Isolated Singularities of Nonlinear Parabolic Inequalities

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Abstract

We study $C^{2,1}$ nonnegative solutions $u(x,t)$ of the nonlinear parabolic inequalities

$$0 \le u_t - \Delta u \le u^\lambda$$

in a punctured neighborhood of the origin in $\mathbf{R}^n \times [0,\infty)$, when $n \ge 1$ and $\lambda > 0$.

We show that a necessary and sufficient condition on λ for such solutions u to satisfy an a priori bound near the origin is $\lambda \le \frac{n+2}{n}$, and in this case, the a priori bound on u is

$$u(x,t) = O(t^{-n/2}) \quad \text{as} \quad (x,t) \to (0,0), \quad t > 0.$$

This a priori bound for u can be improved by imposing an upper bound on the initial condition of u.

1 Introduction

In this paper, we study $C^{2,1}$ nonnegative solutions $u(x,t)$ of the nonlinear parabolic inequalities

$$0 \le u_t - \Delta u \le f(u)$$

in a punctured neighborhood of the origin in $\mathbf{R}^n \times [0,\infty)$, where $f\colon [0,\infty) \to [0,\infty)$ is a given continuous function. In particular, we give nearly optimal conditions on f such that all such solutions satisfy an a priori bound near the origin.

For the sake of clarity, we discuss in this section weaker, but simpler, versions of our main results in Sections 3, 4, and 5.

Our first result is the following theorem, which is an immediate consequence of Theorem 3.1 in Section 3.

Theorem 1.1. *Let $u(x,t)$ be a $C^{2,1}$ nonnegative solution of the differential inequalities*

$$0 \le u_t - \Delta u \le u^{\frac{n+2}{n}} + 1 \tag{1.1}$$

in a punctured neighborhood of the origin in $\mathbf{R}^n \times [0,\infty)$, $n \ge 1$. Then

$$u(x,t) = O(1/t^{n/2}) \quad \textit{as} \quad (x,t) \to (0,0), \quad t > 0. \tag{1.2}$$

Remark 1. Note that there are no initial or boundary conditions imposed on u in Theorem 1.1.

Remark 2. One of the main accomplishments of this paper is the proof of Theorem 1.1 when the nonlinear term on the right side of (1.1) is $u^{\frac{n+2}{n}}$. When the nonlinear term is u^λ, $\lambda < \frac{n+2}{n}$, the proof of Theorem 1.1 is much easier.

Remark 3. Theorem 1.1 is optimal in two ways. First, the exponent $n/2$ on t in (1.2) cannot be decreased because the heat kernel

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & \text{if } t > 0 \\ 0, & \text{if } t \le 0 \end{cases} \tag{1.3}$$

is a $C^{2,1}$ nonnegative solution of (1.1) in $(\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\}$ and $\Phi(0,t)t^{n/2} \to (4\pi)^{-n/2}$ as $t \to 0^+$. And second, the exponent $\frac{n+2}{n}$ on u in (1.1) cannot be increased by the following theorem, which is an immediate consequence of Theorem 4.1 in Section 4.

Theorem 1.2. *Let $\lambda \in \left(\frac{n+2}{n}, \infty\right)$ and let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function. Then there exists a $C^{2,1}$ nonnegative solution $u(x,t)$ of*

$$0 \le u_t - \Delta u \le u^\lambda \quad \text{in} \quad (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\} \tag{1.4}$$

satisfying $u \equiv 0$ in $\mathbf{R}^n \times (-\infty, 0)$ and

$$u(0,t) \ne O(\varphi(t)) \quad \text{as} \quad t \to 0^+.$$

Remark 4. In Theorem 1.1, $u(x,0)$ is required to be finite for each x in some punctured neighborhood of the origin in $\mathbf{R}^n$, but the upper bound $t^{-n/2}$ for $u(x,t)$ is infinite at each point of the hyperplane $t = 0$. It would be desirable to obtain an a priori upper bound for the function $u(x,t)$ in Theorem 1.1 of the form

$$u(x,t) = O(\varphi(x,t)) \quad \text{as} \quad (x,t) \to (0,0), \quad t > 0, \tag{1.5}$$

for some function φ which, like u, is continuous in some punctured neighborhood of the origin in $\mathbf{R}^n \times [0,\infty)$, and in particular, finite when $t = 0$ and $x \ne 0$. However, this is not possible because given such a φ, no matter how large, we can choose a smooth L^1 function $\psi\colon (\mathbf{R}^n - \{0\}) \to (0,\infty)$ such that

$$\limsup_{x \to 0} \frac{\psi(x)}{\varphi(x,0)} = \infty$$

and then take u to be the $C^{2,1}$ positive solution of

$$u_t - \Delta u = 0 \quad \text{in} \quad \mathbf{R}^n \times [0,\infty) - \{(0,0)\}$$

satisfying $u(x,0) = \psi(x)$ for $x \in \mathbf{R}^n - \{0\}$ given by

$$u(x,t) = \int_{\mathbf{R}^n} \Phi(x-y,t)\psi(y)\,dy.$$

Clearly this solution u does not satisfy (1.5).

In the following theorem, which is an immediate consequence of Theorem 5.1 in Section 5, we obtain an a priori upper bound for the function u in Theorem 1.1 of the form (1.5) for some function φ as above by imposing an upper bound on the initial condition of u.

Theorem 1.3. *Let u be as in Theorem 1.1 and suppose*

$$u(x,0) = O\left(\frac{1}{|x|^p}\right) \quad as \quad x \to 0 \tag{1.6}$$

for some constant $p \in [0,n]$. Then for each positive constant q,

$$u(x,t) = O(\varphi(x,t;p,q)) \quad as \quad (x,t) \to (0,0),\ t \geq 0, \tag{1.7}$$

where

$$\varphi(x,t;p,q) = \begin{cases} \dfrac{1}{t^{n/2}} + \dfrac{1}{t^{p/2}}, & t > |x|^2 \\ \dfrac{1}{|x|^p} + \left(\dfrac{t}{|x|^2}\right)^q \dfrac{1}{|x|^n}, & 0 \leq t \leq |x|^2 > 0. \end{cases}$$

Remark 5. The condition in Theorem 1.3 that u satisfy (1.6) for some $p \leq n$ is not a big restriction on u because if $p \geq n$ then there are no functions u satisfying the conditions of Theorem 1.1 and also satisfying

$$u(x,0) \geq \frac{1}{|x|^p} \quad \text{for } |x| \text{ small and positive}$$

because $u(x,0)$ is necessarily summable in some neighborhood of the orgin in $\mathbf{R}^n$ by Theorem 2.1 in the next section.

Remark 6. The term $t^{-p/2}$ in the definition of φ in the region $t > |x|^2$ is not really necessary because $t^{-p/2} \leq t^{-n/2}$ for $0 < t < 1$. We only insert this term so that φ is continuous on the punctured hypersurface $t = |x|^2 > 0$ and therefore continuous on $\mathbf{R}^n \times [0,\infty) - \{(0,0)\}$.

Remark 7. The estimate Theorem 1.3 gives for u in the region $t > |x|^2$ cannot be improved because $u(x,t) = \Phi(x,t)$ satisfies the hypotheses of Theorem 1.3 and $t^{n/2}\Phi(x,t)$ is bounded between positive constants in the region $t > |x|^2$.

Remark 8. The larger we take q in Theorem 1.3, the better our estimate (1.7) for u becomes in the region $0 \leq t \leq |x|^2$.

Remark 9. The graph in the region $0 \leq t \leq |x|^2$ of the term $\left(\frac{t}{|x|^2}\right)^q \frac{1}{|x|^n}$ in the definition of φ has the same basic shape as the graph of $\Phi(x,t)$ in that region. In particular, each term is zero on the punctured hyperplane $t = 0 < |x|$ and each term is a positive constant multiple of $|x|^{-n}$ on the punctured hypersurface $t = |x|^2 > 0$.

Remark 10. Let $\hat{\varphi}(x,t;p)$ be the function obtained from the function $\varphi(x,t;p,q)$ in Theorem 1.3 by replacing the term $\left(\frac{t}{|x|^2}\right)^q \frac{1}{|x|^n}$ in the definition of φ in the region $0 \leq t \leq |x|^2$ with $\Phi(x,t)$. We conjecture that Theorem 1.3 is true when φ is replaced with $\hat{\varphi}$. If this conjecture is true, it would be optimal when $p \in [0,n)$ because

$$u(x,t;p) := \Phi(x,t) + \int_{\mathbf{R}^n} \Phi(x-y,t)\frac{1}{|y|^p}\,dy$$

satisfies the hypotheses of Theorem 1.3 when $p \in [0,n)$ and $u(x,t;p)/\hat{\varphi}(x,t;p)$ is bounded between positive constants in a punctured neighborhood of the origin in $\mathbf{R}^n \times [0,\infty)$.

Remark 11. Theorems 1.1 and 1.3 can be strengthened by relaxing somewhat the regularity conditions on u. Also, Theorem 1.2 can be strengthened by replacing the nonlinear term u^λ in (1.4) with a smaller nonlinear term. The statements and proofs of these stronger versions of Theorems 1.1, 1.2, and 1.3 are given in Sections 3, 4, and 5 respectively.

