Isolated Singularities of Nonlinear Elliptic Inequalities. II. Asymptotic Behavior of Solutions

Steven D. Taliaferro
Mathematics Department
Texas A&M University
College Station, TX 77843-3368
stalia@math.tamu.edu

Abstract

We give conditions on a continuous function $f\colon (0,\infty) \to (0,\infty)$ which guarantee that every C^2 positive solution $u(x)$ of the differential inequalities

$$0 \le -\Delta u \le f(u)$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \ge 2)$ is asymptotically radial (or asymptotically harmonic) as $|x| \to 0^+$.

1 Introduction

It is well-known that if u is positive and harmonic in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \ge 2)$ then either the origin is a removable singularity of u or for some finite positive number m,

$$\lim_{|x|\to 0^+} \frac{u(x)}{\Phi(|x|)} = m, \tag{1.1}$$

where Φ is the fundamental solution of $-\Delta$. In particular, u is *asymptotically radial* as $|x| \to 0^+$, i.e.

$$\lim_{|x|\to 0^+} \frac{u(x)}{\bar{u}(|x|)} = 1, \tag{1.2}$$

where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$.

In this paper we study when similar results hold for C^2 positive solutions u of the differential inequalities

$$0 \le -\Delta u \le f(u) \quad \text{in a punctured neighborhood of the origin} \tag{1.3}$$

where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function.

Specifically, we give essentially optimal conditions on f so that every C^2 positive solution u of (1.3) satisfies (1.2), and in this case we describe the possible behavior of $\bar{u}(|x|)$, and hence of $u(x)$, as $|x| \to 0^+$.

We also give essentially optimal conditions on f so that every C^2 positive solution u of (1.3) satisfies

$$\lim_{|x|\to 0^+} \frac{u(x)}{h(x)} = 1 \tag{1.4}$$

for some function h which is positive and harmonic in a punctured neighborhood of origin. We say a positive function u satisfying (1.4) is *asymptotically harmonic* as $|x| \to 0^+$.

Since (1.4) implies (1.2), the conditions on f for (1.4) to hold will have to be at least as strong as the conditions on f for (1.2) to hold.

As an example of the essential optimality of our results, it follows from Section 2 that every C^2 positive solution $u(x)$ of

$$0 \le -\Delta u \le e^u$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^2$ is asymptotically harmonic as $|x| \to 0^+$; however, if $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying $\lim_{t\to\infty}(\log f(t))/t = \infty$ then (1.3) has C^2 positive solutions u in a punctured neighborhood of the origin in $\boldsymbol{R}^2$ which are not asymptotically radial (and hence not asymptotically harmonic) as $|x| \to 0^+$.

This paper is a continuation of our paper [11] in which we give essentially optimal conditions on f so that every C^2 positive solution u of (1.3) satisfies

$$u(x) = O(\Phi(|x|)) \quad \text{as} \quad |x| \to 0^+.$$

The question as to when such solutions u satisfy (1.2) or (1.4) was left open in that paper (see [11, open question at the bottom of p. 1887 and conjecture on p. 1889]).

Many authors (see for example [1], [2], [3], [4], [5], [6], [7]) have studied the asymptotic behavior at an isolated singularity of solutions of the differential *equation* $-\Delta u = f(u)$ under various conditions on the positive function f. Of particular relevance to our results is a result of Lions [8] which states that every C^2 positive solution of $-\Delta u = u^p$ in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ is asymptotically harmonic as $|x| \to 0^+$ provided $p < n/(n-2)$ (if $n = 2$, $p < \infty$). Note however that in this paper we study differential *inequalities* rather than differential *equations*.

2 Two dimensional results

Our result for positive solutions of (1.3) in two dimensions is the following.

Theorem 2.1. *Let $u(x)$ be a C^2 positive solution of*

$$0 \le -\Delta u \le f(u) \tag{2.1}$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^2$, where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\log f(t) = O(t) \quad \textit{as} \quad t \to \infty. \tag{2.2}$$

Then either u has a C^1 extension to the origin or

$$\lim_{|x|\to 0^+} \frac{u(x)}{\log \frac{1}{|x|}} = m \tag{2.3}$$

for some finite positive number m.

In particular the function u in Theorem 2.1 satisfies (1.4) and hence also (1.2). In [10], we showed that if $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying $\lim\limits_{t\to\infty}(\log f(t))/t = \infty$ then (2.1) has a C^2 positive solution u in a punctured neighborhood of the origin in $\boldsymbol{R}^2$ which satisfies neither (1.4) nor (1.2). Thus the condition (2.2) on f is not only essentially optimal for (1.4) to hold, but also essentially optimal for (1.2) to hold. There is no analogous condition on f in three and higher dimensions, as we discuss in the next section.

Since

$$u(x) = m \log \frac{1}{|x|} + \log \log \frac{1}{|x|}, \qquad m \geq 2,$$

is a C^2 positive solution of $0 \leq -\Delta u \leq e^u$ in a punctured neighborhood of the origin in $\boldsymbol{R}^2$, we see that the conclusion (2.3) of Theorem 2.1 cannot be strengthened to

$$u(x) = m \log \frac{1}{|x|} + O(1) \quad \text{as} \quad |x| \to 0^+ \tag{2.4}$$

for some $m \in (0, \infty)$. However, (2.4) does hold if the condition on u in Theorem 2.1 is slightly strengthened. More precisely, as shown in [9] and [10], if u is a C^2 positive solution in a punctured neighborhood of the origin in $\boldsymbol{R}^2$ of either

$$ae^u \leq -\Delta u \leq e^u \quad \text{or} \quad 0 \leq -\Delta u \leq f(u)$$

where $a \in (0,1)$ is a constant and $f\colon (0, \infty) \to (0, \infty)$ is a continuous function satisfying $\log f(t) = o(t)$ as $t \to \infty$ then either u satisfies (2.4) for some $m \in (0, \infty)$ or u has a C^1 extension to the origin.

Proof of Theorem 2.1. Since u is positive and superharmonic in a punctured neighborhood of the origin, u is bounded below by some positive constant in some smaller punctured neighborhood of the origin. Therefore, using (2.1) and (2.2) and scaling u and x appropriately, we find that it suffices to prove Theorem 2.1 under the assumption that u is a C^2 positive solution of

$$0 \leq -\Delta u \leq e^u \quad \text{in} \quad B_{2r_0}(0) - \{0\} \tag{2.5}$$

for some $r_0 \in (0, 1/4)$.

