Pointwise Bounds and Blow-up for Choquard-Pekar Inequalities at an Isolated Singularity

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Abstract

We study the behavior near the origin in $\mathbb{R}^n, n \geq 3$, of nonnegative functions

$$u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap L^\lambda(\mathbb{R}^n) \tag{0.1}$$

satisfying the Choquard-Pekar type inequalities

$$0 \leq -\Delta u \leq (|x|^{-\alpha} * u^\lambda) u^\sigma \quad \text{in } B_2(0)\backslash\{0\} \tag{0.2}$$

where $\alpha \in (0, n), \lambda > 0$, and $\sigma \geq 0$ are constants and $*$ is the convolution operation in $\mathbb{R}^n$. We provide optimal conditions on α, λ, and σ such that nonnegative solutions u of (0.1,0.2) satisfy pointwise bounds near the origin.

MSC: 35B09; 35B33; 35B40; 35J60; 35J91; 35R45

Keywords: pointwise bound; blow-up; isolated singularity; Choquard-Pekar equation; Riesz potential

1 Introduction

In this paper we study the behavior near the origin in $\mathbb{R}^n, n \geq 3$, of nonnegative functions

$$u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap L^\lambda(\mathbb{R}^n) \tag{1.1}$$

satisfying the Choquard-Pekar type inequalities

$$0 \leq -\Delta u \leq (|x|^{-\alpha} * u^\lambda) u^\sigma \quad \text{in } B_2(0)\backslash\{0\} \tag{1.2}$$

where $\alpha \in (0, n), \lambda > 0$, and $\sigma \geq 0$ are constants and $*$ is the convolution operation in $\mathbb{R}^n$. The regularity condition $u \in L^\lambda(\mathbb{R}^n)$ in (1.1) is natural because one does not want the nonlocal convolution operation on the right hand side of (1.2) to be infinite at every point in $\mathbb{R}^n$.

A motivation for the study of (1.1,1.2) comes from the equation

$$-\Delta u = (|x|^{-\alpha} * u^\lambda)|u|^{\lambda-2}u \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $\alpha \in (0, n)$ and $\lambda > 1$. For $n = 3$, $\alpha = 1$, and $\lambda = 2$, equation (1.3) is known in the literature as the *Choquard-Pekar equation* and was introduced in [17] as a model in quantum theory of a Polaron at rest (see also [3]). Later, the equation (1.3) appears as a model of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [8]. More

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recently, the same equation (1.3) was used in a model of self-gravitating matter (see, e.g., [7, 13]) and it is known in this context as the *Schrödinger-Newton equation*.

The Choquard-Pekar equation (1.3) has been investigated for a few decades by variational methods starting with the pioneering works of Lieb [8] and Lions [9, 10]. More recently, new and improved techniques have been devised to deal with various forms of (1.3) (see, e.g., [11, 12, 14, 15, 16, 20] and the references therein).

Using nonvariational methods, the authors in [15] obtained sharp conditions for the nonexistence of nonnegative solutions to

$$-\Delta u \geq (|x|^{-\alpha} * u^\lambda)u^\sigma$$

in an exterior domain of $\mathbb{R}^n$, $n \geq 3$.

For some very recent results on positive solutions Choquard-Pekar equations which have an isolated singularity at the origin and tend to zero as $|x| \to \infty$ see [2].

In this paper we address the following question.

Question 1. Suppose $\alpha \in (0, n)$ and $\lambda > 0$ are constants. For which nonnegative constants σ, if any, does there exist a continuous function $\varphi : (0, 1) \to (0, \infty)$ such that all nonnegative solutions u of (1.1,1.2) satisfy

$$u(x) = O(\varphi(|x|)) \quad \text{as } x \to 0 \tag{1.4}$$

and what is the optimal such φ when it exists?

We call the function φ in (1.4) a pointwise bound for u as $x \to 0$.

Remark 1. Let $u_\lambda \in C^2(\mathbb{R}^n\backslash\{0\})$ be a nonnegative function such that $u_\lambda = 0$ in $\mathbb{R}^n\backslash B_3(0)$ and

$$u_\lambda(x) = \begin{cases} |x|^{-(n-2)} & \text{if } 0 < \lambda < \frac{n}{n-2} \\ 1 & \text{if } \lambda \geq \frac{n}{n-2} \end{cases} \qquad \text{for } 0 < |x| < 2.$$

Then $u_\lambda \in L^\lambda(\mathbb{R}^n)$ and $-\Delta u_\lambda = 0$ in $B_2(0)\backslash\{0\}$. Hence u_λ is a solution of (1.1,1.2) for all $\alpha \in (0, n)$, $\lambda > 0$, and $\sigma \geq 0$. Thus any pointwise bound for nonnegative solutions u of (1.1,1.2) as $x \to 0$ must be at least as large as $u_\lambda(x)$ and whenever $u_\lambda(x)$ is such a bound it is necessarily optimal. In this case we say u is harmonically bounded at 0.

In order to state our results for Question 1, we define for each $\alpha \in (0, n)$ the continuous, piecewise linear function $g_\alpha : (0, \infty) \to [0, \infty)$ by

$$g_\alpha(\lambda) = \begin{cases} \frac{n}{n-2} & \text{if } 0 < \lambda < \frac{n-\alpha}{n-2} \\ \frac{2n-\alpha}{n-2} - \lambda & \text{if } \frac{n-\alpha}{n-2} \leq \lambda < \frac{n}{n-2} \\ \max\{0, 1 - \frac{\alpha-2}{n}\lambda\} & \text{if } \lambda \geq \frac{n}{n-2}. \end{cases} \tag{1.5}$$

According to the following theorem, if the point (λ, σ) lies below the graph of $\sigma = g_\alpha(\lambda)$ then all nonnegative solutions u of (1.1,1.2) are harmonically bounded at 0.

Theorem 1.1. *Suppose u is a nonnegative solution of (1.1,1.2) where $\alpha \in (0, n), \lambda > 0$, and*

$$0 \leq \sigma < g_\alpha(\lambda).$$

Then u is harmonically bounded at 0, that is, as $x \to 0$,

$$u(x) = \begin{cases} O(|x|^{-(n-2)}) & \text{if } 0 < \lambda < \frac{n}{n-2} \\ O(1) & \text{if } \lambda \geq \frac{n}{n-2}. \end{cases} \tag{1.6}$$

Moreover, if $\lambda \geq \frac{n}{n-2}$ then u has a C^1 extension to the origin, that is, $u = w|_{\mathbb{R}^n\backslash\{0\}}$ for some function $w \in C^1(\mathbb{R}^n)$.

By Remark 1 the bound (1.6) for u is optimal.

By the next theorem, if the point (λ, σ) lies above the graph of $\sigma = g_\alpha(\lambda)$ then there does not exist a pointwise bound for nonnegative solutions of (1.1,1.2) as $x \to 0$.

Theorem 1.2. *Suppose $\alpha, \lambda,$ and σ are constants satisfying*

$$\alpha \in (0, n), \quad \lambda > 0, \quad \text{and} \quad \sigma > g_\alpha(\lambda).$$

Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying

$$\lim_{t \to 0^+} \varphi(t) = \infty.$$

Then there exists a nonnegative solution u of (1.1,1.2) such that

$$u(x) \neq O(\varphi(|x|)) \quad \text{as } x \to 0.$$

Theorems 1.1 and 1.2 completely answer Question 1 when the point (λ, σ) does not lie on the graph of g_α. Concerning the critical case that (λ, σ) lies on the graph of g_α we have the following result.

Theorem 1.3. *Suppose $\alpha \in (0, n)$.*

(i) If $0 < \lambda < \frac{n-\alpha}{n-2}$ and $\sigma = g_\alpha(\lambda)$ then all nonnegative solutions u of (1.1,1.2) are harmonically bounded at 0.

(ii) If $\lambda = \frac{n-\alpha}{n-2}$ and $\sigma = g_\alpha(\lambda)$ then there does not exist a pointwise bound for nonnegative solutions u of (1.1,1.2) as $x \to 0$.

(iii) If $\alpha \in (2, n)$, $\lambda > \frac{n}{\alpha-2}$, and $\sigma = g_\alpha(\lambda)$ then there does not exist a pointwise bound for nonnegative solutions u of (1.1,1.2) as $x \to 0$.