The proofs of Theorems 1.1 and 1.3 rely heavily on the fact that the function u in Theorem 1.1 satisfies equation (2.8) in Theorem 2.1 below. A crucial step in the proofs of Theorems 1.1 and 1.3 is an adaptation and extension to parabolic inequalities of a method of Brezis [4] concerning elliptic equations and based on Moser's iteration. This method is used to obtain an estimate of the form

$$\|u_j\|_{L^{\frac{n+2}{n}p}(\Omega')} \le C\|u_j\|_{L^p(\Omega)}$$

where $p > 1$, $\Omega' \subset \Omega$, C is a constant which does not depend on j, and u_j, $j = 1, 2, \ldots$, is obtained from the function u in Theorem 1.1 by appropriately scaling u about (x_j, t_j) where (x_j, t_j) is a sequence tending to the origin for which (1.2) is violated. Our proofs also require certain estimates for the heat potential which can be found in Appendix A.

For results related to those in this paper, see [1, 2, 3, 5, 7, 9, 12, 14, 15, 16, 19]. An elliptic analog of the results in this paper can be found in [17, 18].

2 Nonnegative solutions of $u_t - \Delta u \ge 0$

For the proof of Theorem 3.1 in Section 3, we will need the following theorem, which gives a description of nonnegative solutions of $u_t - \Delta u \ge 0$ in a punctured neighborhood of the origin in $\mathbf{R}^n \times [0, \infty)$. Brezis and Lions [6] proved a similar result for nonnegative solutions of $-\Delta u \ge 0$ in a punctured neighborhood of the origin in $\mathbf{R}^n$. Their result is also a consequence of Doob's results [8] on superharmonic functions.

Theorem 2.1. *Suppose*

$$u \in C((B_3(0) \times [0, 3)) - \{(0, 0)\}) \tag{2.1}$$

is a nonnegative function such that

$$Hu \in L^1_{\text{loc}}(B_3(0) \times (0, 3)) \tag{2.2}$$

and

$$Hu \ge 0 \quad \text{in} \quad B_3(0) \times (0, 3) \subset \mathbf{R}^n \times \mathbf{R}, \quad n \ge 1, \tag{2.3}$$

where $Hu = u_t - \Delta u$ is the heat operator. Then

$$u(\cdot, 0) \in L^1(B_2(0)) \tag{2.4}$$

and

$$u, Hu \in L^1(B_2(0) \times (0, 2)). \tag{2.5}$$

Moreover, for some finite nonnegative number m and some $h \in C^{2,1}(B_1(0) \times (-1, 1))$ satisfying

$$Hh = 0 \quad \text{in} \quad B_1(0) \times (-1, 1) \tag{2.6}$$

$$h = 0 \quad \text{in} \quad B_1(0) \times (-1, 0] \tag{2.7}$$

we have

$$u = m\Phi + N + v + h \quad \text{in} \quad B_1(0) \times (0, 1), \tag{2.8}$$

where Φ is given by (1.3),

$$N(x, t) := \int_0^2 \int_{|y|<2} \Phi(x - y, t - s)(Hu(y, s))\, dy\, ds, \tag{2.9}$$

and

$$v(x,t) := \int\limits_{|y|<2} \Phi(x-y,t)u(y,0)\,dy. \tag{2.10}$$

Moutoussamy and Véron [13, Theorem 1.1] prove (2.5) under slightly different conditions on the function u. Our proof below of (2.4) and (2.5) is a modification of their proof. However the main part of Theorem 2.1 is the representation formula (2.8).

Proof of Theorem 2.1. Let $\varphi \in C_0^\infty(B_3(0))$ be a nonnegative function satisfying $\varphi \equiv 1$ in $B_2(0)$. Let $t_0 \in (0,2)$ be fixed, and let $\psi_\varepsilon \colon \mathbf{R} \to [0,1]$, ε small and positive, be a one parameter family of smooth functions such that

$$\psi_\varepsilon(t) = \begin{cases} 1, & t_0+\varepsilon \le t \le 2-\varepsilon \\ 0, & t \le t_0-\varepsilon \text{ or } t \ge 2+\varepsilon \end{cases}$$

and $\psi'_\varepsilon \neq 0$ on $(t_0-\varepsilon, t_0+\varepsilon) \cup (2-\varepsilon, 2+\varepsilon)$. Then by (2.2),

$$\int (Hu)\psi_\varepsilon\varphi = -\int u\psi'_\varepsilon\varphi - \int u\psi_\varepsilon\Delta\varphi$$

and letting $\varepsilon \to 0^+$ we obtain

$$\int_{t_0}^2 \int\limits_{|x|<3} Hu(x,t)\varphi(x)\,dx = \int\limits_{|x|<3} u(x,2)\varphi(x)\,dx - \int\limits_{|x|<3} u(x,t_0)\varphi(x)\,dx - \int_{t_0}^2 \int\limits_{2<|x|<3} u(x,t)\Delta\varphi(x)\,dx\,dt.$$

Hence, for $t_0 \in (0,2)$, we have

$$\begin{aligned} \int\limits_{|x|<2} u(x,t_0)\,dx + \int_{t_0}^2 \int\limits_{|x|<2} Hu(x,t)\,dx\,dt &\le \int\limits_{|x|<3} u(x,t_0)\varphi(x)\,dx + \int_{t_0}^2 \int\limits_{|x|<3} Hu(x,t)\varphi(x)\,dx\,dt \\ &\le \int\limits_{|x|<3} u(x,2)\varphi(x)\,dx + \int_0^2 \int\limits_{2<|x|<3} u(x,t)|\Delta\varphi(x)|\,dx\,dt < \infty \end{aligned} \tag{2.11}$$

by (2.1). Thus, letting $t_0 \to 0^+$ and using Fatou's lemma, the monotone convergence theorem, and (2.3), we obtain (2.4) and (2.5). Consequently, letting

$$f = \begin{cases} Hu & \text{in } B_2(0)\times(0,2) \\ 0 & \text{elsewhere in } \mathbf{R}^n \times \mathbf{R} \end{cases} \tag{2.12}$$

we have

$$f \in L^1(\mathbf{R}^n \times \mathbf{R}). \tag{2.13}$$

For $(x,t) \in \mathbf{R}^n \times (0,\infty)$, let v be defined by (2.10) and define

$$v(x,0) = u(x,0) \quad \text{for} \quad x \in B_2(0) - \{0\}.$$

Since (2.1) and (2.4) hold, it is well-known that

$$v \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(B_2(0)\times[0,\infty) - \{(0,0)\})$$

and $Hv = 0$ in $\mathbf{R}^n \times (0,\infty)$. Also

$$\int\limits_{\mathbf{R}^n} v(x,t)\,dx = \int\limits_{|y|<2} u(y,0)\,dy < \infty \quad \text{for} \quad t > 0 \tag{2.14}$$

and thus $v \in L^1(\mathbf{R}^n \times (0,2))$. Define

$$w = \begin{cases} u - v, & \text{in } B_2(0) \times [0,2) - \{(0,0)\} \\ 0, & \text{elsewhere in } \mathbf{R}^n \times \mathbf{R}. \end{cases} \tag{2.15}$$

Then

$$w \in C((B_2(0) \times [0,2)) - \{(0,0)\}) \cap L^1(\mathbf{R}^n \times \mathbf{R}) \tag{2.16}$$

and

$$\begin{array}{lll} Hw = f & \text{in} & \mathcal{D}'(B_2(0) \times (0,2)) \\ w(x,0) = 0 & \text{for} & x \in B_2(0) - \{0\}. \end{array}$$

Also, by (2.11) and (2.14),

$$\int\limits_{|x|<2} |w(x,t)|\,dx \quad \text{is bounded for } 0 < t < 2. \tag{2.17}$$

Let $\Omega = B_1(0) \times (-1,1)$ and define $\Lambda \in \mathcal{D}'(\Omega)$ by $\Lambda = -Hw + f$, that is

$$\Lambda\varphi = \int wH^*\varphi + \int f\varphi \quad \text{for} \quad \varphi \in C_0^\infty(\Omega),$$

where $H^*\varphi := \varphi_t + \Delta\varphi$. By (2.13) and (2.16), Λ is a distribution of order two in Ω. We now show the support of Λ is contained in $\{(0,0)\}$. Let $\varphi \in C_0^\infty(\Omega)$ and suppose $(0,0) \notin \operatorname{supp} \varphi$. Then for some small $r > 0$, $\varphi(x,t) \equiv 0$ for $|x| < r$ and $|t| < r$. Let $t_0 \in (0,r)$ be fixed, and let $\psi_\varepsilon \colon \mathbf{R} \to [0,1]$, ε small and positive, be a one parameter family of smooth nondecreasing functions such that

$$\psi_\varepsilon(t) = \begin{cases} 1, & t > t_0 + \varepsilon \\ 0, & t < t_0 - \varepsilon. \end{cases}$$

Then

$$\begin{aligned} -\int f\varphi\psi_\varepsilon = -\int (Hw)\varphi\psi_\varepsilon &= \int wH^*(\varphi\psi_\varepsilon) \\ &= \int w(\varphi_t\psi_\varepsilon + \varphi\psi_\varepsilon' + \psi_\varepsilon\Delta\varphi) \\ &= \int w\psi_\varepsilon H^*\varphi + \int w\varphi\psi_\varepsilon' \end{aligned}$$

and letting $\varepsilon \to 0^+$ we get

$$-\int_{t_0}^1 \int\limits_{|x|<1} f\varphi\,dx\,dt = \int_{t_0}^1 \int\limits_{|x|<1} wH^*\varphi\,dx\,dt + \int\limits_{r<|x|<1} w(x,t_0)\varphi(x,t_0)\,dx.$$

Next, letting $t_0 \to 0^+$, we obtain

$$-\int f\varphi = \int wH^*\varphi,$$

where we have used the fact that $\lim_{t\to 0^+} w(x,t) = w(x,0) = 0$ uniformly for $r \le |x| \le 1$, which follows from (2.16). So $\Lambda\varphi = 0$ and thus Λ is a distribution of order two whose support is contained in $\{(0,0)\}$. Hence

$$\Lambda = -m\delta + \sum_{1\le|\alpha|\le 2} a_\alpha D^\alpha \delta \tag{2.18}$$

is a linear combination of the delta function and its derivatives of order at most two.