Let $\Omega = B_{r_0}(0)$. As shown in [9], the fact that u is positive and superharmonic in $B_{2r_0}(0) - \{0\}$ implies that

$$u, -\Delta u \in L^1(\Omega) \tag{2.6}$$

and that there exists a nonnegative constant m and a continuous function $h\colon \overline{\Omega} \to \boldsymbol{R}$, which is harmonic in Ω, such that

$$u(x) = m \log \frac{1}{|x|} + N(x) + h(x) \quad \text{for} \quad x \in \overline{\Omega} - \{0\}, \tag{2.7}$$

where

$$N(x) = \frac{1}{2\pi} \int_\Omega \left(\log \frac{1}{|x-y|} \right) (-\Delta u(y)) \, dy \tag{2.8}$$

is the Newtonian potential of $-\Delta u$ in Ω.

It was proved in [11, Theorem 2.3] that

$$u(y) = O\left(\log \frac{1}{2|y|} \right) \quad \text{as} \quad |y| \to 0^+.$$

It therefore follows from (2.5) that there exists a positive constant C such that

$$0 \leq -\Delta u(y) \leq \frac{1}{(2|y|)^C} \quad \text{for} \quad y \in \Omega - \{0\}. \tag{2.9}$$

To complete the proof of Theorem 2.1 we need the following lemma.

Lemma 2.1. *$N(x) = o(\log \frac{1}{|x|})$ as $|x| \to 0^+$.*

Proof. Let $\varepsilon > 0$ and $M = \frac{2}{\varepsilon}\int_\Omega -\Delta u(y)\,dy + 1$. For $|x|$ small and positive we have

$$N(x) = \frac{1}{2\pi}(I(x) + J(x))$$

where

$$\begin{aligned}
I(x) &:= \int_{\substack{|y-x|>\frac{|x|}{2} \\ y\in\Omega}} \left(\log \frac{1}{|x-y|}\right)(-\Delta u(y))\,dy \\
&= \int_{\frac{|x|}{2}<|y-x|<|x|^{1/M}} \left(\log \frac{1}{|x-y|}\right)(-\Delta u(y))\,dy + \int_{\substack{|y-x|>|x|^{1/M} \\ y\in\Omega}} \log\left(\frac{1}{|x-y|}\right)(-\Delta u(y))\,dy \\
&\le \left(\log \frac{2}{|x|}\right) \int_{|y-x|<|x|^{1/M}} -\Delta u(y)\,dy + \frac{1}{M}\left(\log \frac{1}{|x|}\right)\int_\Omega -\Delta u(y)\,dy \\
&\le \frac{2}{M}\left(\log \frac{1}{|x|}\right)\int_\Omega -\Delta u(y)\,dy < \varepsilon \log \frac{1}{|x|}
\end{aligned}$$

and where

$$J(x) := \int_{|y-x|<\frac{|x|}{2}} \left(\log \frac{1}{|x-y|}\right)(-\Delta u(y))\,dy.$$

By (2.9),

$$0 \le -\Delta u(y) \le \frac{1}{|x|^C} \quad \text{for} \quad x, y \in \Omega - \{0\} \quad \text{and} \quad |y-x| < \frac{|x|}{2}. \tag{2.10}$$

Let

$$r(x)^2 = \frac{1}{\pi}E(x)|x|^C$$

where

$$E(x) := \int_{|y-x|<\frac{|x|}{2}} -\Delta u(y)\,dy \to 0 \quad \text{as} \quad |x| \to 0^+$$

by (2.6). Since

$$\int_{|y-x|<r(x)} \frac{dy}{|x|^C} = \frac{\pi r(x)^2}{|x|^C} = \int_{|y-x|<\frac{|x|}{2}} -\Delta u(y)\,dy$$

it follows from (2.10) that

$$\begin{aligned}
J(x) &\le \frac{1}{|x|^C}\int_{|y-x|<r(x)} \left(\log \frac{1}{|x-y|}\right)dy = \frac{1}{|x|^C}\int_{|\zeta|<r(x)} \left(\log \frac{1}{|\zeta|}\right)d\zeta \\
&= \frac{2\pi}{|x|^C}\left(\frac{r(x)^2}{2}\log \frac{1}{r(x)} + \frac{r(x)^2}{4}\right) \\
&= O\left(E(x)\log \frac{1}{E(x)|x|^C}\right) \\
&= o\left(\log \frac{1}{|x|}\right) \quad \text{as} \quad |x| \to 0^+.
\end{aligned}$$

This proves Lemma 2.1. ☐

By Lemma 2.1, Theorem 2.1 is true when the nonnegative constant m in (2.7) is positive. Hence we can assume $m = 0$ and it follows from (2.7), (2.5), and Lemma 2.1 that

$$-\Delta u(y) = O(|y|^{-1/2}) \quad \text{as} \quad |y| \to 0^+.$$

Thus N, and hence u, is bounded in Ω. It follows therefore from (2.5) that $-\Delta u$ is bounded in Ω. Therefore N, and hence u, has a C^1 extension to origin. This completes the proof of Theorem 2.1. ☐

3 Asymptotically radial solutions in three and higher dimensions

The following theorem gives conditions on f such that each C^2 positive solution of (1.3) in three and higher dimensions is asymptotically radial as $|x| \to 0^+$.