If u is a nonnegative solution of (1.1,1.2) where (λ, σ) lies in the first quadrant of the $\lambda\sigma$-plane and $\sigma \neq g_\alpha(\lambda)$ then according to Theorems 1.1 and 1.2 either

(i) u is bounded around the origin and can be extended to a C^1 function in the whole $\mathbb{R}^n$; or

(ii) u can be unbounded around the origin but must satisfy $u = O(|x|^{-(n-2)})$ as $x \to 0$; or

(iii) no pointwise a priori bound exists for u as $x \to 0$, that is solutions can be arbitrarily large around the origin.

The regions in which these three possibilities occur are depicted in Figs. 1–3 below.

Our results in Theorems 1.1–1.3 leave Question 1 unresolved when either

(i) $\alpha \in (2, n)$, $\frac{n-\alpha}{n-2} < \lambda \leq \frac{n}{\alpha-2}$, and $\sigma = g_\alpha(\lambda)$; or

(ii) $\alpha \in (0, 2]$, $\lambda > \frac{n-\alpha}{n-2}$, and $\sigma = g_\alpha(\lambda)$.

We leave these two borderline cases, which are more difficult and not amenable to our methods, as open problems for future research. However, we suspect that (1.1,1.2) in these cases has arbitrarily large solutions around the origin because otherwise by Theorem 1.3 $(\lambda, \sigma) = (\frac{n-\alpha}{n-2}, g_\alpha(\frac{n-\alpha}{n-2}))$ would be an isolated point on the curve $\sigma = g_\alpha(\lambda)$ for which (1.1,1.2) has arbitrarily large solutions, which seems unlikely.

The parabolic version of (1.1,1.2) will be investigated in a future work.

If $\alpha \in (0, n)$ and $\lambda > 0$ then one of the following three conditions holds:

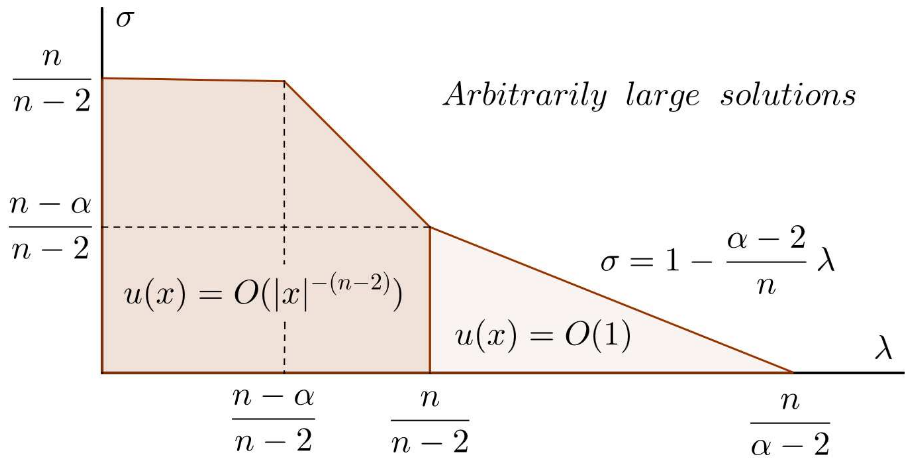

Figure 1: Case $\alpha \in (2, n)$.

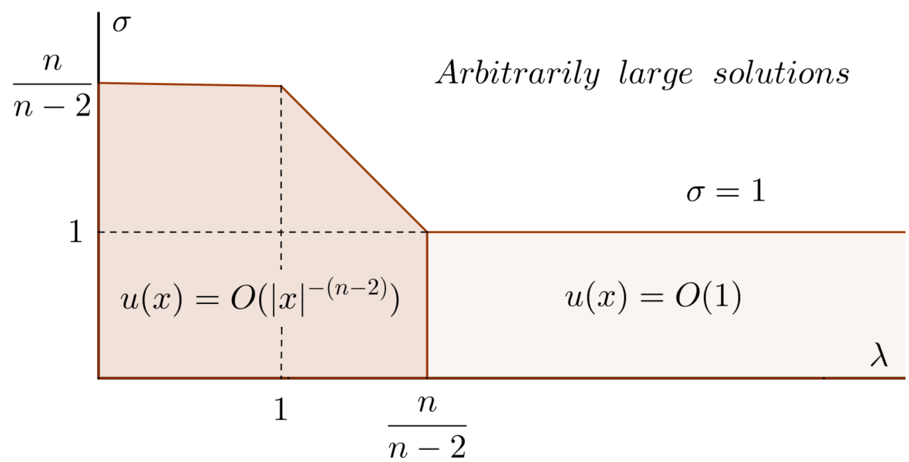

Figure 2: Case $\alpha = 2$.

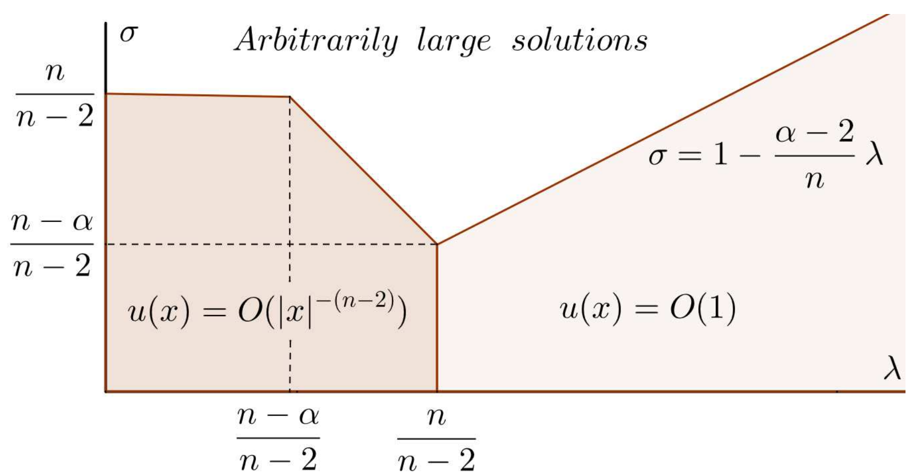

Figure 3: Case $\alpha \in (0, 2)$.

(i) $0 < \lambda < \frac{n-\alpha}{n-2}$;

(ii) $\frac{n-\alpha}{n-2} \le \lambda < \frac{n}{n-2}$;

(iii) $\frac{n}{n-2} \le \lambda < \infty$.

The proofs of Theorems 1.1–1.3 in case (i)(resp. (ii), (iii)) are given in Section 3 (resp. 4, 5). In Section 2 we provide some lemmas needed for these proofs. Our approach relies on an integral representation formula for nonnegative superharmonic functions due to Brezis and Lions [1] (see Lemma 2.1 below) together with various integral estimates for Riesz potentials.

Finally we mention that throughout this paper ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and by *Riesz potential estimates* we mean the estimates given in [6, Lemma 7.12] and [18, Chapter 5, Theorem 1]. See also [5, Appendix C].

2 Preliminary lemmas

In this section we provide some lemmas needed for the proofs of our results in Sections 3–5.

Lemma 2.1. *Suppose u is a nonnegative solution of (1.1,1.2) for some constants $\alpha \in (0,n), \lambda > 0$, and $\sigma \ge 0$. Let $v = u + 1$. Then*

$$v \in C^2(\mathbb{R}^n \backslash \{0\}) \cap L^\lambda(B_2(0)) \tag{2.1}$$

and, for some positive constant C depending on u, the function v satisfies

$$\left.\begin{array}{l} 0 \le -\Delta v \le C[I_{n-\alpha}(v^\lambda)]v^\sigma \\ v \ge 1 \end{array}\right\} \text{ in } B_2(0)\backslash\{0\}, \tag{2.2}$$

where

$$(I_\beta f)(x) := \int_{|y|<1} \frac{f(y)dy}{|x-y|^{n-\beta}} \quad \text{for } \beta \in (0,n). \tag{2.3}$$

Also

$$-\Delta v, v^\mu \in L^1(B_1(0)) \quad \text{for all } \mu \in [1, \frac{n}{n-2}) \tag{2.4}$$

and

$$v(x) = \frac{m}{|x|^{n-2}} + h(x) + C\int_{|y|<1} \frac{-\Delta v(y)dy}{|x-y|^{n-2}} \quad \text{for } 0 < |x| < 1 \tag{2.5}$$

where $m = m(u) \ge 0$ and $C = C(n) > 0$ are constants and h is harmonic and bounded in $B_1(0)$.