We now use a method of Brezis and Lions [6] to show $a_\alpha = 0$ for $1 \le |\alpha| \le 2$. Choose $\varphi \in C_0^\infty(\Omega)$ such that

$$(-1)^{|\alpha|} D^\alpha \varphi(0,0) = a_\alpha \quad \text{for} \quad 1 \le |\alpha| \le 2.$$

For $\varepsilon \in (0,1)$, define $\varphi_\varepsilon \in C_0^\infty(B_\varepsilon(0) \times (-\varepsilon^2, \varepsilon^2))$ by $\varphi_\varepsilon(x,t) = \varphi(x/\varepsilon, t/\varepsilon^2)$. Then for $1 \le |\alpha| \le 2$,

$$D^\alpha \varphi_\varepsilon(0,0) = D^\alpha \varphi(0,0)/\varepsilon^{p_\alpha} \quad \text{for some positive integer } p_\alpha,$$

and so

$$\left(\sum_{1\le|\alpha|\le 2} a_\alpha D^\alpha \delta\right) \varphi_\varepsilon = \sum_{1\le|\alpha|\le 2} \frac{a_\alpha^2}{\varepsilon^{p_\alpha}} \ge \frac{1}{\varepsilon} \sum_{1\le|\alpha|\le 2} a_\alpha^2. \tag{2.19}$$

On the other hand, since $\sup_{\mathbf{R}^n\times\mathbf{R}} |H^*\varphi_\varepsilon| = O(\varepsilon^{-2})$ as $\varepsilon \to 0^+$, we have

$$(\Lambda - f)\varphi_\varepsilon = \int wH^*\varphi_\varepsilon = O\left(\frac{1}{\varepsilon^2}\int_0^{\varepsilon^2} \int\limits_{|x|<\varepsilon} |w(x,t)|\, dx\, dt\right) = O(1) \quad \text{as} \quad \varepsilon \to 0^+$$

by (2.17). Therefore, applying (2.18) to φ_ε and using (2.19) we find that $a_\alpha = 0$ for $1 \le |\alpha| \le 2$ and so $\Lambda = -m\delta$.

By (2.9) and (2.12), we can extend N to a function on $\mathbf{R}^n \times \mathbf{R}$ by the formula

$$N(x,t) = \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y, t-s) f(y,s)\, dy\, ds.$$

Then

$$N \equiv 0 \quad \text{in} \quad \mathbf{R}^n \times (-\infty, 0). \tag{2.20}$$

By (2.13), we have $N \in L^1(\Omega)$ and $HN = f$ in $\mathcal{D}'(\Omega)$. Also $\delta = H\Phi$ in $\mathcal{D}'(\Omega)$. Consequently

$$H(w - N - m\Phi) = -\Lambda + f - f - m\delta = 0 \quad \text{in} \quad \mathcal{D}'(\Omega),$$

which implies

$$w = m\Phi + N + h \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{2.21}$$

for some $C^{2,1}$ solution h of (2.6). It follows therefore from (2.15) that (2.8) holds. By (2.15), (1.3), (2.20), and (2.21), we have (2.7) holds.

Finally, we now show $m \ge 0$. It follows from Fatou's Lemma and (2.1) that

$$\liminf_{t\to 0^+} \int\limits_{|x|<1/2} u(x,t)\, dx \ge \int\limits_{|x|<1/2} u(x,0)\, dx = \lim_{t\to 0^+} \int\limits_{|x|<1/2} v(x,t)\, dx.$$

Also,

$$\int_{\mathbf{R}^n} N(x,t)\, dx = \int_0^t \int_{|y|<2} f(y,s)\, dy\, ds \to 0 \quad \text{as} \quad t \to 0^+$$

by (2.13). So integrating both sides of (2.8) with respect to x on $B_{1/2}(0)$ and then letting $t \to 0^+$, we see that $m \ge 0$. □

3 An a priori bound for solutions at the origin

The main result of this section in the following theorem.

Theorem 3.1. *Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 1$, containing the origin and let $a > 0$ be a constant. Suppose*

$$u \in C^{2,1}(\Omega \times (0,a)) \cap C(\overline{\Omega \times (0,a)} - \{(0,0)\}) \tag{3.1}$$

is a nonnegative solution of

$$0 \leq u_t - \Delta u \leq C(u^{\frac{n+2}{n}} + t^{-\frac{n+2}{2}}) \quad in \quad \Omega \times (0,a), \tag{3.2}$$

where C is a positive constant. Then

$$\max_{x \in \overline{\Omega}} u(x,t) = O(t^{-n/2}) \quad as \quad t \to 0^+. \tag{3.3}$$

Theorem 3.1 is stronger than Theorem 1.1 in two ways. First, the set in Theorem 3.1 where u is $C^{2,1}$ is not required to include the set $(\Omega - \{0\}) \times \{0\}$. And second, the term $t^{-\frac{n+2}{2}}$ in (3.2) is larger than the corresponding term 1 in (1.1).

Proof of Theorem 3.1. By scaling we can assume that

$$B_3(0) \times (0,3) \subset \Omega \times (0,a) \tag{3.4}$$

and that the right side of the second inequality in (3.2) is $u^{\frac{n+2}{n}} + Ct^{-\frac{n+2}{2}}$. (However the constant C cannot be completely removed by scaling.)

By Theorem 2.1,

$$u, Hu \in L^1(B_2(0) \times (0,2)), \qquad u(\cdot,0) \in L^1(B_2(0)), \tag{3.5}$$

and

$$u = m\Phi + N + v + h \quad \text{in} \quad B_1(0) \times (0,1), \tag{3.6}$$

where $m, \Phi, N, v,$ and h are as in Theorem 2.1.

Suppose for contradiction that (3.3) does not hold. Then there exists a sequence $\{(x_j,t_j)\} \subset \overline{\Omega} \times (0,a)$ such that $t_j \to 0$ as $j \to \infty$ and

$$\lim_{j \to \infty} t_j^{n/2} u(x_j,t_j) = \infty. \tag{3.7}$$

By (3.1), $x_j \to 0$ as $j \to \infty$. Clearly

$$v(x,t) \leq \frac{1}{(4\pi t)^{n/2}} \|u(\cdot,0)\|_{L^1(B_2(0))} \quad \text{for} \quad (x,t) \in \mathbf{R}^n \times (0,\infty).$$

For $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ and $r > 0$, let

$$E_r(x,t) := \{(y,s) \in \mathbf{R}^n \times \mathbf{R} : |y - x| < \sqrt{r} \text{ and } t - r < s < t\}.$$

Then for $(x,t) \in \overline{E_{t_j/4}(x_j,t_j)}$ and $(y,s) \in \mathbf{R}^n \times (0,\infty) - E_{t_j}(x_j,t_j)$ we have

$$\Phi(x-y,t-s) \le \max_{0\le\tau<\infty} \Phi\left(\frac{\sqrt{t_j}}{2},\tau\right) = \Phi\left(\frac{\sqrt{t_j}}{2},\frac{t_j}{8n}\right) = \left(\frac{2n}{\pi e t_j}\right)^{n/2},$$

which implies for $(x,t) \in \overline{E_{t_j/4}(x_j,t_j)}$ that

$$\iint\limits_{B_2(0)\times(0,2)-E_{t_j}(x_j,t_j)} \Phi(x-y,t-s)(Hu(y,s))\,dy\,ds \le \left(\frac{2n}{\pi e t_j}\right)^{n/2} \iint\limits_{B_2(0)\times(0,2)} Hu(y,s)\,dy\,ds.$$

It follows therefore from (3.5) and (3.6) that for $(x,t) \in \overline{E_{t_j/4}(x_j,t_j)}$ we have

$$u(x,t) \le \frac{C}{t_j^{n/2}} + \iint\limits_{E_{t_j}(x_j,t_j)} \Phi(x-y,t-s)(Hu(y,s))\,dy\,ds, \tag{3.8}$$

where C is a positive constant which does not depend on j or (x,t).