Theorem 3.1. *Let $u(x)$ be a C^2 positive solution of*

$$0 \leq -\Delta u \leq f(u) \tag{3.1}$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \geq 3)$, where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\limsup_{t\to\infty} \frac{f(t)}{t^{\frac{n}{n-2}}} \leq \ell \tag{3.2}$$

for some finite positive number ℓ. Then

$$\lim_{|x|\to 0^+} \frac{u(x)}{\bar{u}(|x|)} = 1, \tag{3.3}$$

where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$. Moreover, either

(i) *u has a C^1 extension to the origin,*

(ii) *$\lim_{|x|\to 0^+} |x|^{n-2}u(x) = m$ for some finite positive number m, or*

(iii) *u satisfies the following two conditions:*

$$\lim_{|x|\to 0^+} |x|^{n-2}u(x) = 0 \tag{3.4}$$

and

$$\liminf_{|x|\to 0^+} \left(\log \frac{1}{|x|}\right)^{\frac{n-2}{2}} |x|^{n-2}u(x) \geq \left(\frac{n-2}{\sqrt{2\ell}}\right)^{n-2}. \tag{3.5}$$

In [10], we showed that if $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying $\lim_{t\to\infty} f(t)/t^{n/(n-2)} = \infty$ then (3.1) has a C^2 positive solution u in $\boldsymbol{R}^n - \{0\}$, $n \geq 3$, which does not satisfy (3.3). Thus the condition (3.2) on f in Theorem 3.1 is essentially optimal for (3.3) to hold, but too weak to imply (1.4) because for $0 < \sigma \leq (n-2)/2$ the function

$$u_\sigma(x) := \left(\frac{n-2}{\sqrt{2}}\right)^{n-2} \frac{1}{|x|^{n-2}(\log \frac{1}{|x|})^\sigma} \tag{3.6}$$

is a C^2 positive solution of $0 \le -\Delta u \le u^{\frac{n}{n-2}}$ in a punctured neighborhood of the origin and $u_\sigma(x)$ does not satisfy (1.4). This is in contrast to the situation in two dimensions as discussed in the paragraph following Theorem 2.1.

Proof of Theorem 3.1. Choose $r_0 > 0$ such that u is a C^2 positive solution of (3.1) in $B_{2r_0}(0) - \{0\}$ and let $\Omega = B_{r_0}(0)$. Since u is positive and superharmonic in $B_{2r_0}(0) - \{0\}$, it is well-known (see Li [6]) that

$$u, -\Delta u \in L^1(\Omega) \tag{3.7}$$

and that there exists a nonnegative constant m and a continuous function $h\colon \overline{\Omega} \to \boldsymbol{R}$, which is harmonic in Ω, such that

$$u(x) = \frac{m}{|x|^{n-2}} + N(x) + h(x) \quad \text{for} \quad x \in \overline{\Omega} - \{0\}, \tag{3.8}$$

where

$$N(x) = \alpha_n \int_\Omega \frac{-\Delta u(y)}{|x-y|^{n-2}}\, dy, \qquad x \in \boldsymbol{R}^n,$$

is the Newtonian potential of $-\Delta u$ in Ω. Here $\alpha_n = 1/(n(n-2)\omega_n)$, where ω_n is the volume of the unit ball in $\boldsymbol{R}^n$.

Another consequence of the positivity and superharmonicity of u in $B_{2r_0}(0) - \{0\}$ is that u is bounded below by a positive constant in $\Omega - \{0\}$, and thus by (3.1) and (3.2), there exists a positive constant K such that u is a C^2 positive solution of

$$0 \le -\Delta u \le K u^{\frac{n}{n-2}} \quad \text{in} \quad \Omega - \{0\}. \tag{3.9}$$

It was proved in [11, Theorem 2.1] that

$$u(x) = O(|x|^{2-n}) \quad \text{as} \quad |x| \to 0^+. \tag{3.10}$$

It therefore follows from (3.9) that

$$-\Delta u(x) = O(|x|^{-n}) \quad \text{as} \quad |x| \to 0^+. \tag{3.11}$$

A portion of our proof of Theorem 3.1 will consist of two lemmas, the first of which is

Lemma 3.1. *$N(x) = o(|x|^{2-n})$ as $|x| \to 0^+$.*

Proof. For $|x|$ small and positive we have

$$N(x) = \alpha_n(I(x) + J(x))$$

where

$$\begin{aligned}
I(x) &:= \int_{\substack{|y-x|>\frac{|x|}{2} \\ y\in\Omega}} \frac{-\Delta u(y)}{|y-x|^{n-2}}\, dy \\
&= \int_{\frac{|x|}{2}<|y-x|<\sqrt{|x|}} \frac{-\Delta u(y)}{|y-x|^{n-2}}\, dy + \int_{\substack{|y-x|>\sqrt{|x|} \\ y\in\Omega}} \frac{-\Delta u(y)}{|x-y|^{n-2}}\, dy \\
&\le \left(\frac{2}{|x|}\right)^{n-2} \int_{|y-x|<\sqrt{|x|}} -\Delta u(y)\, dy + \frac{1}{|x|^{\frac{n-2}{2}}} \int_\Omega -\Delta u(y)\, dy \\
&= o(|x|^{2-n}) \quad \text{as} \quad |x| \to 0^+
\end{aligned}$$

by (3.7), and where

$$J(x) := \int\limits_{|y-x|<\frac{|x|}{2}} \frac{-\Delta u(y)}{|y-x|^{n-2}}\, dy.$$

By (3.11), there exists $C > 0$ such that

$$-\Delta u(y) \le C|x|^{-n} \quad \text{for } |x| \text{ small and positive and } |y-x| < \frac{|x|}{2}. \tag{3.12}$$

Let

$$r(x) = \left(\frac{1}{C\omega_n} \int\limits_{|y-x|<\frac{|x|}{2}} -\Delta u(y)\ dy \right)^{\frac{1}{n}} |x|.$$

Since

$$\int\limits_{|y-x|<r(x)} C|x|^{-n}\, dy = \int\limits_{|y-x|<\frac{|x|}{2}} -\Delta u(y)\, dy$$

it follows from (3.12) that

$$\begin{aligned} J(x) &\le \int\limits_{|y-x|<r|x|} \frac{C}{|x|^n} \frac{dy}{|y-x|^{n-2}} = \frac{C}{|x|^n} \int\limits_{|\zeta|<r(x)} \frac{d\zeta}{|\zeta|^{n-2}} \\ &= \frac{C}{|x|^n} \frac{n\omega_n}{2} r(x)^2 = o(|x|^{2-n}) \quad \text{as} \quad |x| \to 0^+. \end{aligned}$$

This proves Lemma 3.1. □

By Lemma 3.1, Theorem 3.1 is true when the nonnegative constant m in (3.8) is positive. Hence we can assume $m = 0$, which implies

$$u(x) = N(x) + h(x) \quad \text{for} \quad x \in \overline{\Omega} - \{0\}. \tag{3.13}$$