Proof. (2.1) follows from (1.1) and the definition of v.

For $0 < |x| < 2$ we have

$$\begin{aligned}\int_{|y|>1} \frac{u(y)^\lambda dy}{|x-y|^\alpha} &\le \left(\max_{1\le|y|\le3} u(y)^\lambda\right)\int_{1<|y|<3} \frac{dy}{|x-y|^\alpha} + \int_{|y|>3} u(y)^\lambda dy \\ &\le C \le C \min_{|z|\le2}\int_{|y|<1} \frac{dy}{|z-y|^\alpha},\end{aligned}$$

where, as usual, C is a positive constant whose value may change from line to line. Thus for $0<|x|<2$

$$\int_{\mathbb{R}^n}\frac{u(y)^\lambda dy}{|x-y|^\alpha}\le\int_{|y|<1}\frac{u(y)^\lambda dy}{|x-y|^\alpha}+C\int_{|y|<1}\frac{1^\lambda}{|x-y|^\alpha}dy$$
$$\le C\int_{|y|<1}\frac{(u(y)+1)^\lambda}{|x-y|^\alpha}dy=C[I_{n-\alpha}(v^\lambda)](x).$$

Hence, since $\Delta u=\Delta v$ and $u<v$ we see that (2.2) follows from (1.2). Also (2.1), (2.2), and [1] imply (2.4) with $\mu=1$ and (2.5), which together with Riesz potential estimates, yield the complete statement (2.4). □

The following lemma will be needed for the proof of Theorem 1.2 when $0<\lambda\le\frac{n}{n-2}$.

Lemma 2.2. *Suppose $\alpha\in(0,n)$ and $\lambda\in(0,\frac{n}{n-2}]$. Let $\{x_j\}\subset\mathbb{R}^n, n\ge 3$, and $\{r_j\},\{\varepsilon_j\}\subset(0,1)$ be sequences satisfying*

$$0<4|x_{j+1}|<|x_j|<1/2, \tag{2.6}$$

$$0<r_j<|x_j|/4 \quad \text{and} \quad \sum_{j=1}^{\infty}(\varepsilon_j^\lambda+\varepsilon_j)<\infty. \tag{2.7}$$

Then there exists a nonnegative function

$$u\in C^\infty(\mathbb{R}^n\backslash\{0\})\cap L^\lambda(\mathbb{R}^n) \tag{2.8}$$

such that

$$0\le-\Delta u\le\left\{\begin{array}{ll}\frac{\varepsilon_j}{r_j^n} & \text{if } 0<\lambda<\frac{n}{n-2}\\ \frac{\varepsilon_j}{r_j^n(\log\frac{1}{r_j})^{\frac{n-2}{n}}} & \text{if } \lambda=\frac{n}{n-2}\end{array}\right\}\quad \text{in } B_{r_j}(x_j), \tag{2.9}$$

$$-\Delta u=0\quad \text{in } B_2(0)\backslash(\{0\}\cup\bigcup_{j=1}^{\infty}B_{r_j}(x_j)), \tag{2.10}$$

$$u\ge A\left\{\begin{array}{ll}\frac{\varepsilon_j}{r_j^{n-2}} & \text{if } 0<\lambda<\frac{n}{n-2}\\ \frac{\varepsilon_j}{r_j^{n-2}(\log\frac{1}{r_j})^{\frac{n-2}{n}}} & \text{if } \lambda=\frac{n}{n-2}\end{array}\right\}\quad \text{in } B_{r_j}(x_j), \tag{2.11}$$

and for $x\in B_{r_j}(x_j)$

$$\int_{|y|<1}\frac{u(y)^\lambda dy}{|x-y|^\alpha}\ge B\begin{cases}\varepsilon_j^\lambda & \text{if } 0<\lambda<\frac{n-\alpha}{n-2}\\ \varepsilon_j^\lambda\log\frac{1}{r_j} & \text{if } \lambda=\frac{n-\alpha}{n-2}\\ \varepsilon_j^\lambda r_j^{n-\alpha-(n-2)\lambda} & \text{if } \frac{n-\alpha}{n-2}<\lambda<\frac{n}{n-2}\\ \varepsilon_j^\lambda r_j^{-\alpha}(\log\frac{1}{r_j})^{-1} & \text{if } \lambda=\frac{n}{n-2}\end{cases} \tag{2.12}$$

where $A=A(n)$ and $B=B(n,\lambda,\alpha)$ are positive constants.

Proof. Let $\psi:\mathbb{R}^n\to[0,1]$ be a C^∞ function whose support is $\overline{B_1(0)}$. Define $\psi_j,f_j:\mathbb{R}^n\to[0,\infty)$ by $\psi_j(y)=\psi(\eta)$ where $y=x_j+r_j\eta$ and $f_j=M_j\psi_j$ where $M_j=\frac{\varepsilon_j}{r_j^n\delta_j}$ and

$$\delta_j=\begin{cases}1 & \text{if } 0<\lambda<\frac{n}{n-2}\\ (\log\frac{1}{r_j})^{\frac{n-2}{n}} & \text{if } \lambda=\frac{n}{n-2}.\end{cases}$$

Since

$$\int_{\mathbb{R}^n} f_j(y)dy = M_j \int_{\mathbb{R}^n} \psi(\eta) r_j^n d\eta \le \varepsilon_j \int_{\mathbb{R}^n} \psi(\eta) d\eta$$

and by (2.6) and $(2.7)_1$ the supports $\overline{B_{r_j}(x_j)}$ of the functions f_j are disjoint and contained in $B_{3/4}(0)$ we see by $(2.7)_2$ that

$$f := \sum_{j=1}^{\infty} f_j \in C^\infty(\mathbb{R}^n \backslash \{0\}) \cap L^1(\mathbb{R}^n) \quad \text{and} \quad \operatorname{supp}(f) \subset B_1(0). \tag{2.13}$$

Defining

$$v_j(y) = \int_{\mathbb{R}^n} \frac{f_j(z)dz}{|y-z|^{n-2}}$$

and making the change of variables

$$x = x_j + r_j \xi, \quad y = x_j + r_j \eta, \quad \text{and} \quad z = x_j + r_j \zeta,$$

we find for $\beta \in [0, n)$ and $R \in [\frac{1}{2}, 2]$ that

$$\begin{aligned} \int_{|y-x_j|<R} \frac{v_j(y)^\lambda dy}{|x-y|^\beta} &= \int_{|\eta|<R/r_j} \frac{\left(\int_{\mathbb{R}^n} \frac{M_j \psi(\zeta) r_j^n d\zeta}{r_j^{n-2} |\eta-\zeta|^{n-2}} \right)^\lambda}{r_j^\beta |\xi - \eta|^\beta} r_j^n d\eta \\ &= \varepsilon_j^\lambda \delta_j^{-\lambda} r_j^{n-\beta-(n-2)\lambda} \int_{|\eta|<R/r_j} \frac{\left(\int_{\mathbb{R}^n} \frac{\psi(\zeta) d\zeta}{|\eta-\zeta|^{n-2}} \right)^\lambda}{|\xi-\eta|^\beta} d\eta. \end{aligned} \tag{2.14}$$

Also

$$0 < C_1(n) < \frac{\int_{\mathbb{R}^n} \frac{\psi(\zeta)d\zeta}{|\eta-\zeta|^{n-2}}}{\frac{1}{|\eta|^{n-2}+1}} < C_2(n) < \infty \quad \text{for } \eta \in \mathbb{R}^n. \tag{2.15}$$

Taking $\beta = 0$ and $R = 2$ in (2.14) and using (2.15) we get

$$\int_{|y-x_j|<2} v_j(y)^\lambda dy \le C \varepsilon_j^\lambda \delta_j^{-\lambda} r_j^{n-(n-2)\lambda} \int_{|\eta|<2/r_j} \left(\frac{1}{|\eta|^{n-2}+1} \right)^\lambda d\eta \le C(n,\lambda) \varepsilon_j^\lambda \tag{2.16}$$

because $\lambda(n-2) \le n$. Defining

$$v(x) := \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-2}} = \frac{1}{n(n-2)\omega_n} \sum_{j=1}^{\infty} v_j(x) \quad \text{for } x \in \mathbb{R}^n$$

and using (2.16) we get for $1 \le \lambda \le \frac{n}{n-2}$ that

$$n(n-2)\omega_n \|v\|_{L^\lambda(B_1(0))} \le \sum_{j=1}^{\infty} \|v_j\|_{L^\lambda(B_1(0))} \le \sum_{j=1}^{\infty} \|v_j\|_{L^\lambda(B_2(x_j))} \le C \sum_{j=1}^{\infty} \varepsilon_j < \infty$$

by (2.7).