Substituting $(x,t)=(x_j,t_j)$ in (3.8) and using (3.7) we obtain

$$t_j^{n/2} \iint\limits_{E_{t_j}(x_j,t_j)} \Phi(x_j-y,t_j-s)(Hu(y,s))\,dy\,ds \to \infty \quad \text{as} \quad j\to\infty. \tag{3.9}$$

Also, by (3.5) we have

$$\iint\limits_{E_{t_j}(x_j,t_j)} Hu(y,s)\,dy\,ds \to 0 \quad \text{as} \quad j\to\infty. \tag{3.10}$$

For each positive integer j, define $f_j\colon E_1(0,0)\to[0,\infty)$ by

$$f_j(\xi,\tau) = t_j^{\frac{n+2}{2}} Hu(x_j+\sqrt{t_j}\xi, t_j+t_j\tau).$$

Making the change of variables $y = x_j + \sqrt{t_j}\eta$, $s = t_j + t_j\zeta$ in (3.10), (3.9), and (3.8) and using (3.2) we obtain

$$\iint\limits_{E_1(0,0)} f_j(\eta,\zeta)\,d\eta\,d\zeta \to 0 \quad \text{as} \quad j\to\infty, \tag{3.11}$$

$$\iint\limits_{E_1(0,0)} \Phi(-\eta,-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta \to \infty \quad \text{as} \quad j\to\infty, \tag{3.12}$$

and

$$f_j(\xi,\tau)^{\frac{n}{n+2}} \le C + \iint\limits_{E_1(0,0)} \Phi(\xi-\eta,\tau-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta \tag{3.13}$$

for $(\xi,\tau)\in\overline{E_{1/4}(0,0)}$, where C is a positive constant which does not depend on j or (ξ,τ).

Define $u_j(\xi,\tau)$ for $(\xi,\tau)\in\mathbf{R}^n\times(-1,\infty)$ to be the right side of (3.13). It follows from (3.1) and (3.6) that $N\in C^{2,1}(B_1(0)\times(0,1))$ and $HN = Hu$ in $B_1(0)\times(0,1)$. Hence

$$u_j \in C^{2,1}(E_1(0,0)) \cap C(\mathbf{R}^n\times(-1,\infty))$$

and

$$Hu_j = f_j \quad \text{in} \quad E_1(0,0). \tag{3.14}$$

Thus, by (3.13),

$$Hu_j \le u_j^{\frac{n+2}{n}} \quad \text{in} \quad E_{1/4}(0,0). \tag{3.15}$$

The rest of this proof is an adaptation and extension to parabolic inequalities of some methods of Brezis [4] concerning elliptic equations. Let $0 < R < \frac{1}{16}$ and $\lambda > 1$ be constants, and let $\varphi \in C_0^\infty(B_{\sqrt{2R}}(0) \times (-2R,\infty))$ satisfy $\varphi \equiv 1$ on $E_R(0,0)$ and $\varphi \ge 0$ on $\mathbf{R}^n \times \mathbf{R}$. Since

$$\nabla u_j \cdot \nabla(u_j^{\lambda-1}\varphi^2) = \frac{4(\lambda-1)}{\lambda^2}|\nabla(u_j^{\lambda/2}\varphi)|^2 - \frac{\lambda-2}{\lambda^2}\nabla u_j^\lambda \cdot \nabla\varphi^2 - \frac{4(\lambda-1)}{\lambda^2}u_j^\lambda|\nabla\varphi|^2,$$

we have for $-2R < t < 0$ that

$$\begin{aligned}\int\limits_{|x|<\sqrt{2R}} (-\Delta u_j)u_j^{\lambda-1}\varphi^2\,dx &= \int\limits_{|x|<\sqrt{2R}} \nabla u_j \cdot \nabla(u_j^{\lambda-1}\varphi^2)\,dx \\ &\ge \frac{4(\lambda-1)}{\lambda^2} \int\limits_{|x|<\sqrt{2R}} |\nabla(u_j^{\lambda/2}\varphi)|^2 dx - C_2 \int\limits_{|x|<\sqrt{2R}} u_j^\lambda\,dx,\end{aligned} \tag{3.16}$$

where $C_2 = C_2(n,R,\lambda)$ is a positive constant whose value may change from line to line. Thus by the parabolic Sobolev inequality (see [11, Theorem 6.9]),

$$\iint\limits_{E_{2R}(0,0)} (-\Delta u_j)u_j^{\lambda-1}\varphi^2\,dx\,dt \ge C_1 \frac{\iint\limits_{E_{2R}(0,0)} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt}{\max\limits_{-2R\le t\le 0}\left(\int\limits_{|x|<\sqrt{2R}} u_j^\lambda\varphi^2\,dx\right)^{2/n}} - C_2 \iint\limits_{E_{2R}(0,0)} u_j^\lambda\,dx\,dt,$$

where $C_1 = C_1(n,R,\lambda)$ is a positive constant whose value may change from line to line. (Here, C_1 only depends on n and λ, but later it will also depend on R.)

Also, for $|x| < \sqrt{2R}$ we have

$$\begin{aligned}\int_{-2R}^0 \frac{\partial u_j}{\partial t}u_j^{\lambda-1}\varphi^2\ dt &= \frac{1}{\lambda}\int_{-2R}^0 \frac{\partial u_j^\lambda}{\partial t}\varphi^2\ dt \\ &= \frac{1}{\lambda}\left[u_j(x,0)^\lambda\varphi(x,0)^2 - \int_{-2R}^0 u_j^\lambda\frac{\partial\varphi^2}{\partial t}dt\right] \ge -C_2\int_{-2R}^0 u_j^\lambda\ dt\end{aligned} \tag{3.17}$$

and thus

$$\iint\limits_{E_{2R}(0,0)} (Hu_j)u_j^{\lambda-1}\varphi^2\,dx\,dt \ge C_1 \frac{\iint\limits_{E_{2R}(0,0)} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt}{\max\limits_{-2R\le t\le 0}\left(\int\limits_{|x|<\sqrt{2R}} u_j^\lambda\varphi^2\,dx\right)^{2/n}} - C_2 \iint\limits_{E_{2R}(0,0)} u_j^\lambda\,dx\,dt. \tag{3.18}$$

On the other hand,

$$\begin{aligned}\iint\limits_{E_{2R}(0,0)} (Hu_j)u_j^{\lambda-1}\varphi^2\,dx\,dt &= \iint\limits_{E_{2R}(0,0)} \frac{Hu_j}{u_j}u_j^{\lambda}\varphi^2\,dx\,dt\\ &\le \left(\iint\limits_{E_{2R}(0,0)} \left(\frac{Hu_j}{u_j}\right)^{\frac{n+2}{2}} dx\,dt\right)^{\frac{2}{n+2}} \left(\iint\limits_{E_{2R}(0,0)} (u_j^{\lambda}\varphi^2)^{\frac{n+2}{n}} dx\,dt\right)^{\frac{n}{n+2}}\\ &\le \left(\iint\limits_{E_{2R}(0,0)} f_j\,dx\,dt\right)^{\frac{2}{n+2}} \left(\iint\limits_{E_{2R}(0,0)} (u_j^{\lambda}\varphi^2)^{\frac{n+2}{n}} dx\,dt\right)^{\frac{n}{n+2}} \end{aligned} \tag{3.19}$$

because

$$\left(\frac{Hu_j}{u_j}\right)^{\frac{n+2}{2}} = (Hu_j)\left(\frac{Hu_j}{u_j^{\frac{n+2}{n}}}\right)^{\frac{n}{2}} \le f_j$$

by (3.14) and (3.15). Since by (3.14),

$$\begin{aligned}\frac{\partial}{\partial t}u_j^{\lambda}\varphi^2 &= \lambda u_j^{\lambda-1}\frac{\partial u_j}{\partial t}\varphi^2 + u_j^{\lambda}2\varphi\frac{\partial\varphi}{\partial t}\\ &= \lambda u_j^{\lambda-1}\varphi^2[(\Delta u_j)+f_j] + 2u_j^{\lambda}\varphi\frac{\partial\varphi}{\partial t},\end{aligned}$$

it follows from (3.16) that for $-2R \le t < 0$ we have

$$\frac{\partial}{\partial t}\int\limits_{|x|<\sqrt{2R}} u_j^{\lambda}\varphi^2\,dx \le C_2\int\limits_{|x|<\sqrt{2R}} u_j^{\lambda}\,dx + \lambda\int\limits_{|x|<\sqrt{2R}} u_j^{\lambda-1}\varphi^2 f_j\,dx,$$

and thus

$$\begin{aligned}\max_{-2R\le t\le 0}\int\limits_{|x|<\sqrt{2R}} u_j^{\lambda}\varphi^2\,dx &\le C_2\iint\limits_{E_{2R}(0,0)} u_j^{\lambda}\,dx\,dt + \lambda\iint\limits_{E_{2R}(0,0)} u_j^{\lambda-1}\varphi^2 f_j\,dx\,dt\\ &\le C_2\iint\limits_{E_{2R}(0,0)} u_j^{\lambda}\,dx\,dt + \lambda\left(\iint\limits_{E_{2R}(0,0)} f_j\,dx\,dt\right)^{\frac{2}{n+2}}\left(\iint\limits_{E_{2R}(0,0)} (u_j^{\lambda}\varphi^2)^{\frac{n+2}{n}} dx\,dt\right)^{\frac{n}{n+2}}\\ &\le C_2\left[\iint\limits_{E_{2R}(0,0)} u_j^{\lambda}\,dx\,dt + \left(\iint\limits_{E_{2R}(0,0)} (u_j^{\lambda}\varphi^2)^{\frac{n+2}{n}} dx\,dt\right)^{\frac{n}{n+2}}\right]\end{aligned} \tag{3.20}$$

by (3.14), (3.19) and (3.11).