Thus, by Lemma 3.1,

$$u(x) = o(|x|^{2-n}) \quad \text{as} \quad |x| \to 0^+. \tag{3.14}$$

We will now prove (3.3). Let $\varepsilon \in (0, 1/2)$ be fixed. For $x \in \Omega - \{0\}$ let

$$\Omega_x = \{y \in \boldsymbol{R}^n : \varepsilon|x| \le |y| \le |x|/\varepsilon\},$$

$$N_1(x) = \alpha_n \int\limits_{\Omega_x \cap \Omega} \frac{-\Delta u(y)}{|y-x|^{n-2}}\, dy, \qquad \text{and} \qquad N_2(x) = \alpha_n \int\limits_{\Omega - \Omega_x} \frac{-\Delta u(y)}{|y-x|^{n-2}}\, dy.$$

Lemma 3.2. *For some positive constant $C = C(n, \Omega, \varepsilon)$ we have*

$$\sup_{y\in\Omega_x} u(y) \le C \inf_{y\in\Omega_x} u(y) \quad \textit{for} \quad |x| \quad \textit{small and positive.}$$

Proof. Choose $x_0 \in \Omega$ such that $\Omega_{x_0} \subset\subset \Omega - \{0\}$. For $0 < \delta < 1$, define $v_\delta \colon \Omega_{x_0} \to \boldsymbol{R}$ by

$$v_\delta(\xi) = u(y), \quad y = \delta\xi \in \Omega_{\delta x_0}.$$

Then for $\xi \in \Omega_{x_0}$,

$$\begin{aligned}\left|\frac{-\Delta v_\delta(\xi)}{v_\delta(\xi)}\right| &= \frac{-\delta^2 \Delta u(y)}{u(y)} \le \delta^2 K u(y)^{\frac{2}{n-2}} \\ &= \frac{K}{|\xi|^2}(|y|^{n-2}u(y))^{\frac{2}{n-2}} \le \frac{K}{(\varepsilon|x_0|)^2}(|y|^{n-2}u(y))^{\frac{2}{n-2}}.\end{aligned}$$

Hence

$$\sup_{\xi\in\Omega_{x_0}}\left|\frac{-\Delta v_\delta(\xi)}{v_\delta(\xi)}\right| \le \frac{K}{(\varepsilon|x_0|)^2}\sup_{y\in\Omega_{\delta x_0}}(|y|^{n-2}u(y))^{\frac{2}{n-2}} \to 0 \quad \text{as} \quad \delta \to 0^+$$

by (3.14). Thus by Harnack's inequality, there exists a constant $C = C(n, \Omega, \Omega_{x_0}) > 0$ such that for δ small and positive we have

$$\begin{aligned}\sup_{y\in\Omega_{\delta x_0}} u(y) &= \sup_{\xi\in\Omega_{x_0}} v_\delta(\xi) \le C \inf_{\xi\in\Omega_{x_0}} v_\delta(\xi) \\ &= C \inf_{y\in\Omega_{\delta x_0}} u(y).\end{aligned}$$

This proves Lemma 3.2. □

By (3.14), we find for $x \in \Omega - \{0\}$ that $g(x) := |x|^2 \sup_{\Omega_x\cap\Omega} u^{\frac{2}{n-2}} \to 0$ as $|x| \to 0^+$. It follows therefore from (3.9) and Lemma 3.2 that for $|x|$ small and positive we have

$$\begin{aligned}N_1(x) &\le \alpha_n \int_{\Omega_x\cap\Omega} \frac{Ku(y)^{\frac{2}{n-2}}u(y)}{|x-y|^{n-2}}\,dy \\ &\le \alpha_n \frac{Kg(x)}{|x|^2} Cu(x) \int_{|y|<\frac{|x|}{\varepsilon}} |y|^{2-n}\,dy \\ &= \alpha_n \frac{n\omega_n}{2\varepsilon^2} KCg(x)u(x) \\ &= o(u(x)) \quad \text{as} \quad |x| \to 0^+. \end{aligned} \tag{3.15}$$

By (3.13), (3.15), and the fact that u is bounded below by a positive constant in $\Omega - \{0\}$, there exists a positive constant c such that

$$N_2(x) + h(x) = u(x) - N_1(x) \ge c \quad \text{for} \quad |x| \quad \text{small and positive.} \tag{3.16}$$

For $x, \xi \in \boldsymbol{R}^n - \{0\}$ and $|x| = |\xi|$ it is easy to check that

$$\left|\frac{|y-\xi|}{|y-x|} - 1\right| \le \frac{2\varepsilon}{1-\varepsilon} < 4\varepsilon \quad \text{for} \quad y \in \boldsymbol{R}^n - \Omega_x$$

by considering separately the two cases $|y| < \varepsilon|x|$ and $|y| > |x|/\varepsilon$. Thus

$$|N_2(x) - N_2(\xi)| < [(1+4\varepsilon)^{n-2} - 1]N_2(\xi) \quad \text{for} \quad x, \xi \in \Omega - \{0\} \quad \text{and} \quad |x| = |\xi|. \tag{3.17}$$

Also, for $x, \xi \in \Omega - \{0\}$ we have

$$\frac{N_2(x) + h(x)}{N_2(\xi) + h(\xi)} - 1 = \frac{(N_2(x) - N_2(\xi)) + (h(x) - h(\xi))}{N_2(\xi) + h(\xi)} \tag{3.18}$$

$$= \frac{\frac{N_2(x)-N_2(\xi)}{N_2(\xi)} + \frac{h(x)-h(\xi)}{N_2(\xi)}}{1 + \frac{h(\xi)}{N_2(\xi)}} \tag{3.19}$$

where the last equation holds if and only if $N_2(\xi) \neq 0$. Using (3.16), (3.17), (3.18), and (3.19) it is easy to check by considering separately the three cases $h(0) = 0$, $N_2(\xi) > 2|h(0)| > 0$, and $N_2(\xi) \leq 2|h(0)| > 0$ that

$$\limsup_{|x|=|\xi|\to 0^+} \left| \frac{N_2(x) + h(x)}{N_2(\xi) + h(\xi)} - 1 \right| \leq \delta$$

where

$$\delta = 2[(1 + 4\varepsilon)^{n-2} - 1] \max\left(1, \frac{|h(0)|}{c}\right). \tag{3.20}$$

Thus, since

$$\frac{u(x)}{u(\xi)} = \frac{N_2(x) + h(x)}{N_2(\xi) + h(\xi)} B(x, \xi),$$

where

$$B(x, \xi) := \frac{1 - \frac{N_1(\xi)}{u(\xi)}}{1 - \frac{N_1(x)}{u(x)}} \to 1 \quad \text{as} \quad |x| = |\xi| \to 0^+,$$

we have

$$\limsup_{|x|=|\xi|\to 0^+} \left| \frac{u(x)}{u(\xi)} - 1 \right| \leq \delta.$$

Hence, since ε is an arbitrary number in the interval $(0, 1/2)$, it follows from the definition (3.20) of δ that (3.3) holds.