If $\lambda \in (0,1)$ then using (2.16) we see that

$$n(n-2)\omega_n \|v\|_{L^\lambda(B_1(0)} = \|\sum_{j=1}^\infty v_j\|_{L^\lambda(B_1(0))} = \left(\int_{B_1(0)} \left(\sum_{j=1}^\infty v_j(y)\right)^\lambda dy\right)^{1/\lambda}$$
$$\leq \left(\int_{B_1(0)\subset B_2(x_j)} \sum_{j=1}^\infty v_j(y)^\lambda dy\right)^{1/\lambda}$$
$$\leq \left(\sum_{j=1}^\infty \int_{B_2(x_j)} v_j(y)^\lambda dy\right)^{1/\lambda} \leq \left(\sum_{j=1}^\infty C\varepsilon_j^\lambda\right)^{1/\lambda} < \infty$$

by (2.7). Thus by (2.13)

$$v \in C^\infty(\mathbb{R}^n \backslash \{0\}) \cap L^\lambda(B_1(0)) \tag{2.17}$$

and

$$-\Delta v = f \quad \text{ in } \mathbb{R}^n \backslash \{0\}. \tag{2.18}$$

Taking $\beta = \alpha \in (0,n)$ and $R = \frac{1}{2}$ in (2.14) and using (2.15) we find for $|x - x_j| < r_j$ (i.e. $|\xi| < 1$) that

$$(n(n-2)\omega_n)^\lambda \int_{|y|<1} \frac{v(y)^\lambda dy}{|x-y|^\alpha} \geq \int_{|y-x_j|<1/2} \frac{v_j(y)^\lambda dy}{|x-y|^\alpha} \geq C\varepsilon_j^\lambda \delta_j^{-\lambda} r_j^{n-\alpha-(n-2)\lambda} I_j(\xi)$$

where

$$I_j(\xi) := \int_{|\eta|<\frac{1}{2r_j}} \frac{\left(\frac{1}{|\eta|^{n-2}+1}\right)^\lambda}{|\xi-\eta|^\alpha} d\eta$$
$$\geq \int_{|\eta|<2} \frac{\left(\frac{1}{2^{n-2}+1}\right)^\lambda}{|\xi-\eta|^\alpha} d\eta + \int_{2<|\eta|<\frac{1}{2r_j}} \frac{\left(\frac{1}{2|\eta|^{n-2}}\right)^\lambda}{|\xi-\eta|^\alpha} d\eta$$
$$\geq C(n,\lambda,\alpha) + \left(\frac{2}{3}\right)^\alpha \frac{1}{2^\lambda} \int_{2<|\eta|<\frac{1}{2r_j}} \frac{1}{|\eta|^{(n-2)\lambda+\alpha}} d\eta$$
$$\geq C(n,\lambda,\alpha) \begin{cases} \frac{1}{r_j^{n-\alpha-(n-2)\lambda}} & \text{if } 0 < \lambda < \frac{n-\alpha}{n-2} \\ \log \frac{1}{r_j} & \text{if } \lambda = \frac{n-\alpha}{n-2} \\ 1 & \text{if } \frac{n-\alpha}{n-2} < \lambda < \frac{n}{n-2} \\ 1 & \text{if } \lambda = \frac{n}{n-2}. \end{cases}$$

Thus for $|x - x_j| < r_j$ we have

$$\int_{|y|<1} \frac{v(y)^\lambda dy}{|x-y|^\alpha} \geq C(n,\lambda,\alpha) \begin{cases} \varepsilon_j^\lambda & \text{if } 0 < \lambda < \frac{n-\alpha}{n-2} \\ \varepsilon_j^\lambda \log \frac{1}{r_j} & \text{if } \lambda = \frac{n-\alpha}{n-2} \\ \varepsilon_j^\lambda r_j^{n-\alpha-(n-2)\lambda} & \text{if } \frac{n-\alpha}{n-2} < \lambda < \frac{n}{n-2} \\ \varepsilon_j^\lambda r_j^{-\alpha} (\log \frac{1}{r_j})^{-1} & \text{if } \lambda = \frac{n}{n-2}. \end{cases} \tag{2.19}$$

Also, for $|x - x_j| < r_j$ we have

$$\begin{aligned} v(x) &\geq \frac{1}{n(n-2)\omega_n} \int_{B_{r_j}(x_j)} \frac{f(y)dy}{|x-y|^{n-2}} \\ &= C(n) \int_{B_{r_j}(x_j)} \frac{M_j \psi_j(y)}{|x-y|^{n-2}} dy \\ &= C(n) \int_{|\eta|<1} \frac{M_j \psi(\eta) r_j^n}{r_j^{n-2} |\xi - \eta|^{n-2}} d\eta \\ &= C(n) \frac{\varepsilon_j}{r_j^{n-2}\delta_j} \int_{|\eta|<1} \frac{\psi(\eta)d\eta}{|\xi-\eta|^{n-2}} \geq A \frac{\varepsilon_j}{r_j^{n-2}\delta_j} \end{aligned} \tag{2.20}$$

where

$$A = C(n) \min_{|\xi|\leq 1} \int_{|\eta|<1} \frac{\psi(\eta)d\eta}{|\xi-\eta|^{n-2}} > 0.$$

Finally, letting $u = \chi v$ where $\chi \in C^\infty(\mathbb{R}^n \to [0,1])$ satisfies $\chi = 1$ in $B_2(0)$ and $\chi = 0$ in $\mathbb{R}^n \backslash B_3(0)$, it follows from (2.17)–(2.20) that u satisfies (2.8)–(2.12). □

The following lemma will be needed for the proof of Theorem 1.2 when $\lambda > \frac{n}{n-2}$.

Lemma 2.3. *Suppose $\alpha \in (0,n)$ and $\lambda > \frac{n}{n-2}$. Let $\{x_j\} \subset \mathbb{R}^n$, $n \geq 3$, and $\{r_j\}, \{\varepsilon_j\} \subset (0,1)$ be sequences satisfying*

$$0 < 4|x_{j+1}| < |x_j| < 1/2, \tag{2.21}$$

$$0 < r_j < |x_j|/4 \quad \text{and} \quad \sum_{j=1}^{\infty} \varepsilon_j < \infty. \tag{2.22}$$

Then there exists a positive function

$$u \in C^\infty(\mathbb{R}^n \backslash \{0\}) \cap L^\lambda(\mathbb{R}^n) \tag{2.23}$$

such that

$$0 \leq -\Delta u \leq \frac{\varepsilon_j}{r_j^{2+n/\lambda}} \quad \text{in } B_{r_j}(x_j) \tag{2.24}$$

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^n \backslash (\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j)) \tag{2.25}$$

$$u \geq \frac{A\varepsilon_j}{r_j^{n/\lambda}} \quad \text{in } B_{r_j}(x_j) \tag{2.26}$$

and

$$\int_{|y-x_j|<r_j} \frac{u(y)^\lambda dy}{|x-y|^\alpha} \geq \frac{B\varepsilon_j^\lambda}{r_j^\alpha} \quad \text{for } x \in B_{r_j}(x_j) \tag{2.27}$$

where $A = A(n)$ and $B = B(n,\lambda,\alpha)$ are positive constants.