If

$$\left(\iint\limits_{E_{2R}(0,0)} (u_j^{\lambda}\varphi^2)^{\frac{n+2}{n}} dx\,dt\right)^{\frac{n}{n+2}} \ge \iint\limits_{E_{2R}(0,0)} u_j^{\lambda}\,dx\,dt \tag{3.21}$$

then by (3.20),

$$\max_{-2R\le t\le 0}\left(\int\limits_{|x|<\sqrt{2R}} u_j^\lambda \varphi^2\,dx\right)^{2/n} \le C_2\left(\iint\limits_{E_{2R}(0,0)} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt\right)^{\frac{2}{n+2}}$$

and hence by (3.18) we have

$$\iint\limits_{E_{2R}(0,0)} (Hu_j)u_j^{\lambda-1}\varphi^2\,dx\,dt \ge C_1\left(\iint\limits_{E_{2R}(0,0)} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt\right)^{\frac{n}{n+2}} - C_2\iint\limits_{E_{2R}(0,0)} u_j^\lambda\,dx\,dt.$$

Thus by (3.19) and (3.11),

$$\left(\iint\limits_{E_{2R}(0,0)} (u_j^\lambda\varphi^2)^{\frac{n+2}{n}}\,dx\,dt\right)^{\frac{n}{n+2}} \le C_2\iint\limits_{E_{2R}(0,0)} u_j^\lambda\,dx\,dt,$$

which clearly holds even when (3.21) does not hold. Consequently,

$$\iint\limits_{E_R(0,0)} u_j^{\lambda\frac{n+2}{n}}\,dx\,dt \le C_2\left(\iint\limits_{E_{2R}(0,0)} u_j^\lambda\,dx\,dt\right)^{\frac{n+2}{n}}. \tag{3.22}$$

It follows from the definition of u_j, (3.11), and Theorem A.1 in Appendix A that

$$\limsup_{j\to\infty}\iint\limits_{E_1(0,0)} u_j^{\frac{n+1}{n}}\,dx\,dt < \infty. \tag{3.23}$$

Starting with (3.23) and using (3.22) a finite number of times we find that for each $p > 1$ there exists $\varepsilon > 0$ such that the sequence u_j is bounded in $L^p(E_\varepsilon(0,0))$ and thus the same is true for the sequence f_j by (3.14) and (3.15). Hence by Theorem A.1 in Appendix A,

$$\limsup_{j\to\infty}\iint\limits_{E_\varepsilon(0,0)} \Phi(-\eta,-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta < \infty \tag{3.24}$$

for some $\varepsilon > 0$. Also, by (3.11),

$$\lim_{j\to\infty}\iint\limits_{E_1(0,0)-E_\varepsilon(0,0)} \Phi(-\eta,-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta = 0. \tag{3.25}$$

Adding (3.24) and (3.25) we contradict (3.12). □

4 Arbitrarily large solutions at the origin

The main result of this section is the following theorem which gives conditions on a nonnegative function f such that there exists a smooth nonnegative solution $u(x,t)$ of

$$0 \le u_t - \Delta u \le f(u) \quad \text{in} \quad (\mathbf{R}^n\times\mathbf{R}) - \{(0,0)\} \tag{4.1}$$

which is arbitrarily large at the origin.

Theorem 4.1. *Suppose $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function such that for some $s_0 > 0$ we have $f(s) > 0$ for $s \geq s_0$ and*

$$\int_{s_0}^{\infty} \left(\int_{s}^{\infty} \frac{d\bar{s}}{f(\bar{s})} \right)^{n/2} ds < \infty. \tag{4.2}$$

Let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function. Then there exists a C^∞ nonnegative solution $u(x,t)$ of (4.1) satisfying $u \equiv 0$ in $\mathbf{R}^n \times (-\infty, 0)$ and

$$u(0,t) \neq O(\varphi(t)) \quad \text{as} \quad t \to 0^+. \tag{4.3}$$

Theorem 4.1 implies Theorem 1.2 in the introduction because the functions $f(s) = s^\lambda$, $\lambda > \frac{n+2}{n}$, satisfy the conditions on f in Theorem 4.1.

Actually the smaller (at ∞) functions

$$f(s) = s^{\frac{n+2}{n}} (\log(1+s))^\beta, \qquad \beta > \frac{2}{n},$$

also satisfy the conditions on f in Theorem 4.1.

Proof of Theorem 4.1. For $s \geq s_0/\alpha_n$, let

$$g(s) = -\frac{1}{\alpha_n} \int_{\alpha_n s}^{\infty} \frac{d\bar{s}}{f(\bar{s})},$$

where α_n is defined in (4.19) below. By (4.2),

$$\int_{s_0/\alpha_n}^{\infty} (-g(s))^{n/2} ds < \infty \tag{4.4}$$

and thus $\liminf_{s\to\infty} s(-g(s))^{n/2} = 0$. Hence there exists a strictly decreasing sequence $\{t_j\}_{j=1}^\infty$ of positive real numbers, which tends to zero, such that

$$-g\left(\frac{\beta_n}{t_j^{n/2}} \right) \Big/ t_j \to 0 \quad \text{as} \quad j \to \infty, \tag{4.5}$$

where

$$\beta_n = \frac{1}{(8\pi)^{n/2} 2e}. \tag{4.6}$$

Let $w_j(t)$ be the solution of the initial value problem

$$w_j' = f(\alpha_n w_j), \qquad w_j(t_j) = \frac{\beta_n}{t_j^{n/2}}. \tag{4.7}$$

Since $g'(s) = 1/f(\alpha_n s)$, we have

$$\begin{aligned} t - t_j = \int_{t_j}^{t} d\tau &= \int_{t_j}^{t} g'(w_j(\tau)) w_j'(\tau)\, d\tau \\ &= g(w_j(t)) - g(w_j(t_j)). \end{aligned}$$

Let $T_j = t_j - g(w_j(t_j))$. Then

$$T_j - t = -g(w_j(t)) \tag{4.8}$$

and thus

$$\lim_{t\to T_j^-} w_j(t) = \infty. \tag{4.9}$$

Also,

$$\frac{T_j - t_j}{t_j} = \frac{-g(w_j(t_j))}{t_j} = \frac{-g(\beta_n/t_j^{n/2})}{t_j} \to 0 \quad \text{as} \quad j\to\infty \tag{4.10}$$

by (4.5).

Thanks to (4.9), we can choose $a_j \in (t_j, T_j)$ such that

$$\frac{w_j(a_j)}{\varphi(a_j)} \to \infty \quad \text{as} \quad j\to\infty. \tag{4.11}$$

Let $h_j(s) = \sqrt{4(a_j - s)}$ and $H_j(s) = \sqrt{4(a_j + \varepsilon_j - s)}$ where $\varepsilon_j > 0$ satisfies

$$a_j + 2\varepsilon_j < T_j, \quad t_j - \varepsilon_j > t_j/2, \quad \text{and} \quad w_j(t_j - \varepsilon_j) > \frac{w_j(t_j)}{2}. \tag{4.12}$$

Define

$$\begin{aligned}\omega_j &= \{(y,s)\in \mathbf{R}^n\times\mathbf{R}\colon\ |y| < h_j(s) \quad\text{and}\quad t_j < s < a_j\}\\ \Omega_j &= \{(y,s)\in \mathbf{R}^n\times\mathbf{R}\colon\ |y| < H_j(s) \quad\text{and}\quad t_j - \varepsilon_j < s < a_j + \varepsilon_j\}.\end{aligned}$$

Let $\chi_j\colon\ \mathbf{R}^n\times\mathbf{R}\to[0,1]$ be a C^∞ function such that $\chi_j\equiv 1$ in ω_j and $\chi_j\equiv 0$ in $\mathbf{R}^n\times\mathbf{R} - \Omega_j$. Define v_j, $u_j\colon \mathbf{R}^n\times\mathbf{R}\to[0,\infty)$ by

$$\begin{aligned} v_j(y,s) &= \chi_j(y,s)w_j'(s)\\ u_j(x,t) &= \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s)v_j(y,s)\,dy\,ds.\end{aligned}$$

We can assume f is C^∞ because given any function f satisfying the conditions of Theorem 4.1 there exists a C^∞ function $\hat f \le f$ satisfying the same conditions. Thus w_j, v_j, and u_j are C^∞ and

$$\begin{aligned} \frac{\partial u_j}{\partial t} + \Delta u_j &= v_j \quad \text{in } \mathbf{R}^n\times\mathbf{R}\\ u_j &\equiv 0 \qquad \text{in } \mathbf{R}^n\times(-\infty,0).\end{aligned} \tag{4.13}$$

Letting p be a fixed number larger than $\frac{n+2}{2}$, say $p = n+2$, we have by Theorem A.1 in Appendix A that

$$\begin{aligned}\left\| \iint\limits_{\Omega_j-\omega_j} \Phi(x-y,t-s)w_j'(s)\,dy\,ds \right\|_{L^\infty(\mathbf{R}^n\times(0,1))} &\le C_n\|w_j'(s)\|_{L^p(\Omega_j-\omega_j)}\\ &\le w_j(t_j)\end{aligned} \tag{4.14}$$

provided we decrease ε_j if necessary.