Averaging (3.9), increasing the constant K if necessary, and using (3.3) and the positivity of u in $B_{2r_0}(0) - \{0\}$ we see that

$$0 \leq -\Delta \bar{u} \leq K \bar{u}^{\frac{n}{n-2}} \quad \text{in} \quad \Omega - \{0\}.$$

Furthermore, it follows from (3.14) that $r^{n-2}\bar{u}(r) \to 0$ as $r \to 0^+$. Thus, applying Lemma 6.1 to $\bar{u}$, and using (3.3), (3.9), (3.13), and the fact that N has a C^1 extension to the origin when $-\Delta u$ is bounded in Ω, we see that either (i) of Theorem 3.1 holds or (iii) of Theorem 3.1 holds with ℓ replaced with K. However, if ε is any positive number and (3.5) holds with ℓ replaced with K then by sufficiently decreasing the radius r_0 of Ω and using (3.3) and (3.2) we see that u is a C^2 positive solution of

$$0 \leq -\Delta u \leq (\ell + \varepsilon) u^{\frac{n}{n-2}} \quad \text{in} \quad \Omega - \{0\}$$

and thus u satisfies (3.5) with ℓ replaced with $\ell + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, (3.5) holds as stated. This completes the proof of Theorem 3.1. □

4 Asymptotically harmonic solutions in three and higher dimensions

As discussed in the paragraph following Theorem 3.1, the condition (3.2) on f in Theorem 3.1 is too weak to imply (1.4). In the following theorem, we strengthen the condition (3.2) on f in Theorem 3.1 in order to strengthen the conclusion (3.3) of Theorem 3.1 to (1.4), or equivalently, to rule out possibility (iii) of Theorem 3.1.

We use the following notation:

$$\log_1 := \log \qquad \log_2 := \log\circ\log \qquad \log_3 := \log\circ\log\circ\log \qquad \text{etc.}$$

Theorem 4.1. *Let u be a C^2 positive solution of*

$$0 \le -\Delta u \le \frac{u^{\frac{n}{n-2}}}{(\log_1 u)(\log_2 u)\dots(\log_{q-1} u)(\log_q u)^\beta} \tag{4.1}$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \ge 3)$ where $\beta \in (1,\infty)$ and q is a positive integer. Then either (i) or (ii) of Theorem 3.1 hold.

Theorem 4.1 is essentially optimal because a solution of (4.1) when $\beta = 1$ is

$$u(|x|) = \frac{1}{|x|^{n-2}\log_{q+2}\frac{1}{|x|}}$$

which satisfies neither (i) nor (ii) of Theorem 3.1.

Proof of Theorem 4.1. By Theorem 3.1, u satisfies (3.3). Thus, by averaging (4.1), we see that it suffices to prove Theorem 4.1 when u is radial.

Under the change of variables (6.3) and (6.8) used in the proof of Lemma 6.1, we have

$$0 \le -(v''(t)+v'(t)) \le \frac{v(t)^{\frac{n}{n-2}}}{(\log_1 e^t v(t))\dots(\log_{q-1} e^t v(t))(\log_q e^t v(t))^\beta} \tag{4.2}$$

for t large and positive.

Suppose for contradiction that u satisfies (iii) of Theorem 3.1 with $\ell = 1$. Then

$$v(t) > \frac{1}{2}\left(\frac{n-2}{2t}\right)^{\frac{n-2}{2}} \qquad \text{for } t \text{ large and positive} \tag{4.3}$$

and

$$\lim_{t\to\infty} v(t) = 0. \tag{4.4}$$

Hence, for $j = 1, 2, \dots, q$,

$$\log_j(e^t v(t)) = (\log_{j-1} t)(1+o(1)) \quad \text{as} \quad t\to\infty.$$

It follows therefore from (4.2) that

$$0 \le -(v''(t)+v'(t)) \le g(t)v(t)^{\frac{n}{n-2}} \tag{4.5}$$

for t large and positive, where

$$g(t) = \frac{2}{t(\log_1 t)\dots(\log_{q-2} t)(\log_{q-1} t)^\beta}.$$

Multiplying (4.5) by e^t and integrating the resulting inequalities from t_0 to t, where t_0 is positive and large, we obtain

$$0 \le -v'(t) \le e^{-t} I(t) + Ce^{-t} \quad \text{for} \quad t \ge t_0 \tag{4.6}$$

where C is a positive constant and

$$I(t) := \int_{t_0}^{t} e^{\tau} g(\tau) v(\tau)^{\frac{n}{n-2}} \, d\tau.$$

Integrating $I(t)$ by parts we get

$$I(t) = (e^{\tau} g(\tau) v(\tau)^{\frac{n}{n-2}}) \Big|_{\tau=t_0}^{\tau=t} + J(t) \tag{4.7}$$

where

$$\begin{aligned} J(t) &:= -\int_{t_0}^{t} e^{\tau} (g(\tau) v(\tau)^{\frac{n}{n-2}})' \, d\tau \\ &= -\int_{t_0}^{t} e^{\tau} g(\tau) \frac{g'(\tau)}{g(\tau)} v(\tau)^{\frac{n}{n-2}} \, d\tau - \frac{n}{n-2} \int_{t_0}^{t} e^{\tau} g(\tau) v(\tau)^{\frac{n}{n-2}} \frac{v'(\tau)}{v(\tau)} \, d\tau. \end{aligned} \tag{4.8}$$