Proof. Let $\psi : \mathbb{R}^n \to [0,1]$ be a C^∞ function whose support is $\overline{B_1(0)}$. Define $\psi_j, f_j : \mathbb{R}^n \to [0,\infty)$ by $\psi_j(y) = \psi(\eta)$ where $y = x_j + r_j\eta$ and $f_j = M_j\psi_j$ where $M_j = \frac{\varepsilon_j}{r_j^{2+n/\lambda}}$. Since

$$\int_{\mathbb{R}^n} f_j(y)dy = M_j \int_{\mathbb{R}^n} \psi(\eta) r_j^n d\eta = \varepsilon_j r_j^{\frac{(n-2)\lambda-n}{\lambda}} \int_{\mathbb{R}^n} \psi(\eta)d\eta \leq \varepsilon_j \int_{\mathbb{R}^n} \psi(\eta)d\eta$$

and by (2.21) and $(2.22)_1$ the supports $\overline{B_{r_j}(x_j)}$ of the functions f_j are disjoint and contained in $B_{3/4}(0)$ we see by $(2.22)_2$ that

$$f := \sum_{j=1}^{\infty} f_j \in C^{\infty}(\mathbb{R}^n \backslash \{0\}) \cap L^1(\mathbb{R}^n) \quad \text{and} \quad \text{supp}(f) \subset B_1(0). \tag{2.28}$$

Defining

$$u_j(y) = \int_{\mathbb{R}^n} \frac{f_j(z)dz}{|y-z|^{n-2}}$$

and making the change of variables

$$x = x_j + r_j\xi, \quad y = x_j + r_j\eta, \quad \text{and} \quad z = x_j + r_j\zeta,$$

we find for $\beta \in [0, n)$ that

$$\begin{aligned}\int_{\mathbb{R}^n \text{ or } B_{r_j}(x_j)} \frac{u_j(y)^\lambda dy}{|x-y|^\beta} &= \int_{\mathbb{R}^n \text{ or } |\eta|<1} \frac{\left(\int_{\mathbb{R}^n} \frac{M_j \psi(\zeta) r_j^n d\zeta}{r_j^{n-2}|\eta-\zeta|^{n-2}}\right)^\lambda}{r_j^\beta |\xi-\eta|^\beta} r_j^n d\eta \\ &= \varepsilon_j^\lambda r_j^{-\beta} \int_{\mathbb{R}^n \text{ or } |\eta|<1} \frac{\left(\int_{\mathbb{R}^n} \frac{\psi(\zeta)d\zeta}{|\eta-\zeta|^{n-2}}\right)^\lambda}{|\xi-\eta|^\beta} d\eta. \end{aligned} \tag{2.29}$$

Also

$$0 < C_1(n) < \frac{\int_{\mathbb{R}^n} \frac{\psi(\zeta)d\zeta}{|\eta-\zeta|^{n-2}}}{\frac{1}{|\eta|^{n-2}+1}} < C_2(n) < \infty \quad \text{for } \eta \in \mathbb{R}^n. \tag{2.30}$$

Taking $\beta = 0$ in (2.29) and using (2.30) we get

$$\int_{\mathbb{R}^n} u_j(y)^\lambda dy \le C(n,\lambda)\varepsilon_j^\lambda \int_{\mathbb{R}^n} \left(\frac{1}{|\eta|^{n-2}+1}\right)^\lambda d\eta \le C(n,\lambda)\varepsilon_j^\lambda \tag{2.31}$$

because $\lambda > n/(n-2)$. Defining

$$u(x) := \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-2}} = \frac{1}{n(n-2)\omega_n} \sum_{j=1}^{\infty} u_j(x) \quad \text{for } x \in \mathbb{R}^n \tag{2.32}$$

and using (2.31) we get

$$n(n-2)\omega_n \|u\|_{L^\lambda(\mathbb{R}^n)} \le \sum_{j=1}^{\infty} \|u_j\|_{L^\lambda(\mathbb{R}^n)} \le C \sum_{j=1}^{\infty} \varepsilon_j < \infty$$

by (2.22). Thus (2.28) and (2.32) imply (2.23) and $-\Delta u = f$ in $\mathbb{R}^n \backslash \{0\}$. Hence (2.24) and (2.25) hold.

Taking $\beta = \alpha \in (0, n)$ in (2.29) and using (2.30) we get

$$\min_{x \in B_{r_j}(x_j)} \int_{|y-x_j|<r_j} \frac{u(y)^\lambda dy}{|x-y|^\alpha} \ge C(n,\lambda)\varepsilon_j^\lambda r_j^{-\alpha} \min_{|\xi|\le 1} \int_{|\eta|<1} \frac{\left(\frac{1}{|\eta|^{n-2}+1}\right)^\lambda}{|\xi-\eta|^\alpha} d\eta \ge C(n,\lambda,\alpha)\varepsilon_j^\lambda r_j^{-\alpha}$$

which proves (2.27).

Finally, for $|x - x_j| < r_j$ we have

$$
\begin{aligned}
u(x) &\geq \frac{1}{n(n-2)\omega_n} \int_{B_{r_j}(x_j)} \frac{f(y)dy}{|x-y|^{n-2}} \\
&= C(n) \int_{B_{r_j}(x_j)} \frac{M_j \psi_j(y)}{|x-y|^{n-2}} dy \\
&= C(n) \int_{|\eta|<1} \frac{M_j \psi(\eta) r_j^n d\eta}{r_j^{n-2} |\xi - \eta|^{n-2}} d\eta \\
&= C(n) \frac{\varepsilon_j}{r_j^{n/\lambda}} \int_{|\eta|<1} \frac{\psi(\eta) d\eta}{|\xi - \eta|^{n-2}} \geq \frac{A\varepsilon_j}{r_j^{n/\lambda}}
\end{aligned}
$$

where

$$A = C(n) \min_{|\xi| \leq 1} \int_{|\eta|<1} \frac{\psi(\eta) d\eta}{|\xi - \eta|^{n-2}} > 0.$$

This proves (2.26). ☐

Lemma 2.4. *Suppose for some constants $\alpha \in (0, n)$, $\lambda > 0$, and $\sigma \geq 0$ that u is a nonnegative solution of (1.1,1.2) and $u(x) = O(1)$ as $x \to 0$. Then u has a C^1 extension to the origin, that is, $u = w|_{\mathbb{R}^n \setminus \{0\}}$ for some function $w \in C^1(\mathbb{R}^n)$.*

Proof. Let $v = u + 1$. Then by Lemma 2.1, v satisfies (2.5). Since u, and hence v, is bounded in $B_1(0) \setminus \{0\}$, the constant m in (2.5) is zero and by (1.1,1.2) $-\Delta u$, and hence $-\Delta v$, is bounded in $B_1(0) \setminus \{0\}$. It therefore follows from (2.5) that v, and hence u, has a C^1 extension to the origin. ☐

3 The case $0 < \lambda < \frac{n-\alpha}{n-2}$

In this section we prove Theorem 1.1–1.3 when $0 < \lambda < \frac{n-\alpha}{n-2}$. For these values of λ, the following theorem implies Theorems 1.1 and 1.3.

Theorem 3.1. *Suppose u is a nonnegative solution of (1.1,1.2) for some constants $\alpha \in (0, n)$,*

$$0 < \lambda < \frac{n-\alpha}{n-2} \quad \text{and} \quad 0 \leq \sigma \leq \frac{n}{n-2}. \tag{3.1}$$

Then

$$u(x) = O(|x|^{2-n}) \quad \text{as } x \to 0. \tag{3.2}$$

Proof. Let $v = u + 1$. Then by Lemma 2.1 we have that (2.1)–(2.5) hold. To prove (3.2), it clearly suffices to prove

$$v(x) = O(|x|^{2-n}) \quad \text{as } x \to 0. \tag{3.3}$$

Choose $\varepsilon \in (0, 1)$ such that

$$\lambda < \frac{n-\alpha}{n-2+\varepsilon}. \tag{3.4}$$

By (2.4) we have $v \in L^{\frac{n}{n-2+\varepsilon}}(B_1(0))$ which implies

$$v^\lambda \in L^{\frac{n}{(n-2+\varepsilon)\lambda}}(B_1(0)).$$

Thus, since (3.4) implies

$$\frac{\lambda(n-2+\varepsilon)}{n} < \frac{n-\alpha}{n},$$

we have by Riesz potential estimates that

$$I_{n-\alpha}(v^\lambda) \in L^\infty(B_1(0)).$$

Hence by (2.1) and (2.2), v is a C^2 positive solution of

$$0 \le -\Delta v \le Cv^\sigma \quad \text{in } B_1(0)\backslash\{0\}.$$

Thus by (3.1) and [19, Theorem 2.1], v satisfies (3.3). □

Our next result implies Theorem 1.2 when $0 < \lambda < \frac{n-\alpha}{n-2}$.

