, γ, and δ depend only on n, p, and Ω.

We note here for future reference that (6.2) implies

$$\frac{1}{1-(p-1)n/2} = \frac{n+1}{1-n(n+1)a/2} > \frac{2}{p-1} \quad \text{for} \quad p-1 < \frac{2}{n} \tag{6.8}$$

and

$$\frac{n+2}{1+2/(p-1)} = 1 + \frac{(n+1)a}{p+1}. \tag{6.9}$$

Let $x_0 \in \partial\Omega$ and

$$D = \{(x,t) \in \Omega \times (0,T) \colon \ |x - x_0| < \sqrt{t}\}.$$

Then by the third paragraph after Theorem 1.4 there exists a nonnegative solution $u_0(x,t)$ of

$$\begin{aligned} Hu_0 &= 0 \quad \text{in } \Omega \times (0,\infty) \\ u_0 &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

where H is as in Lemma 2.2, such that

$$\frac{u_0(x,t)}{\left(\frac{\rho(x)}{\sqrt{t}} \wedge 1\right)/\sqrt{t}^{\,n+1}} = \frac{u_0(x,t)}{\rho(x)/\sqrt{t}^{\,n+2}} > 8\delta \quad \text{for} \quad (x,t) \in D. \tag{6.10}$$

Choose a sequence of positive numbers $\{\rho_j\}_{j=1}^{\infty}$ such that

$$\rho_{j+1} < \rho_j/4 \quad \text{for} \quad j \geq 1. \tag{6.11}$$

Let $x_j = x_0 + \rho_j\eta$, where η is the inward unit normal to $\partial\Omega$ at x_0, and define $t_j > 0$ by

$$\rho_j = \sqrt{t_j}^{\,1+\frac{(n+1)a}{p+1}}.$$

By taking a subsequence of ρ_j if necessary, we can assume $(x_j, t_j) \in D$ and

$$\rho_j = \rho(x_j) = |x_j - x_0| \qquad \text{for} \qquad j \geq 1. \tag{6.12}$$

Thus (6.4) holds.

Choose

$$a_j > \frac{\delta}{\rho_j^{2/(p-1)}} \tag{6.13}$$

such that

$$\frac{a_j}{\psi(x_j, t_j)} \to \infty \quad as \quad j \to \infty. \tag{6.14}$$

Since decreasing g increases the right side of (6.5), we can assume in addition to (6.1) that

$$\frac{g(\rho)}{\rho} = O\left(\log \frac{1}{\rho}\right) \quad \text{as} \quad \rho \to 0^+. \tag{6.15}$$

Let $b_j = \sqrt{\rho_j g(\rho_j)}$. Then by (6.1),

$$\frac{g(\rho_j)}{b_j} = \frac{b_j}{\rho_j} = \sqrt{\frac{g(\rho_j)}{\rho_j}} \to \infty \quad \text{as} \quad j \to \infty, \tag{6.16}$$

and thus by (6.15),

$$\frac{b_j}{\rho_j} = o\left(\frac{g(\rho_j)}{\rho_j}\right) = o\left(\log \frac{1}{\rho_j}\right) \quad \text{as} \quad j \to \infty. \tag{6.17}$$

Taking a subsequence of ρ_j, we can by (6.16) assume

$$\frac{\rho_j}{b_j} < \frac{1}{2^j} \quad \text{for} \quad j \geq 1. \tag{6.18}$$

Let $w_j(s)$ be the solution of

$$w_j'(s) = \frac{\gamma}{p-1} w_j(s)^p$$

satisfying

$$w_j(t_j) = \begin{cases} \left(\frac{1}{b_j}\right)^{\frac{1}{1-(p-1)n/2}} & \text{if } p-1 < \frac{2}{n} \\ e^{1/b_j} & \text{if } p-1 = \frac{2}{n} \\ a_j & \text{if } p-1 > \frac{2}{n}. \end{cases} \tag{6.19}$$

Then

$$t_j - t = \frac{1}{\gamma}\left[\frac{1}{w_j(t)^{p-1}} - \frac{1}{w_j(t_j)^{p-1}}\right] \quad \text{for} \quad t \leq t_j. \tag{6.20}$$