Also, for $(x,t)\in\Omega_j$ we have

$$|x| \le \sqrt{4(a_j - t_j + 2\varepsilon_j)} \le \sqrt{4(T_j - t_j)}$$

by (4.12); and thus using (4.12) again we obtain

$$\max_{(x,t)\in\Omega_j} \frac{|x|^2}{t} \le \frac{4(T_j - t_j)}{t_j - \varepsilon_j} \le \frac{8(T_j - t_j)}{t_j} \to 0 \quad \text{as} \quad j \to \infty \tag{4.15}$$

by (4.10).

In order to obtain a lower bound for u_j in Ω_j, note first that for $t_j - \varepsilon_j \le s \le t \le a_j + \varepsilon_j$ and $|x| \le H_j(t)$ we have

$$\int\limits_{|y|<H_j(s)} \Phi(x-y, t-s)\, dy = \frac{1}{\pi^{n/2}} \int\limits_{|z - \frac{x}{\sqrt{4(t-s)}}| < \frac{H_j(s)}{\sqrt{4(t-s)}}} e^{-|z|^2} dz \tag{4.16}$$

$$\ge \frac{1}{\pi^{n/2}} \int\limits_{|z - \frac{H_j(s)e_1}{\sqrt{4(t-s)}}| < \frac{H_j(s)}{\sqrt{4(t-s)}}} e^{-|z|^2} dz \quad \text{where } e_1 = (1, 0, \dots, 0) \tag{4.17}$$

$$\ge \alpha_n \tag{4.18}$$

where

$$\alpha_n := \frac{1}{\pi^{n/2}} \int\limits_{|z-e_1|<1} e^{-|z|^2}\, dz \in (0,1). \tag{4.19}$$

Some of the steps in the above calculation need some explanation. Equation (4.16) is obtained by making the change of variables $z = \frac{x-y}{\sqrt{4(t-s)}}$. Since $|x| \le H_j(t) \le H_j(s)$, the center of the ball of integration in (4.16) is closer to the origin than the center of the ball of integration in (4.17). Thus, since the integrand $e^{-|z|^2}$ is a decreasing function of $|z|$, we obtain (4.17). Since $H_j(s) \ge \sqrt{4(t-s)}$, the ball of integration in (4.17) contains the ball of integration in (4.19) and hence inequality (4.18) holds.

For $(x,t) \in \Omega_j$ we have

$$\iint\limits_{\Omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds = \int_{t_j-\varepsilon_j}^{t} w_j'(s) \left(\int\limits_{|y|<H_j(s)} \Phi(x-y, t-s)\, dy \right) ds$$
$$\ge \alpha_n (w_j(t) - w_j(t_j - \varepsilon_j)) \ge \alpha_n w_j(t) - w_j(t_j)$$

by (4.18) and (4.19). It therefore follows from (4.14) that for $(x,t) \in \Omega_j$ we have

$$u_j(x,t) \ge \iint\limits_{\omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds$$
$$= \iint\limits_{\Omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds - \iint\limits_{\Omega_j - \omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds$$
$$\ge \alpha_n w_j(t) - 2w_j(t_j). \tag{4.20}$$

Also,

$$
\begin{aligned}
\iint\limits_{\mathbf{R}^n\times\mathbf{R}} v_j(y,s)\,dy\,ds &\le \iint\limits_{\Omega_j} w_j'(s)\,dy\,ds \\
&= \int_{t_j-\varepsilon_j}^{a_j+\varepsilon_j} w_j'(s)\left(\int\limits_{|y|<H_j(s)} dy\right) ds \\
&= \omega_n \int_{t_j-\varepsilon_j}^{a_j+\varepsilon_j} w_j'(s)(H_j(s))^n ds \\
&\le \omega_n \int_{t_j-\varepsilon_j}^{T_j} w_j'(s)[4(T_j-s)]^{n/2} ds, \qquad \text{by (4.12)} \\
&= 4^{n/2}\omega_n \int_{t_j-\varepsilon_j}^{T_j} [-g(w_j(s))]^{n/2} w_j'(s)\,ds, \qquad \text{by (4.8)} \\
&\le 4^{n/2}\omega_n \int_{w_j(t_j)/2}^{\infty} [-g(w)]^{n/2} dw, \qquad \text{by (4.12).}
\end{aligned}
$$

It therefore follows from (4.4) that

$$
\iint\limits_{\mathbf{R}^n\times\mathbf{R}} \sum_{j=1}^{\infty} v_j(y,s)\,dy\,ds < \infty
$$

provided we take a subsequence of v_j if necessary. Hence the function $u\colon (\mathbf{R}^n\times\mathbf{R})-\{(0,0)\} \to [0,\infty)$ defined by

$$
\begin{aligned}
u(x,t) &= \Phi(x,t) + \iint\limits_{\mathbf{R}^n\times\mathbf{R}} \Phi(x-y,t-s)\sum_{j=1}^{\infty} v_j(y,s)\,dy\,ds \\
&= \Phi(x,t) + \sum_{j=1}^{\infty} u_j(x,t)
\end{aligned}
$$

is C^∞ and by (4.13) we have

$$
\begin{aligned}
u_t - \Delta u = \sum_{j=1}^{\infty} v_j \qquad & \text{in } (\mathbf{R}^n\times\mathbf{R})-\{(0,0)\} \\
u \equiv 0 \qquad & \text{in } \mathbf{R}^n\times(-\infty,0).
\end{aligned} \tag{4.21}
$$

Also, for $(x,t)\in\Omega_j$ it follows from (4.10), (4.15), (4.7), and (4.6) that $\Phi(x,t)\ge 2w_j(t_j)$, provided we take a subsequence of Ω_j if necessary, and thus for $(x,t)\in\Omega_j$ we have by (4.20) that

$$
\begin{aligned}
u(x,t) &\ge \Phi(x,t) + u_j(x,t) \\
&\ge \Phi(x,t) + (\alpha_n w_j(t) - 2w_j(t_j)) \\
&\ge \alpha_n w_j(t).
\end{aligned} \tag{4.22}
$$

Hence, for $(x,t)\in\Omega_j$,

$$
(u_t-\Delta u)(x,t) = v_j(x,t) \le w_j'(t) = f(\alpha_n w_j(t)) \le f(u(x,t)). \tag{4.23}
$$

Inequality (4.23) also holds for $(x,t) \in (\mathbf{R}^n \times \mathbf{R}) - \bigcup_{j=1}^{\infty} \Omega_j$ because $u_t - \Delta u \equiv 0$ there by (4.21). Thus (4.1) holds.

Finally, by (4.22) and (4.11),

$$\frac{u(0,a_j)}{\varphi(a_j)} > \frac{\alpha_n w_j(a_j)}{\varphi(a_j)} \to \infty \quad \text{as} \quad j \to \infty.$$

Hence (4.3) holds. □

5 Initial conditions

As discussed in the introduction, we now improve the upper bound (1.2) for u in Theorem 1.1 by imposing an upper bound on the initial condition of u. Our first such result is the following proposition.

Proposition 5.1. *Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 1$, containing the origin and let $a > 0$ be a constant. Suppose*

$$u \in C^{2,1}(\Omega \times (0,a)) \cap C(\overline{\Omega \times (0,a)} - \{(0,0)\}) \tag{5.1}$$

is a nonnegative solution of

$$0 \leq u_t - \Delta u \leq f(u) \quad in \quad \Omega \times (0,a), \tag{5.2}$$

where $f\colon\ [0,\infty) \to [0,\infty)$ is a continuous function satisfying

$$f(s) = O(s^{\frac{n+2}{n}}) \quad as \quad s \to \infty.$$

If

$$u(x,0) = O\left(\frac{1}{|x|^n}\right) \quad as \quad x \to 0 \tag{5.3}$$

then

$$\max_{x \in \overline{\Omega}} \frac{u(x,t)}{U(x,t)} = O(1) \quad as \quad t \to 0^+ \tag{5.4}$$

where

$$U(x,t) = \begin{cases} \dfrac{1}{t^{n/2}}, & |x| < \sqrt{t} \\ \dfrac{1}{|x|^n}, & |x| \geq \sqrt{t}. \end{cases}$$

Note that $U\colon \mathbf{R}^n \times [0,\infty) - \{(0,0)\} \to (0,\infty)$ is continuous and $\lim_{\substack{(x,t)\to(0,0) \\ t \geq 0}} U(x,t) = \infty$.