Theorem 3.2. *Suppose $\alpha, \lambda,$ and σ are constants satisfying $\alpha \in (0, n)$*

$$0 < \lambda < \frac{n-\alpha}{n-2} \quad \textit{and} \quad \sigma > \frac{n}{n-2}.$$

Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying

$$\lim_{t\to 0^+} \varphi(t) = \infty.$$

Then there exists a nonnegative solution u of (1.1,1.2) such that

$$u(x) \ne O(\varphi(|x|)) \quad \textit{as } x \to 0. \tag{3.5}$$

Proof. Let $\{x_j\} \subset \mathbb{R}^n$ and $\{r_j\}, \{\varepsilon_j\} \subset (0,1)$ be sequences satisfying (2.6) and (2.7). Holding x_j and ε_j fixed and decreasing r_j to a sufficiently small positive number we can assume

$$\frac{A\varepsilon_j}{r_j^{n-2}} > j\varphi(|x_j|) \quad \text{for } j = 1, 2, ... \tag{3.6}$$

and

$$r_j^{(n-2)\sigma - n} < A^\sigma B \varepsilon_j^{\lambda+\sigma-1} \quad \text{for } j = 1, 2, ... \tag{3.7}$$

where A and B are as in Lemma 2.2.

Let u be as in Lemma 2.2. By (2.10), u satisfies (1.2) in $B_2(0)\backslash(\{0\} \cup \cup_{j=1}^\infty B_{r_j}(x_j))$. Also, for $x \in B_{r_j}(x_j)$, it follows from (2.9), (3.7), (2.12), and (2.11) that

$$\begin{aligned} 0 \le -\Delta u \le \frac{\varepsilon_j}{r_j^n} &= \frac{r_j^{(n-2)\sigma-n}}{A^\sigma B \varepsilon_j^{\lambda+\sigma-1}} (B\varepsilon_j^\lambda) \left(\frac{A\varepsilon_j}{r_j^{n-2}}\right)^\sigma \\ &\le (|x|^{-\alpha} * u^\lambda) u^\sigma. \end{aligned}$$

Thus u satisfies (1.2) in $B_2(0)\backslash\{0\}$. Finally by (2.11) and (3.6) we have

$$u(x_j) \ge \frac{A\varepsilon_j}{r_j^{n-2}} > j\varphi(|x_j|)$$

and thus (3.5) holds. □

4 The case $\frac{n-\alpha}{n-2} \le \lambda < \frac{n}{n-2}$

In this section we prove Theorems 1.1–1.3 when $\frac{n-\alpha}{n-2} \le \lambda < \frac{n}{n-2}$. For these values of λ, the result below implies Theorem 1.1.

Theorem 4.1. *Suppose u is a nonnegative solution of (1.1,1.2) for some constants $\alpha \in (0,n)$,*

$$\frac{n-\alpha}{n-2} \le \lambda < \frac{n}{n-2} \quad \text{and} \quad 0 \le \sigma < \frac{2n-\alpha}{n-2} - \lambda. \tag{4.1}$$

Then

$$u(x) = O(|x|^{2-n}) \quad \text{as } x \to 0. \tag{4.2}$$

Proof. Let $v = u+1$. Then by Lemma 2.1 we have that (2.1)–(2.5) hold. To prove (4.2), it clearly suffices to prove

$$v(x) = O(|x|^{-(n-2)}) \quad \text{as } x \to 0. \tag{4.3}$$

Since increasing λ or σ increases the right side of the second inequality in $(2.2)_1$, we can assume instead of (4.1) that

$$\frac{n-\alpha}{n-2} < \lambda < \frac{n}{n-2}, \quad \sigma > 0, \quad \text{and} \quad 1 < \lambda + \sigma < \frac{2n-\alpha}{n-2}. \tag{4.4}$$

Since the increased value of λ is less than $\frac{n}{n-2}$, it follows from (2.4) that (2.1) still holds.

By (4.4) there exists $\varepsilon = \varepsilon(n,\lambda,\sigma,\alpha) \in (0,1)$ such that

$$\left(\frac{n+2-\alpha}{n+2-\alpha-\varepsilon}\right)\frac{n-\alpha}{n-2} < \lambda < \frac{n}{n-2+\varepsilon} \quad \text{and} \quad \lambda + \sigma < \frac{2n-\alpha}{n-2+\varepsilon} \tag{4.5}$$

which implies

$$\sigma < \frac{2n-\alpha}{n-2+\varepsilon} - \lambda < \frac{2n-\alpha}{n-2+\varepsilon} - \frac{n-\alpha}{n-2} < \frac{n}{n-2+\varepsilon}. \tag{4.6}$$

Suppose for contradiction that (4.3) is false. Then there is a sequence $\{x_j\} \subset B_{1/2}(0)\backslash\{0\}$ such that $x_j \to 0$ as $j \to \infty$ and

$$\lim_{j\to\infty} |x_j|^{n-2} v(x_j) = \infty. \tag{4.7}$$

Since for $|x - x_j| < |x_j|/4$

$$\int_{\substack{|y-x_j|>|x_j|/2 \\ |y|<1}} \frac{-\Delta v(y)}{|x-y|^{n-2}} dy \le \left(\frac{4}{|x_j|}\right)^{n-2} \int_{|y|<1} -\Delta v(y) dy,$$

it follows from (2.4) and (2.5) that

$$v(x) \le C\left[\frac{1}{|x_j|^{n-2}} + \int_{|y-x_j|<|x_j|/2} \frac{-\Delta v(y)}{|x-y|^{n-2}} dy\right] \quad \text{for } |x - x_j| < \frac{|x_j|}{4}. \tag{4.8}$$

Substituting $x = x_j$ in (4.8) and using (4.7) we find that

$$|x_j|^{n-2} \int_{|y-x_j|<|x_j|/2} \frac{-\Delta v(y)}{|x_j - y|^{n-2}} dy \to \infty \quad \text{as } j \to \infty. \tag{4.9}$$

Also by (2.4) we have

$$\int_{|y-x_j|<|x_j|/2} -\Delta v(y)dy \to 0 \quad \text{as } j \to \infty. \tag{4.10}$$

Defining $f_j(\eta) = -r_j^n \Delta v(x_j + r_j\eta)$ where $r_j = |x_j|/8$ and making the change of variables $y = x_j + r_j\eta$ in (4.10) and (4.9) we get

$$\int_{|\eta|<4} f_j(\eta)d\eta \to 0 \quad \text{as } j \to \infty \tag{4.11}$$

and

$$\int_{|\eta|<4} \frac{f_j(\eta)d\eta}{|\eta|^{n-2}} \to \infty \quad \text{as } j \to \infty. \tag{4.12}$$

Let

$$N(y) = \int_{|z|<1} \frac{-\Delta v(z)dz}{|y-z|^{n-2}} \quad \text{for } 0 < |y| < 1.$$

By (2.4) and Riesz potential estimates, $N \in L^{\frac{n}{n-2+\varepsilon}}(B_1(0))$. Thus $N^\lambda \in L^{\frac{n}{\lambda(n-2+\varepsilon)}}(B_1(0))$. Hence by Hölder's inequality and (4.5) we have for $R \in (0,1]$ and $|x - x_j| < R|x_j|/8$ that

$$\begin{aligned}
\int_{|y|<1} \frac{N(y)^\lambda dy}{|x-y|^\alpha} - \int_{|y-x_j|<R|x_j|/4} \frac{N(y)^\lambda dy}{|x-y|^\alpha} &= \int_{|y-x_j|>R|x_j|/4, |y|<1} \frac{N(y)^\lambda dy}{|x-y|^\alpha} \\
&\le \left(\|N^\lambda\|_{L^{\frac{n}{\lambda(n-2+\varepsilon)}}(B_1(0))} \right) \left(\int_{|y-x_j|>R|x_j|/4} \frac{dy}{|x-y|^{\alpha q}} \right)^{1/q} \quad \text{where } \frac{\lambda(n-2+\varepsilon)}{n} + \frac{1}{q} = 1 \\
&\le C \left(\int_{|y-x_j|>R|x_j|/4} \frac{dy}{|y-x_j|^{\alpha q}} \right)^{1/q} \qquad (4.13) \\
&= C \frac{1}{|x_j|^{(n-2+\varepsilon)\lambda - (n-\alpha)}} \qquad (4.14)
\end{aligned}$$

where $C > 0$ depends on R but not on j. In (4.13) we used the fact that

$$\frac{|y-x|}{|y-x_j|} \ge \frac{|y-x_j| - |x-x_j|}{|y-x_j|} = 1 - \frac{|x-x_j|}{|y-x_j|} > 1 - \frac{1}{2} = \frac{1}{2}$$

for $|x - x_j| < R|x_j|/8$ and $|y - x_j| > R|x_j|/4$.