By taking a subsequence of ρ_j, it follows from (6.8), (6.13), and (6.17) that

$$\frac{\delta}{\rho_j^{2/(p-1)}} < w_j(t_j) \quad \text{for} \quad j \geq 1.$$

Thus there is a unique $\tau_j < t_j$ such that

$$w_j(\tau_j) = \frac{\delta}{\rho_j^{2/(p-1)}} \tag{6.21}$$

and by (6.20),

$$t_j - \tau_j \leq \frac{1}{\gamma} \frac{1}{w_j(\tau_j)^{p-1}} = \frac{\rho_j^2}{\gamma \delta^{p-1}} < \frac{\rho_j^2}{4} < \frac{t_j}{4} \tag{6.22}$$

by (6.7) and (6.4). Hence there exists $\varepsilon_j > 0$ such that

$$\sqrt{t_j - \tau_j + 2\varepsilon_j} < \frac{\rho_j}{2} \quad \text{and} \quad t_j + \varepsilon_j < 2^{\frac{2}{n+2}} t_j. \tag{6.23}$$

Let $h_j(s) = \sqrt{t_j - s}$ and $H_j(s) = \sqrt{t_j + \varepsilon_j - s}$. Then by (6.23)

$$H_j(\tau_j - \varepsilon_j) < \frac{\rho_j}{2}. \tag{6.24}$$

Define

$$\begin{aligned} D_j &= \{(y,s) \in \mathbb{R}^n \times \mathbb{R} \colon \ |y - x_j| < h_j(s) \quad \text{and} \quad \tau_j < s < t_j\} \\ E_j &= \{(y,s) \in \mathbb{R}^n \times \mathbb{R} \colon \ |y - x_j| < H_j(s) \quad \text{and} \quad \tau_j - \varepsilon_j < s < t_j + \varepsilon_j\}. \end{aligned}$$

Then by (6.24),

$$\frac{\rho_j}{2} < \rho(x) < \frac{3\rho_j}{2} \quad \text{for} \quad (x,t) \in E_j. \tag{6.25}$$

Thus by (6.11),

$$E_j \cap E_k = \emptyset \quad \text{for} \quad 1 \le j < k. \tag{6.26}$$

For $(x,t) \in E_j$, we have using (6.12) and (6.24) that

$$|x - x_0| \le |x - x_j| + |x_j - x_0| \le \frac{\rho_j}{2} + \rho_j = \frac{3}{2}\rho_j,$$

and using (6.23) that

$$t > \tau_j - \varepsilon_j > t_j - \frac{\rho_j^2}{4} = \rho_j^2 \left(\frac{t_j}{\rho_j^2} - \frac{1}{4} \right).$$

Therefore, taking a subsequence of ρ_j, it follows from (6.4) that

$$D_j \subset E_j \subset D \qquad \text{for} \quad j \ge 1.$$

Hence for $(x,t) \in E_j$, we obtain from (6.10), (6.23), (6.25), (6.4), (6.9), and (6.21) that

$$\begin{aligned} u_0(x,t) &\ge 8\delta \frac{\rho(x)}{\sqrt{t}^{n+2}} \ge 8\delta \frac{\rho_j/2}{2\sqrt{t_j}^{n+2}} \\ &= 2\delta \frac{\rho_j}{\sqrt{t_j}^{n+2}} = 2\delta \frac{\rho_j}{\rho_j^{1+2/(p-1)}} \\ &= \frac{2\delta}{\rho_j^{2/(p-1)}} = 2w_j(\tau_j). \end{aligned} \tag{6.27}$$

Using (3.17), Hölder's inequality, and the well-known fact that

$$G(x,y,\tau) \le \Phi(x - y, \tau) \quad \text{for} \quad (x,y) \in \Omega, \tau > 0,$$

we find for $(x,t) \in E_j$ that

$$\begin{aligned} \iint\limits_{E_j \setminus D_j} G(x,y,t-s) w_j'(s)\, dy\, ds &\le \left(\iint\limits_{\mathbb{R}^n \times (0,1)} \Phi(x - y, t - s)^{\frac{n+1}{n}}\, dy\, ds \right)^{\frac{n}{n+1}} \left(\iint\limits_{E_j \setminus D_j} w_j'(s)^{n+1}\, dy\, ds \right)^{\frac{1}{n+1}} \\ &\le C(n) \left(\iint\limits_{E_j \setminus D_j} w_j'(s)^{n+1}\, dy\, ds \right)^{\frac{1}{n+1}} \\ &\le w_j(\tau_j) \end{aligned} \tag{6.28}$$

provided we decrease ε_j if necessary.