Proof of Proposition 5.1. By scaling we can assume (3.4) holds and $f(s) = s^{\frac{n+2}{n}} + 1$. By Theorem 2.1, statements (3.5) and (3.6) hold.

Suppose for contradiction that (5.4) does not hold. Then there exists a sequence $\{(x_j, \hat{t}_j)\} \subset \overline{\Omega} \times (0,a)$ such that $\hat{t}_j \to 0$ as $j \to \infty$ and

$$\lim_{j\to\infty} \frac{u(x_j,\hat{t}_j)}{U(x_j,\hat{t}_j)} = \infty. \tag{5.5}$$

If $(x,t) \in \mathbf{R}^n \times (0,\infty)$ and $t \geq \varepsilon |x|^2$ for some $\varepsilon \in (0,1)$ then

$$U(x,t) \geq U\left(\left(\frac{t}{\varepsilon}\right)^{1/2}, t\right) = \left(\frac{\varepsilon}{t}\right)^{n/2}.$$

It follows therefore from (5.5) and Theorem 3.1 that

$$\frac{\hat{t}_j}{|x_j|^2} \to 0 \quad \text{as} \quad j \to \infty. \tag{5.6}$$

Also, by (5.1), $x_j \to 0$ as $j \to \infty$. Let

$$t_j = \frac{|x_j|^2}{9} \tag{5.7}$$

and for $(x,t) \in \mathbf{R}^n \times (0,\infty)$ and $r > 0$, let

$$\mathcal{E}_r(x,t) = \{(y,s) \in \mathbf{R}^n \times \mathbf{R} \colon \; |y - x| < \sqrt{r} \text{ and } 0 < s < t\}.$$

Then for $(x,t) \in \mathcal{E}_{t_j}(x_j,t_j)$ and $(y,s) \in \mathbf{R}^n \times (0,\infty) - \mathcal{E}_{4t_j}(x_j,t_j)$ we have

$$\begin{aligned} \max\{\Phi(x-y,t-s), \Phi(x,t)\} &\leq \max_{0 \leq \tau < \infty} \Phi(\sqrt{t_j}, \tau) \\ &= \Phi\left(\sqrt{t_j}, \frac{t_j}{2n}\right) = \left(\frac{n}{2\pi e t_j}\right)^{n/2}, \end{aligned}$$

which implies for $(x,t) \in \mathcal{E}_{t_j}(x_j,t_j)$ that

$$\iint\limits_{B_2(0)\times(0,2) - \mathcal{E}_{4t_j}(x_j,t_j)} \Phi(x-y,t-s) Hu(y,s)\, dy\, ds \leq \left(\frac{n}{2\pi e t_j}\right)^{n/2} \iint\limits_{B_2(0)\times(0,2)} Hu(y,s)\, dy\, ds.$$

Also, for $(x,t) \in \mathcal{E}_{t_j}(x_j,t_j)$ we have

$$\begin{aligned} v(x,t) &= \int\limits_{\substack{|y-x_j| > 2\sqrt{t_j} \\ |y|<2}} \Phi(x-y,t) u(y,0)\, dy + \int\limits_{|y-x_j| < 2\sqrt{t_j}} \Phi(x-y,t) u(y,0)\, dy \\ &\leq \left(\frac{n}{2\pi e t_j}\right)^{n/2} \int\limits_{|y|<2} u(y,0)\, dy + \frac{C}{(\sqrt{t_j})^n} \int\limits_{y \in \mathbf{R}^n} \Phi(x-y,t)\, dy \end{aligned}$$

by (5.3) where C is a positive constant which does not depend on j or (x,t).

It follows therefore from (3.5) and (3.6) that for $(x,t) \in \mathcal{E}_{t_j}(x_j,t_j)$ we have

$$u(x,t) \leq \frac{C}{t_j^{n/2}} + \iint\limits_{\mathcal{E}_{4t_j}(x_j,t_j)} \Phi(x-y,t-s) Hu(y,s)\, dy\, ds \tag{5.8}$$

where C is a positive constant which does not depend on j or (x,t).

Using (5.6) and (5.7) we obtain

$$U(x_j, \hat{t}_j) = \frac{1}{|x_j|^n} = \frac{1}{(9t_j)^{n/2}}$$

and thus by (5.5),

$$t_j^{n/2} u(x_j, \hat{t}_j) \to \infty \quad \text{as} \quad j \to \infty. \tag{5.9}$$

By (5.6) and (5.7), $(x_j, \hat{t}_j) \in \mathcal{E}_{t_j}(x_j, t_j)$ for large j.

Substituting $(x,t) = (x_j, \hat{t}_j)$ in (5.8) and using (5.9) we obtain

$$t_j^{n/2} \iint\limits_{\mathcal{E}_{4t_j}(x_j,t_j)} \Phi(x_j - y, \hat{t}_j - s) Hu(y,s)\, dy\, ds \to \infty \quad \text{as} \quad j \to \infty. \tag{5.10}$$

Also, by (3.5) we have

$$\iint\limits_{\mathcal{E}_{4t_j}(x_j,t_j)} Hu(y,s)\, dy\, ds \to 0 \quad \text{as} \quad j \to \infty. \tag{5.11}$$

For each positive integer j, define $f_j \colon \mathcal{E}_4(0,1) \to [0,\infty)$ by

$$f_j(\xi,\tau) = t_j^{\frac{n+2}{2}} Hu(x_j + \sqrt{t_j}\, \xi, t_j \tau).$$

Making the change of variables $y = x_j + \sqrt{t_j}\ \eta$, $s = t_j \zeta$ in (5.11), (5.10), and (5.8) and using (5.2) and the first sentence of this proof, we obtain

$$\iint\limits_{\mathcal{E}_4(0,1)} f_j(\eta,\zeta)\, d\eta\, d\zeta \to 0 \quad \text{as} \quad j \to \infty, \tag{5.12}$$

$$\iint\limits_{\mathcal{E}_4(0,1)} \Phi(-\eta, \tau_j - \zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta \to \infty \quad \text{as} \quad j \to \infty, \tag{5.13}$$

where $\tau_j = \hat{t}_j / t_j \in (0,1)$, and

$$f_j(\xi,\tau)^{\frac{n}{n+2}} \le C + \iint\limits_{\mathcal{E}_4(0,1)} \Phi(\xi - \eta, \tau - \zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta \tag{5.14}$$

for $(\xi,\tau) \in \mathcal{E}_1(0,1)$, where C is a positive constant which does not depend on j or (ξ,τ).

Define $u_j(\xi,\tau)$ for $(\xi,\tau) \in \mathbf{R}^n \times \mathbf{R}$ to be the right side of (5.14). Since $f_j \in L^\infty(\mathcal{E}_4(0,1))$,

$$u_j \in C(\mathbf{R}^n \times \mathbf{R}) \quad \text{and} \quad u_j(\xi,0) = C = \min_{\mathbf{R}^n \times \mathbf{R}} u_j. \tag{5.15}$$

It follows from (5.1) and (3.6) that

$$N \in C^{2,1}(B_1(0) \times (0,1)) \quad \text{and} \quad HN = Hu \quad \text{in} \quad B_1(0) \times (0,1).$$

Hence

$$u_j \in C^{2,1}(\mathcal{E}_4(0,1))$$

and

$$Hu_j = f_j \quad \text{in} \quad \mathcal{E}_4(0,1).$$

Thus by (5.14),

$$Hu_j \le u_j^{\frac{n+2}{n}} \quad \text{in} \quad \mathcal{E}_1(0,1).$$

Let $0 < R < \frac{1}{4}$ and $\lambda > 1$ be constants, and let $\varphi \in C^\infty(\mathbf{R}^n)$ be a nonnegative function satisfying $\varphi \equiv 1$ for $|x| \le \sqrt{R}$ and $\varphi \equiv 0$ for $|x| \ge \sqrt{2R}$. The proof of Proposition 5.1 is now completed by replacing $\max\limits_{-2R\le t\le 0}$, integrals with respect to t from $-2R$ to 0, $E_{2R}(0,0)$, and $E_R(0,0)$ in the proof of Theorem 3.1 with $\max\limits_{0\le t\le 1}$, integrals with respect to t from 0 to 1, $\mathcal{E}_{2R}(0,1)$, and $\mathcal{E}_R(0,1)$, respectively. However we should point out that (3.17) is replaced with

$$\int_0^1 \frac{\partial u_j}{\partial t} u_j^{\lambda-1}\varphi^2\, dt = \frac{\varphi^2}{\lambda}(u_j(x,1)^\lambda - u_j(x,0)^\lambda) \ge 0$$

by (5.15). Other similar simplications hold because φ does not depend on t. □

The main result of this section is the following theorem, which immediately implies Theorem 1.3 in the introduction.