But $g'(\tau)/g(\tau) = O(1/\tau)$ as $\tau \to \infty$ and by Remark 6.1 and equation (4.4) we have

$$\frac{v'(\tau)}{v(\tau)} = O(v(\tau)^{\frac{2}{n-2}}) = o(1) \quad \text{as} \quad \tau \to \infty.$$

Hence, by increasing t_0,

$$J(t) \le \frac{1}{2} I(t) \quad \text{for} \quad t \ge t_0. \tag{4.9}$$

It follows from (4.3) that $e^{\tau} g(\tau) v(\tau)^{\frac{n}{n-2}} \to \infty$ as $\tau \to \infty$. Thus, by (4.9) and (4.7), there exists $t_1 > t_0$ such that

$$\frac{1}{2} I(t) \le 2e^t g(t) v(t)^{\frac{n}{n-2}} \quad \text{for} \quad t \ge t_1$$

and it follows therefore from (4.6) and (4.3) that

$$\begin{aligned} 0 < -v'(t) &\le 4g(t) v(t)^{\frac{n}{n-2}} + Ce^{-t} \\ &\le 8g(t) v(t)^{\frac{n}{n-2}} \end{aligned} \tag{4.10}$$

for $t \ge t_1$, by increasing t_1 if necessary. Multiplying (4.10) by $v(t)^{-\frac{n}{n-2}}$ and integrating from t_1 to t we get

$$\infty \leftarrow \frac{n-2}{2} \left(\frac{1}{v(t)^{\frac{2}{n-2}}} - \frac{1}{v(t_1)^{\frac{2}{n-2}}} \right) \le 8 \int_{t_1}^{\infty} g(t) \, dt < \infty.$$

This contradiction shows that u does not satisfy (iii) of Theorem 3.1 with $\ell = 1$ and thus by Theorem 3.1, u satisfies either (i) or (ii) of Theorem 3.1. ☐

5 Oscillating solutions in three and higher dimensions

Possibilities (i) and (ii) in Theorem 3.1 give a more precise description of the behavior of u near the origin than possibility (iii) does and it is natural to ask whether (iii) in Theorem 3.1 can be replaced with a more precise statement. The answer, by the following theorem, is essentially no.

Theorem 5.1. *Let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function such that $\varphi(r)$ tends to zero (perhaps very slowly) as $r \to 0^+$. Then there exists a C^2 positive radial solution u of*

$$0 \le -\Delta u \le u^{\frac{n}{n-2}} \tag{5.1}$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \ge 3)$ which satisfies (3.4),

$$\limsup_{|x|\to 0^+} \frac{|x|^{n-2}u(x)}{\varphi(|x|)} \ge 1$$

and

$$\liminf_{|x|\to 0^+} \left(\log \frac{1}{|x|}\right)^{\frac{n-2}{2}} |x|^{n-2}u(x) = \left(\frac{n-2}{\sqrt{2}}\right)^{n-2}.$$

Less precisely, but perhaps more clearly, Theorem 5.1 says there exists a C^2 positive solution of (5.1) in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \ge 3)$ which oscillates between the upper and lower bounds (3.4) and (3.5) of possibility (iii) of Theorem 3.1 as $|x| \to 0^+$.

Proof of Theorem 5.1. Under the change of variables (6.3) and (6.8) used in the proof of Lemma 6.1, proving Theorem 5.1 is equivalent to proving the existence of a positive C^1 solution $w(v)$ of (6.14), (6.15) such that some positive solution of

$$-\frac{dv}{dt} = w(v), \quad t \text{ large and positive,}$$

satisfies $\lim_{t\to\infty} v(t) = 0$,

$$\limsup_{t\to\infty} \frac{v(t)}{g(t)} \ge 1, \tag{5.2}$$

and

$$\liminf_{t\to\infty} t^{\frac{n-2}{2}} v(t) = \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} \tag{5.3}$$

where $g\colon [1,\infty) \to (0,\infty)$ is any prescribed continuous function which tends to zero (perhaps very slowly) as $t \to \infty$.

We can assume g is C^2, $g(1) > 1$, and

$$0 < g''(t) < -g'(t) < 2g(t)^{\frac{n}{n-2}} \to 0 \quad \text{as} \quad t \to \infty \tag{5.4}$$

because there exist functions g satisfying these conditions which are larger at ∞ than any given positive continuous function which tends to 0 as $t \to \infty$.

Let $\varepsilon \in (0,1)$. We inductively define a strictly decreasing sequence $\{v_j\}_{j=0}^{\infty}$ of positive real numbers which converges to zero and a continuous piecewise smooth function $w\colon (0,1] \to (0,1]$ as follows:

Let $v_0 = 1$ and $w(1) = 1$. Then $w(v_0) = v_0^{\frac{n}{n-2}}$. Assume inductively that $v_0 > v_1 > \cdots > v_{4j} > 0$ have been defined, $w(v)$ has been defined for $v_{4j} \leq v \leq 1$, and $w(v_{4j}) = v_{4j}^{\frac{n}{n-2}}$. We now proceed to define v_{4j+1}, v_{4j+2}, v_{4j+3}, $v_{4(j+1)}$, and $w(v)$ for $v_{4(j+1)} \leq v \leq v_{4j}$.

Let $w(v) = v^{\frac{n}{n-2}}$ for $v_{4j+1} \leq v \leq v_{4j}$, where $v_{4j+1} \in (0, v_{4j})$ will be specified momentarily.