Since, by (2.4),

$$N(y) \le C \left[\frac{1}{|x_j|^{n-2}} + \int_{|z-x_j|<R|x_j|/2} \frac{-\Delta v(z)dz}{|y-z|^{n-2}} \right] \quad \text{for } |y-x_j| < R|x_j|/4,$$

we see for $x \in \mathbb{R}^n$ that

$$\int_{|y-x_j|<R|x_j|/4} \frac{N(y)^\lambda dy}{|x-y|^\alpha} \le C \left[\frac{1}{|x_j|^{(n-2)\lambda-(n-\alpha)}} + \int_{|y-x_j|<R|x_j|/4} \frac{\left(\int_{|z-x_j|<R|x_j|/2} \frac{-\Delta v(z)dz}{|y-z|^{n-2}} \right)^\lambda}{|x-y|^\alpha} dy \right].$$

It therefore follows from (2.5), (4.4), [4, Corollary 3.7], and (4.14) that for $|x - x_j| < R|x_j|/8$ we have

$$\begin{aligned}\int_{|y|<1} \frac{v(y)^\lambda dy}{|x-y|^\alpha} &\leq C\left[\int_{|y|<1} \frac{dy}{|x-y|^\alpha |y|^{(n-2)\lambda}} + \int_{|y|<1} \frac{N(y)^\lambda dy}{|x-y|^\alpha}\right] \\ &\leq C\left[\frac{1}{|x_j|^{(n-2+\varepsilon)\lambda-(n-\alpha)}} + \int_{|y-x_j|<R|x_j|/4} \frac{\left(\int_{|z-x_j|<R|x_j|/2} \frac{-\Delta v(z)dz}{|y-z|^{n-2}}\right)^\lambda}{|x-y|^\alpha} dy\right]\end{aligned}$$

where $C > 0$ depends on R but not on j.

We see therefore from (2.2), (2.4), and (2.5) that for $|x - x_j| < R|x_j|/8$ and $R \in (0, 1]$ we have

$$\begin{aligned}-\Delta v(x) \leq & C\left[\frac{1}{|x_j|^{(n-2+\varepsilon)\lambda-(n-\alpha)}} + \int_{|y-x_j|<R|x_j|/4} \frac{\left(\int_{|z-x_j|<R|x_j|/2} \frac{-\Delta v(z)dz}{|y-z|^{n-2}}\right)^\lambda}{|x-y|^\alpha} dy\right] \\ &\times \left[\frac{1}{|x_j|^{(n-2)\sigma}} + \left(\int_{|y-x_j|<R|x_j|/2} \frac{-\Delta v(y)dy}{|x-y|^{n-2}}\right)^\sigma\right].\end{aligned}$$

Hence under the change of variables

$$f_j(\xi) = -r_j^n \Delta v(x), \quad x = x_j + r_j\xi, \quad y = x_j + r_j\eta, \quad z = x_j + r_j\zeta, \quad r_j = |x_j|/8$$

we obtain from (4.5) that

$$\begin{aligned}f_j(\xi) &= -r_j^n \Delta v(x) \leq -r_j^{(n-2+\varepsilon)(\lambda+\sigma)-(n-\alpha)} \Delta v(x) \\ &\leq C\left[1 + \int_{|\eta|<4R} \frac{\left(\int_{|\zeta|<4R} \frac{f_j(\zeta)d\zeta}{|\eta-\zeta|^{n-2}}\right)^\lambda}{|\xi-\eta|^\alpha} d\eta\right]\left[1 + \left(\int_{|\eta|<4R} \frac{f_j(\eta)d\eta}{|\xi-\eta|^{n-2}}\right)^\sigma\right] \qquad (4.15)\end{aligned}$$

for $|\xi| < R$ where $C > 0$ depends on R but not on j.

To complete the proof of Theorem 4.1 we will need the following lemma.

Lemma 4.1. *Suppose the sequence*

$$\{f_j\} \text{ is bounded in } L^p(B_{4R}(0)) \qquad (4.16)$$

for some constants $p \in [1, \frac{n}{2}]$ *and* $R \in (0, 1]$. *Then there exists a positive constant* $C_0 = C_0(n, \lambda, \sigma, \alpha)$ *such that the sequence*

$$\{f_j\} \text{ is bounded in } L^q(B_R(0)) \qquad (4.17)$$

for some $q \in (p, \infty)$ *satisfying*

$$\frac{1}{p} - \frac{1}{q} \geq C_0. \qquad (4.18)$$

Proof. For $R \in (0, 1]$ we formally define operators N_R and I_R by

$$(N_R f)(\xi) = \int_{|\eta|<4R} \frac{f(\eta)d\eta}{|\xi-\eta|^{n-2}} \quad \text{and} \quad (I_R f)(\xi) = \int_{|\eta|<4R} \frac{f(\eta)d\eta}{|\xi-\eta|^\alpha}.$$

Define p_2 by

$$\frac{1}{p} - \frac{1}{p_2} = \frac{2-\varepsilon}{n} \tag{4.19}$$

where ε is as in (4.5). Then $p_2 \in (p, \infty)$ and thus by Riesz potential estimates we have

$$\|(N_R f_j)^\lambda\|_{p_2/\lambda} = \|N_R f_j\|_{p_2}^\lambda \le C\|f_j\|_p^\lambda \tag{4.20}$$

and

$$\|(N_R f_j)^\sigma\|_{p_2/\sigma} = \|N_R f_j\|_{p_2}^\sigma \le C\|f_j\|_p^\sigma \tag{4.21}$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(B_{4R}(0))}$. Since

$$\frac{1}{p_2} = \frac{1}{p} - \frac{2-\varepsilon}{n} \le 1 - \frac{2-\varepsilon}{n} = \frac{n-2+\varepsilon}{n}$$

we see by (4.5) that

$$\frac{p_2}{\lambda} > 1. \tag{4.22}$$

Now there are two cases to consider.

Case I. Suppose

$$\frac{p_2}{\lambda} < \frac{n}{n-\alpha}. \tag{4.23}$$

Define p_3 and q by

$$\frac{\lambda}{p_2} - \frac{1}{p_3} = \frac{n-\alpha}{n} \tag{4.24}$$

and

$$\frac{1}{q} := \frac{1}{p_3} + \frac{\sigma}{p_2} = \frac{\lambda+\sigma}{p_2} - \frac{n-\alpha}{n}. \tag{4.25}$$

It follows from (4.22)–(4.25), (4.19), and (4.4) that

$$1 < \frac{p_2}{\lambda} < p_3 < \infty, \quad q > 0, \tag{4.26}$$

and

$$\begin{aligned}\frac{1}{p} - \frac{1}{q} &= \frac{1}{p} - \left((\lambda+\sigma)\left(\frac{1}{p} - \frac{2-\varepsilon}{n}\right) - \frac{n-\alpha}{n}\right)\\ &= \frac{(2-\varepsilon)(\lambda+\sigma)+(n-\alpha)}{n} - \frac{\lambda+\sigma-1}{p}\\ &\ge \frac{(2-\varepsilon)(\lambda+\sigma)+(n-\alpha)-n(\lambda+\sigma-1)}{n}\\ &= \frac{2n-\alpha-(n-2+\varepsilon)(\lambda+\sigma)}{n}.\end{aligned}$$

Thus (4.18) holds by (4.5).

By (4.24), (4.26), (4.20), and Riesz potential estimates we find that

$$\begin{aligned}\|(I_R((N_R f_j)^\lambda))^q\|_{p_3/q} &= \|I_R((N_R f_j)^\lambda)\|_{p_3}^q\\ &\le C\|(N_R f_j)^\lambda\|_{p_2/\lambda}^q\\ &\le C\|f_j\|_p^{\lambda q}.\end{aligned}$$

Also by (4.21) we get

$$\|(N_R f_j)^{\sigma q}\|_{\frac{p_2}{\sigma q}} = \|(N_R f_j)^{\sigma}\|^q_{p_2/\sigma} \le C\|f_j\|^{\sigma q}_p.$$

It therefore follows from (4.15), (4.25), Hölder's inequality, and (4.16) that (4.17) holds.