Let $\chi_j : \mathbb{R}^n \times \mathbb{R} \to [0,1]$ be a C^∞ function such that $\chi_j \equiv 1$ in D_j and $\chi_j \equiv 0$ in $\mathbb{R}^n \times \mathbb{R} \setminus E_j$. Define $v_j, u_j : \overline{\Omega} \times \mathbb{R} \to [0,\infty)$ by

$$\begin{aligned} v_j(y,s) &= \chi_j(y,s) w_j'(s) \\ u_j(x,t) &= \iint\limits_{\Omega \times (0,\infty)} G(x,y,t-s) v_j(y,s)\, dy\, ds. \end{aligned}$$

Then $v_j, u_j \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$ and

$$\begin{aligned} Hu_j &= v_j \quad \text{in } \overline{\Omega} \times \mathbb{R} \\ u_j &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}. \end{aligned} \tag{6.29}$$

Let $(x,t), (y,s) \in E_j$ with $s < t$. Then by (6.23) and (6.25)

$$\sqrt{t-s} \le \sqrt{t_j - \tau_j + 2\varepsilon_j} \le \frac{\rho_j}{2} \le \rho(x) \wedge \rho(y)$$

and hence

$$\frac{\rho(x)}{\sqrt{t-s}} \wedge 1 = 1 = \frac{\rho(y)}{\sqrt{t-s}} \wedge 1$$

and thus by (6.6),

$$G(x,y,t-s) \ge \frac{1}{\alpha (t-s)^{n/2}} e^{\frac{-\alpha |x-y|^2}{t-s}}.$$

Hence, for $\tau_j - \varepsilon_j \le s < t \le t_j + \varepsilon_j$ and $(x,t) \in E_j$, we have

$$\begin{aligned} \int\limits_{|y-x_j| \le H_j(s)} G(x,y,t-s)\, dy &\ge \frac{\omega_n}{2\alpha} \frac{1}{\omega_n (t-s)^{n/2}} \int\limits_{|y-x| < \sqrt{t-s}} e^{\frac{-\alpha|y-x|^2}{t-s}} dy \\ &\ge \frac{\omega_n}{2\alpha} e^{-\alpha} \ge \beta \end{aligned}$$

where we have used (6.7) and the fact that at least half the ball $B_{\sqrt{t-s}}(x)$ is contained in $B_{H_j(s)}(x_j)$. Thus for $(x,t) \in E_j$,

$$\begin{aligned} \iint\limits_{(y,s) \in E_j} G(x,y,t-s) w_j'(s)\, dy\, ds &= \int_{\tau_j - \varepsilon_j}^{t} w_j'(s) \left(\int\limits_{|y-x_j| \le H_j(s)} G(x,y,t-s)\, dy \right) ds \\ &\ge \beta (w_j(t) - w_j(\tau_j - \varepsilon_j)) \ge \beta w_j(t) - w_j(\tau_j) \end{aligned}$$

by (6.7). Hence, for $(x,t) \in E_j$ we have

$$\begin{aligned} u_j(x,t) &\ge \iint\limits_{D_j} G(x,y,t-s) w_j'(s)\, dy\, ds \\ &= \iint\limits_{E_j} G(x,y,t-s) w_j'(s)\, dy\, ds - \iint\limits_{E_j \setminus D_j} G(x,y,t-s) w_j'(s)\, dy\, ds \\ &\ge \beta w_j(t) - 2 w_j(\tau_j) \end{aligned} \tag{6.30}$$

by (6.28). By deceasing ε_j if necessary, we have

$$\iint\limits_{E_j \setminus D_j} w_j'(s)\rho(y)\,dy\,ds < 1/2^j.$$

Thus using (6.25), (6.20), (6.19), and taking a subsequence of ρ_j when necessary we obtain