The inverse $t(v)$ of the unique solution $v(t)$ of the initial value problem

$$-\frac{dv}{dt} = w(v), \qquad v(1) = v_0 = 1 \tag{5.5}$$

is

$$t(v) = 1 + \int_v^{v_0} \frac{d\bar{v}}{w(\bar{v})}, \qquad v_{4j+1} \leq v \leq v_0$$

and hence

$$\begin{aligned} t(v_{4j+1}) &= t(v_{4j}) + \int_{v_{4j+1}}^{v_{4j}} \frac{d\bar{v}}{\bar{v}^{\frac{n}{n-2}}} \\ &= t(v_{4j}) + \frac{n-2}{2}\left(\frac{1}{v_{4j+1}^{\frac{2}{n-2}}} - \frac{1}{v_{4j}^{\frac{2}{n-2}}}\right). \end{aligned} \tag{5.6}$$

Thus by choosing $v_{4j+1} \in (0, v_{4j}/2)$ sufficiently small and letting $t_{4j+1} = t(v_{4j+1})$ we have $t_{4j+1} > 4j+1$ and

$$t_{4j+1} \leq \frac{(n-2)(1+\varepsilon)}{2} \frac{1}{v_{4j+1}^{\frac{2}{n-2}}}.$$

Hence

$$v(t_{4j+1}) \leq \left(\frac{(n-2)(1+\varepsilon)}{2}\right)^{\frac{n-2}{2}} \frac{1}{t_{4j+1}^{\frac{n-2}{2}}}. \tag{5.7}$$

Let

$$\hat{w}(v) = \frac{-1}{2G'(v)} \tag{5.8}$$

where $t = G(v)$ is the inverse of $v = g(t)$. Thanks to (5.4) we have for $0 < v \leq 1$ that

$$0 < \hat{w}(v) < v^{\frac{n}{n-2}} \quad \text{and} \quad 0 < \hat{w}'(v) < \frac{1}{2} \tag{5.9}$$

which imply $\hat{w}(v)$ is a solution of (6.14), (6.15).

Let $v_{4j+2} \in (0, v_{4j+1})$ be the v-coordinate of a point of intersection of the graph of $\hat{w}(v)$ with the line in the vw-plane of slope one passing through $(v_{4j+1}, v_{4j+1}^{\frac{n}{n-2}})$. By (5.9) there exists such a point of intersection. For $v_{4j+2} \leq v \leq v_{4j+1}$, define

$$w(v) = v_{4j+1}^{\frac{n}{n-2}} + v - v_{4j+1}.$$

Thus the graph of $w(v)$, $v_{4j+2} \leq v \leq v_{4j+1}$, in the vw-plane is a line segment of slope one and $w(v_{4j+2}) = \hat{w}(v_{4j+2})$.

Let $w(v) = \hat{w}(v)$ for $v_{4j+3} \leq v \leq v_{4j+2}$ where v_{4j+3} will be specified momentarily. Analogous to (5.6), the inverse $t(v)$ of the unique solution $v(t)$ of the initial value problem (5.5) satisfies

$$\begin{aligned} t(v_{4j+3}) &= t(v_{4j+2}) + \int_{v_{4j+3}}^{v_{4j+2}} \frac{d\bar{v}}{\hat{w}(\bar{v})} \\ &= t(v_{4j+2}) + 2G(v_{4j+3}) - 2G(v_{4j+2}) \end{aligned}$$

because of (5.8). By choosing $v_{4j+3} \in (0, v_{4j+2}/2)$ sufficiently small and letting $t_{4j+3} = t(v_{4j+3})$ we have $t_{4j+3} > 4j+3$ and $t_{4j+3} \geq G(v_{4j+3})$. Hence

$$g(t_{4j+3}) \leq v_{4j+3} = v(t_{4j+3}). \tag{5.10}$$

For $v_{4j+4} \leq v \leq v_{4j+3}$ let the graph of $w(v)$ be the line segment of the slope zero joining the point $(v_{4j+3}, \hat{w}(v_{4j+3}))$ on the graph of $\hat{w}(v)$ to a point $(v_{4j+4}, v_{4j+4}^{\frac{n}{n-2}})$.

Since $\hat{w}(v)$ is a solution of (6.14), (6.15), so is $w(v)$, and it follows from (5.10) that (5.2) holds. Furthermore, by (5.7) equation (5.3) holds with the equal sign replaced with $\leq$. But by Theorem 3.1, equation (5.3) holds with the equal sign replaced with $\geq$. Thus (5.3) holds as stated.

The function $w(v)$, $0 < v < 1$, is continuous and $w'(v)$ is piecewise continuous. But we need w to be C^1 and this can be achieved by rounding off the corners of the graph of $w(v)$ in any one of several standard ways. This completes the proof of Theorem 5.1. □

6 Radial solutions in three and higher dimensions

In Sections 3, 4, and 5 we will need the following lemma concerning positive radial solutions of (1.3) when $f(t)$ is a positive multiple of $t^{\frac{n}{n-2}}$.

Lemma 6.1. *Let $u(|x|)$ be a C^2 positive radial solution of*

$$0 \leq -\Delta u \leq \ell u^{\frac{n}{n-2}} \tag{6.1}$$

in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \geq 3)$ where ℓ is a positive number. Then either

(i) *$u(r)$ tends to some finite positive number as $r \to 0^+$,*

(ii) *$r^{n-2}u(r)$ tends to some finite positive number as $r \to 0^+$, or*

(iii) *u satisfies the following two conditions:*

$$\lim_{r\to 0^+} r^{n-2}u(r) = 0$$

and

$$\liminf_{r\to 0^+} \left(\log \frac{1}{r}\right)^{\frac{n-2}{2}} r^{n-2}u(r) \geq \left(\frac{n-2}{\sqrt{2\ell}}\right)^{n-2}. \tag{6.2}$$

Proof. By scaling u we see that it suffices to prove Lemma 6.1 when $\ell = 1$.

Making the change of independent variable

$$s = \left(\frac{n-2}{r}\right)^{n-2} \tag{6.3}$$

in inequalities (6.1) we find that $u(s)$ is a positive solution of

$$0 \leq -\frac{d^2u}{ds^2} \leq \frac{1}{s}\left(\frac{u}{s}\right)^{\frac{n}{n-2}} \quad \text{for large } s > 0. \tag{6.4}$$

Thus, for some $m_0 \in [0, \infty)$, $u'(s) \searrow m_0$ as $s \to \infty$. In particular $u'(s) \geq 0$ for large $s > 0$. Hence, for some $u_0 \in (0, \infty]$, $\lim_{s\to\infty} u(s) = u_0$. If $u_0 \in (0, \infty)$ then (i) holds. Consequently we can assume

$$\lim_{s\to\infty} u(s) = \infty. \tag{6.5}$$

Thus, by L'Hospital's rule,

$$\lim_{s\to\infty} \frac{u(s)}{s} = \lim_{s\to\infty} u'(s) = m_0.$$

If $m_0 \in (0,\infty)$ then (ii) holds. So we can assume

$$\lim_{s\to\infty} \frac{u(s)}{s} = \lim_{s\to\infty} u'(s) = 0. \tag{6.6}$$

Hence, to complete the proof of Lemma 6.1, it suffices to show u satisfies (6.2), which written in terms of s is

$$\liminf_{s\to\infty} (\log s)^{\frac{n-2}{2}} \frac{u(s)}{s} \geq \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}. \tag{6.7}$$