Theorem 5.1. *Let Ω, a, and u be as in the first two sentences of Proposition 5.1 and suppose*

$$u(x,0) = O\left(\frac{1}{|x|^p}\right) \quad \text{as} \quad x \to 0 \tag{5.16}$$

for some constant $p \in [0,n]$. Then for each positive constant q,

$$\max_{x\in\overline{\Omega}} \frac{u(x,t)}{\varphi(x,t;p,q)} = O(1) \quad \text{as} \quad t \to 0^+ \tag{5.17}$$

where

$$\varphi(x,t;p,q) = \begin{cases} \dfrac{1}{t^{n/2}} + \dfrac{1}{t^{p/2}}, & |x| \le \sqrt{t} \\ \dfrac{1}{|x|^p} + \left(\dfrac{t}{|x|^2}\right)^q \dfrac{1}{|x|^n}, & |x| > \sqrt{t}. \end{cases}$$

For the proof of Theorem 5.1, we will use Proposition 5.1 and the following lemma whose trivial proof we omit.

Lemma 5.1. *Let $q > 0$. Then for some constant $C = C(n,q) > 0$ we have*

$$\Phi\left(\frac{x}{2},\tau\right) \le \frac{C}{|x|^n}\left(\frac{t}{|x|^2}\right)^q$$

for $x \in \mathbf{R}^n - \{0\}$ and $0 < \tau \le t$.

Proof of Theorem 5.1. Since, for some constant $C > 0$,

$$u(y,0) \le \frac{C}{|x|^p} \quad \text{for} \quad |y-x| < \frac{|x|}{2} \quad \text{and} \quad 0 < |x| < 1,$$

it follows from (3.5) that for $0 < |x| < 1$ and $t > 0$ we have

$$\begin{aligned} v(x,t) &= \int\limits_{|y-x|<\frac{|x|}{2}} \Phi(x-y,t)u(y,0)\,dy + \int\limits_{\substack{|y-x|>\frac{|x|}{2} \\ |y|<2}} \Phi(x-y,t)u(y,0)\,dy \\ &\le \frac{C}{|x|^p}\int\limits_{\mathbf{R}^n} \Phi(x-y,t)\,dy + \Phi\left(\frac{x}{2},t\right)\int\limits_{|y|<2} u(y,0)\,dy \\ &\le C\left[\frac{1}{|x|^p} + \Phi\left(\frac{x}{2},t\right)\right], \end{aligned}$$

where here and in what follows C is a constant which does not depend on (x,t).

For $x \in \mathbf{R}^n - \{0\}$ and $t > 0$, let

$$Q(x,t) = \{(y,s) \in \mathbf{R}^n \times \mathbf{R}\colon\ |y-x| < \frac{|x|}{2} \quad \text{and} \quad 0 < s < t\}.$$

It follows from (3.5) that for $0 < |x| < 1$ and $t > 0$ we have

$$\begin{aligned}\iint\limits_{B_2(0)\times(0,2)-Q(x,t)} \Phi(x-y,t-s)Hu(y,s)\,dy\,ds &\le \left[\max_{0\le\tau\le t} \Phi\left(\frac{x}{2},\tau\right)\right] \iint\limits_{B_2(0)\times(0,2)} Hu(y,s)\,dy\,ds \\ &\le C \max_{0\le\tau\le t} \Phi\left(\frac{x}{2},\tau\right).\end{aligned}$$

Thus by (3.6), we have for $0 < |x| < 1$ and $0 < t < 1$ that

$$u(x,t) \le C\left[\frac{1}{|x|^p} + \max_{0\le\tau\le t} \Phi\left(\frac{x}{2},\tau\right) + \iint\limits_{Q(x,t)} \Phi(x-y,t-s)Hu(y,s)\,dy\,ds\right]. \tag{5.18}$$

Suppose, inductively, for some constants $A > 1$, $r \in (0,1)$, and $q \ge 0$ satisfying $Ar^2 < 1$ we have

$$u(x,t) \le C\left[\frac{1}{|x|^p} + \frac{1}{|x|^n}\left(\frac{t}{|x|^2}\right)^q\right] \quad \text{for} \quad 0 < |x| < r \text{ and } 0 < t < A|x|^2. \tag{5.19}$$

(Note that by Proposition 5.1, given any large constant $A > 1$, (5.19) holds with $r = 1/\sqrt{2A}$ and $q = 0$.) Then for

$$0 < |x| < \frac{2}{3}r, \quad 0 < t < \frac{A}{4}|x|^2, \quad \text{and} \quad (y,s) \in Q(x,t)$$

we have

$$\frac{|x|}{2} < |y| < \frac{3}{2}|x| < r \quad \text{and} \quad 0 < s < t < \frac{A}{4}|x|^2 < A|y|^2$$

and thus by (5.19),

$$\begin{aligned}u(y,s) &\le C\left[\frac{1}{|y|^p} + \frac{1}{|y|^n}\left(\frac{s}{|y|^2}\right)^q\right] \\ &\le C\left[\frac{2^p}{|x|^p} + \frac{2^{n+2q}}{|x|^n}\left(\frac{t}{|x|^2}\right)^q\right],\end{aligned}$$

which implies

$$Hu(y,s) \le u(y,s)^{\frac{n+2}{n}} + 1 \le C\left[\frac{1}{|x|^{p\frac{n+2}{n}}} + \frac{1}{|x|^{n+2}}\left(\frac{t}{|x|^2}\right)^{q\frac{n+2}{n}}\right],$$

and hence

$$\begin{aligned}\iint\limits_{Q(x,t)} \Phi(x-y,t-s)Hu(y,s)\,dy\,ds &\le Ct\left[\frac{1}{|x|^{p\frac{n+2}{n}}} + \frac{1}{|x|^{n+2}}\left(\frac{t}{|x|^2}\right)^{q\frac{n+2}{n}}\right] \\ &= C\left[\frac{t}{|x|^2}\frac{1}{|x|^{p\frac{n+2}{n}-2}} + \frac{1}{|x|^n}\left(\frac{t}{|x|^2}\right)^{q\frac{n+2}{n}+1}\right] \\ &\le C\left[\frac{A}{4}\frac{1}{|x|^p} + \frac{1}{|x|^n}\left(\frac{t}{|x|^2}\right)^{q\frac{n+2}{n}+1}\right].\end{aligned}$$

It therefore follows from (5.18) and Lemma 5.1 that

$$u(x,t) \le C\left[\frac{1}{|x|^p} + \frac{1}{|x|^n}\left(\frac{t}{|x|^2}\right)^{q\frac{n+2}{n}+1}\right] \quad \text{for} \quad 0 < |x| < \frac{2}{3}r \text{ and } 0 < t < \frac{A}{4}|x|^2. \tag{5.20}$$

That is (5.19) implies (5.20). Thus given $q_0 > 0$ and iterating the above argument, starting with (5.19) with $q = 0$ and $A = A(q_0)$ sufficiently large, we see that there exists $r_0 > 0$ such that

$$u(x,t) \le C\left[\frac{1}{|x|^p} + \frac{1}{|x|^n}\left(\frac{t}{|x|^2}\right)^{q_0}\right] \quad \text{for} \quad 0 < |x| < r_0 \text{ and } 0 < t < |x|^2.$$

Consequently, Theorem 5.1 follows from Proposition 5.1. □

A Estimates for the heat potential

In this appendix, we give estimates for the heat potential

$$(Vf)(x,t) = \iint\limits_{\mathbf{R}^n \times (a,b)} \Phi(x-y, t-s) f(y,s)\, dy\, ds,$$

where $a < b$, $n \ge 1$, and Φ is given by (1.3).

These estimates, which are needed for the proofs of Theorems 3.1 and 4.1, are the analog of estimates given in [10, Lemma 7.12] for the Riesz potential.

Theorem A.1. *Let $p, q \in [1,\infty]$ satisfy*

$$0 \le \delta := \frac{1}{p} - \frac{1}{q} < \frac{2}{n+2}.$$

Then V maps $L^p(\mathbf{R}^n \times (a,b))$ continuously into $L^q(\mathbf{R}^n \times (a,b))$ and for $f \in L^p(\mathbf{R}^n \times (a,b))$,

$$\|Vf\|_{L^q(\mathbf{R}^n \times (a,b))} \le M \|f\|_{L^p(\mathbf{R}^n \times (a,b))},$$

where

$$M = \left(\frac{(1-\delta)^{\frac{n}{2}} (b-a)^{1-\frac{n\delta}{2(1-\delta)}}}{(4\pi)^{\frac{n\delta}{2(1-\delta)}} \left(1 - \frac{n\delta}{2(1-\delta)}\right)}\right)^{1-\delta}.$$

Proof. Let $r = (1-\delta)^{-1}$. Then $1 \le r < (n+2)/n$ and for $s < t$ we have

$$\int_{\mathbf{R}^n} \Phi(x-y, t-s)^r\, dy = \int_{\mathbf{R}^n} \Phi(x-y, t-s)^r\, dx = \frac{1}{(4\pi)^{(r-1)n/2} r^{n/2} (t-s)^{(r-1)n/2}}.$$

Hence, letting $\Omega = \mathbf{R}^n \times (a,b)$, $X = (x,t)$, and $Y = (y,s)$, we have

$$\|\Phi(X - \cdot)\|_{L^r(\Omega)} \le M \qquad \text{for all } X \in \Omega$$

and

$$\|\Phi(\cdot - Y)\|_{L^r(\Omega)} \le M \qquad \text{for all } Y \in \Omega.$$

The proof is now completed by mimicking the proof of Lemma 7.12 in [10]. □

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