$$\begin{aligned}
\iint\limits_{\Omega\times\mathbb{R}} v_j(y,s)\rho(s)\,dy\,ds - \frac{1}{2^j} &\le \iint\limits_{D_j} w_j'(s)\rho(y)\,dy\,ds + \iint\limits_{E_j\setminus D_j} w_j'(s)\rho(y)\,dy\,ds - \frac{1}{2^j}\\
&\le \iint\limits_{D_j} w_j'(s)\rho(y)\,dy\,ds = \int_{\tau_j}^{t_j} w_j'(s)\left(\int\limits_{|y-x_j|<h_j(s)} \rho(y)\,dy\right)ds\\
&\le 2\omega_n\rho_j \int_{\tau_j}^{t_j} w_j'(s)(t_j-s)^{n/2}ds\\
&= 2\omega_n \frac{\rho_j}{\gamma^{n/2}} \int_{\tau_j}^{t_j}\left(\frac{1}{w_j(s)^{p-1}} - \frac{1}{w_j(t_j)^{p-1}}\right)^{n/2} w_j'(s)\,ds\\
&\le \frac{2\omega_n}{\gamma^{n/2}}\rho_j \int_{w_j(\tau_j)}^{w_j(t_j)} w^{-(p-1)n/2}dw\\
&\le c(n,p)\frac{\rho_j}{\gamma^{n/2}}\begin{cases} w_j(t_j)^{1-(p-1)n/2} = \dfrac{1}{b_j}, & \text{if } p-1<\dfrac{2}{n}\\ \log\dfrac{w_j(t_j)}{w_j(\tau_j)} < \log w_j(t_j) = \dfrac{1}{b_j}, & \text{if } p-1=\dfrac{2}{n}\\ \dfrac{1}{w_j(\tau_j)^{(p-1)n/2-1}} < 1 < \dfrac{1}{b_j}, & \text{if } p-1>\dfrac{2}{n}\end{cases}\\
&\le \frac{c(n,p)\rho_j}{\gamma^{n/2}b_j} \le \frac{c(n,p)}{\gamma^{n/2}}\frac{1}{2^j}
\end{aligned}$$

by (6.18). Thus

$$\iint\limits_{\Omega\times\mathbb{R}} \sum_{j=1}^{\infty} v_j(y,s)\rho(y)\,dy\,ds < \infty.$$

Hence the function $u\colon\ \overline{\Omega}\times(0,\infty)\to[0,\infty)$ defined by

$$\begin{aligned}
u(x,t) &= u_0(x,t) + \iint\limits_{\Omega\times(0,\infty)} G(x,y,t-s)\sum_{j=1}^{\infty} v_j(y,s)\,dy\,ds\\
&= u_0(x,t) + \sum_{j=1}^{\infty} u_j(x,t)
\end{aligned}$$

is in $C^{2,1}(\overline{\Omega}\times(0,\infty))$ and by (6.29) we have

$$\begin{aligned}
Hu &= \sum_{j=1}^{\infty} v_j && \text{in } \overline{\Omega}\times(0,\infty)\\
u &= 0 && \text{on } \partial\Omega\times(0,\infty).
\end{aligned} \tag{6.31}$$

Also, by (6.27) and (6.30), for $(x,t) \in E_j$, we have

$$u(x,t) \geq u_0(x,t) + u_j(x,t) \geq \beta w_j(t). \tag{6.32}$$

Hence, for $(x,t) \in E_j$, it follows from (6.26) that

$$\begin{aligned} Hu(x,t) = v_j(x,t) \leq w_j'(t) &= \frac{\gamma}{p-1} w_j(t)^p \\ &\leq \frac{\gamma}{p-1} \beta^{-p} u(x,t)^p = u(x,t)^p \end{aligned} \tag{6.33}$$

by (6.7). Inequality (6.33) also holds for $(x,t) \in (\Omega \times (0,\infty)) \setminus \bigcup_{j=1}^{\infty} E_j$ because $Hu \equiv 0$ there by (6.31). Thus (6.3) holds. Finally, by (6.32),

$$u(x_j, t_j) \geq \beta w_j(t_j)$$

and it therefore follows from (6.19), (6.16), (6.14), and (6.8) that (6.5) holds. □

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