Making the change of variables

$$u(s) = sv(t), \qquad t = \log s \tag{6.8}$$

in (6.4), (6.5), and (6.6) we find that $v(t)$ is a positive solution of

$$0 \leq -(v''(t) + v'(t)) \leq v(t)^{\frac{n}{n-2}} \quad \text{for large } t > 0 \tag{6.9}$$

$$\lim_{t\to\infty} e^t v(t) = \infty \tag{6.10}$$

and

$$\lim_{t\to\infty} v(t) = 0 = \lim_{t\to\infty} v'(t) \tag{6.11}$$

and to complete the proof of Lemma 6.1 it suffices to prove

$$\liminf_{t\to\infty} t^{\frac{n-2}{2}} v(t) \geq \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} \tag{6.12}$$

which is equivalent to (6.7) under the change of variables (6.8).

It follows from the first equation of (6.11) and the positivity of v that $v'(t_0) < 0$ for some $t_0 > 0$ and it follows from the first inequality of (6.9) that

$$v'(t) \leq e^{t_0} v'(t_0) e^{-t} < 0 \quad \text{for} \quad t \geq t_0.$$

Thus

$$w := -\frac{dv}{dt} \tag{6.13}$$

can be viewed as a function of v instead of t and it follows from (6.9) and (6.11) that w is a positive solution of

$$1 - \frac{v^{\frac{n}{n-2}}}{w} \leq \frac{dw}{dv} \leq 1 \quad \text{for small } v > 0 \tag{6.14}$$

$$\lim_{v\to 0^+} w = 0. \tag{6.15}$$

To complete the proof of Lemma 6.1, we need the following lemma.

Lemma 6.2. *Let A and q be fixed positive constants. Suppose, for some strictly decreasing sequence v_j of real numbers tending to zero we have $w(v_j) = Av_j^q$. If $q = 1$ then $A = 1$. If $q = n/(n-2)$ then $A \leq 1$.*

Proof. For some subsequences $\hat{v}_j$ and $\bar{v}_j$ of v_j we have

$$w'(\hat{v}_j) \geq Aq\hat{v}_j^{q-1} \tag{6.16}$$

and

$$Aq\bar{v}_j^{q-1} \geq w'(\bar{v}_j) \geq 1 - \frac{\bar{v}_j^{\frac{n}{n-2}}}{w(\bar{v}_j)} = 1 - \frac{\bar{v}_j^{\frac{n}{n-2}-q}}{A} \tag{6.17}$$

by (6.14).

If $q = 1$ then by (6.14), (6.16), and (6.17),

$$1 \geq w'(\hat{v}_j) \geq A \geq 1 - \frac{\bar{v}_j^{\frac{2}{n-2}}}{A} \to 1 \quad \text{as} \quad j \to \infty$$

and thus $A = 1$.

If $q = \frac{n}{n-2}$ then by (6.17), $1 - \frac{1}{A} \leq 0$ and thus $A \leq 1$. □

Continuing with the proof of Lemma 6.1, let $\varepsilon \in (0, 1/2)$. By Lemma 6.2, one and only one of the following three possibilities holds:

$$(1-\varepsilon)v < w(v) \leq v \qquad \text{for small } v > 0, \tag{6.18}$$

$$\frac{v^{\frac{n}{n-2}}}{1-\varepsilon} < w(v) < (1-\varepsilon)v \quad \text{for small } v > 0, \tag{6.19}$$

or

$$0 < w(v) < \frac{v^{\frac{n}{n-2}}}{1-\varepsilon} \qquad \text{for small } v > 0. \tag{6.20}$$

We now show neither (6.18) nor (6.19) can hold. Suppose for contradiction (6.18) holds. Then by (6.14)

$$\frac{dw}{dv} > 1 - \frac{v^{\frac{n}{n-2}}}{(1-\varepsilon)v} > 1 - 2v^{\frac{2}{n-2}} \quad \text{for small } v > 0.$$

Integrating from 0 to v and using (6.15) we get

$$-\frac{dv}{dt} = w \geq v - \frac{2(n-2)}{n} v^{\frac{n}{n-2}} \quad \text{for large } t > 0$$

which together with (6.11) implies $v(t) = O(e^{-t})$ as $t \to \infty$ which in turn contradicts (6.10). Hence (6.18) is impossible.

Suppose for contradiction (6.19) holds. Then by (6.14), $\frac{dw}{dv} \geq \varepsilon$ for small $v > 0$ and thus by (6.15), $w > \varepsilon v$ for small $v > 0$ and hence again by (6.14)

$$\frac{dw}{dv} \geq 1 - \frac{v^{\frac{n}{n-2}-1}}{\varepsilon} \to 1 \quad \text{as} \quad v \to 0^+$$

which contradicts the second inequality of (6.19). Thus (6.20) holds. Replacing $w(v)$ with $-\frac{dv}{dt}$ in (6.20) we obtain

$$0 < -\frac{dv}{dt} < \frac{v^{\frac{n}{n-2}}}{1-\varepsilon} \quad \text{for large } t > 0 \tag{6.21}$$

from which we easily deduce that

$$\liminf_{t\to\infty} \frac{v(t)}{\left(\frac{n-2}{2}\frac{1}{t}\right)^{\frac{n-2}{2}}} \geq (1-\varepsilon)^{\frac{n-2}{2}}.$$

Since $\varepsilon \in (0, 1/2)$ is arbitrary we obtain (6.12) and the proof of Lemma 6.1 is complete. □

Remark 6.1. The proof of Lemma 6.1 shows that if $u(|x|)$ is a C^2 positive radial solution of (6.1) with $\ell = 1$ in a punctured neighborhood of the origin in $\boldsymbol{R}^n$ $(n \geq 3)$ which satisfies neither (i) nor (ii), and $v(t)$ is defined in terms of u by (6.3) and (6.8) then $v(t)$ satisfies (6.21). This fact is used in the proof of Theorem 4.1.

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