Case II. Suppose

$$\frac{p_2}{\lambda} \ge \frac{n}{n-\alpha}. \tag{4.27}$$

Then by Riesz potential estimates, (4.16), and (4.20) we find that the sequence

$$\{I_R((N_R f_j)^{\lambda})\} \text{ is bounded in } L^{\gamma}(B_{4R}(0)) \text{ for all } \gamma \in (1,\infty). \tag{4.28}$$

Let $\hat{q} = p_2/\sigma$. Then by (4.19),

$$\frac{1}{p} - \frac{1}{\hat{q}} = \frac{1}{p} - \frac{\sigma}{p_2} = \frac{2-\varepsilon}{n} + \frac{1-\sigma}{p_2}.$$

Thus for $\sigma \le 1$ we have

$$\frac{1}{p} - \frac{1}{\hat{q}} \ge \frac{2-\varepsilon}{n} > 0$$

and for $\sigma > 1$ it follows from (4.27) and (4.5) that

$$\begin{aligned}\frac{1}{p} - \frac{1}{\hat{q}} &\ge \frac{2-\varepsilon}{n} - \frac{\sigma - 1}{\frac{n\lambda}{n-\alpha}} \\ &\ge \frac{2-\varepsilon}{n} - \frac{\frac{2n-\alpha}{n-2} - \lambda - 1}{\frac{n\lambda}{n-\alpha}} \\ &= \frac{n+2-\alpha-\varepsilon}{n\lambda}\left(\lambda - \left(\frac{n+2-\alpha}{n+2-\alpha-\varepsilon}\right)\frac{n-\alpha}{n-2}\right) > 0.\end{aligned}$$

Thus defining $q \in (p, \hat{q})$ by

$$\frac{1}{q} = \frac{\frac{1}{p} + \frac{1}{\hat{q}}}{2}$$

we have for $\sigma > 0$ that

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{\hat{q}}\right) \ge C_0(n,\lambda,\sigma,\alpha) > 0.$$

That is (4.18) holds.

Since $q\sigma/p_2 < \hat{q}\sigma/p_2 = 1$ there exists $\gamma \in (q,\infty)$ such that

$$\frac{q}{\gamma} + \frac{q\sigma}{p_2} = 1. \tag{4.29}$$

Also

$$\|(I_R((N_R f_j)^{\lambda}))^q\|_{\gamma/q} = \|I_R((N_R f_j)^{\lambda})\|^q_{\gamma}$$

and by (4.21)

$$\|(N_R f_j)^{\sigma q}\|_{\frac{p_2}{\sigma q}} = \|(N_R f_j)^{\sigma}\|^q_{p_2/\sigma} \le C\|f_j\|^{\sigma q}_p.$$

It therefore follows from (4.15), (4.29), Hölder's inequality, (4.28), and (4.16) that (4.17) holds. □

We return now to the proof of Theorem 4.1. By (4.11) the sequence

$$\{f_j\} \text{ is bounded in } L^1(B_4(0)). \tag{4.30}$$

Starting with this fact and iterating Lemma 4.1 a finite number of times (m times is enough if $m > 1/C_0$) we see that there exists $R_0 \in (0,1)$ such that the sequence

$$\{f_j\} \text{ is bounded in } L^p(B_{4R_0}(0))$$

for some $p > n/2$. Hence by Riesz potential estimates the sequence $\{N_{R_0} f_j\}$ is bounded in $L^\infty(B_{4R_0}(0))$. Thus (4.15) implies the sequence

$$\{f_j\} \text{ is bounded in } L^\infty(B_{R_0}(0)). \tag{4.31}$$

Since

$$\int_{|\eta|<4} \frac{f_j(\eta)d\eta}{|\eta|^{n-2}} \le \int_{|\eta|<R_0} \frac{f_j(\eta)d\eta}{|\eta|^{n-2}} + \frac{1}{R_0^{n-2}} \int_{R_0 \le |\eta| < 4} f_j(\eta)d\eta$$

we see that (4.30) and (4.31) contradict (4.12). This contradiction completes the proof of Theorem 4.1. □

The following theorem implies Theorems 1.2 and 1.3 when

$$\frac{n-\alpha}{n-2} \le \lambda < \frac{n}{n-2}.$$

Theorem 4.2. *Suppose $\alpha, \lambda,$ and σ are constants satisfying*

$$\alpha \in (0,n), \qquad \frac{n-\alpha}{n-2} \le \lambda \le \frac{n}{n-2}$$

and

$$\begin{array}{ll} \sigma \ge \dfrac{n}{n-2} & \textit{if } \lambda = \dfrac{n-\alpha}{n-2}; \\ \sigma > \dfrac{2n-\alpha}{n-2} - \lambda & \textit{if } \dfrac{n-\alpha}{n-2} < \lambda < \dfrac{n}{n-2}; \\ \sigma > \dfrac{n-\alpha}{n-2} & \textit{if } \lambda = \dfrac{n}{n-2}. \end{array}$$

Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying

$$\lim_{t \to 0^+} \varphi(t) = \infty.$$

Then there exists a nonnegative solution u of (1.1,1.2) such that

$$u(x) \ne O(\varphi(|x|)) \qquad \textit{as } x \to 0. \tag{4.32}$$

Proof. Let $\{x_j\} \subset \mathbb{R}^n$ and $\{r_j\}, \{\varepsilon_j\} \subset (0,1)$ be sequences satisfying (2.6) and (2.7). Holding x_j and ε_j fixed and decreasing r_j to a sufficiently small positive number we can assume for $j = 1, 2, ...$ that

$$j\varphi(|x_j|) \le \begin{cases} \frac{A\varepsilon_j}{r_j^{n-2}} & \text{if } \frac{n-\alpha}{n-2} \le \lambda < \frac{n}{n-2} \\ \dfrac{A\varepsilon_j}{r_j^{n-2}\left(\log \frac{1}{r_j}\right)^{\frac{n-2}{n}}} & \text{if } \lambda = \frac{n}{n-2} \end{cases} \tag{4.33}$$

and

$$A^\sigma B\varepsilon_j^{\lambda+\sigma-1} \geq \begin{cases} r_j^{(n-2)\sigma-n}\left(\log\frac{1}{r_j}\right)^{-1} & \text{if } \lambda=\frac{n-\alpha}{n-2} \\ r_j^{(n-2)\sigma-(2n-\alpha-(n-2)\lambda)} & \text{if } \frac{n-\alpha}{n-2}<\lambda<\frac{n}{n-2} \\ r_j^{(n-2)\sigma-(n-\alpha)}\left(\log\frac{1}{r_j}\right)^{\frac{(n-2)\sigma+2}{n}} & \text{if } \lambda=\frac{n}{n-2} \end{cases} \tag{4.34}$$

where A and B are as in Lemma 2.2. Let u be as in Lemma 2.2. By (2.10), u satisfies (1.2) in $B_2(0)\backslash(\{0\}\cup\cup_{j=1}^\infty B_{r_j}(x_j))$. Also, for $x\in B_{r_j}(x_j)$, it follows from (2.9), (4.34), (2.12), and (2.11) that for $\frac{n-\alpha}{n-2}\leq\lambda<\frac{n}{n-2}$ we have

$$\begin{aligned} 0 &\leq -\Delta u \leq \frac{\varepsilon_j}{r_j^n} \\ &= \begin{cases} \frac{r_j^{(n-2)\sigma-n}\left(\log\frac{1}{r_j}\right)^{-1}}{A^\sigma B\varepsilon_j^{\lambda+\sigma-1}}(B\varepsilon_j^\lambda\log\frac{1}{r_j})\left(A\frac{\varepsilon_j}{r_j^{n-2}}\right)^\sigma & \text{if } \lambda=\frac{n-\alpha}{n-2} \\ \frac{r_j^{(n-2)\sigma-(2n-\alpha-(n-2)\lambda)}}{A^\sigma B\varepsilon_j^{\lambda+\sigma-1}}\left(B\varepsilon_j^\lambda r_j^{(n-\alpha)-(n-2)\lambda}\right)\left(A\frac{\varepsilon_j}{r_j^{n-2}}\right)^\sigma & \text{if } \frac{n-\alpha}{n-2}<\lambda<\frac{n}{n-2} \end{cases} \\ &\leq (|x|^\alpha * u^\lambda)u^\sigma, \end{aligned}$$

and, for $\lambda=\frac{n}{n-2}$, we have

$$\begin{aligned} 0 &\leq -\Delta u \leq \frac{\varepsilon_j}{r_j^n(\log\frac{1}{r_j})^{\frac{n-2}{n}}} \\ &= \frac{r_j^{(n-2)\sigma-(n-\alpha)}(\log\frac{1}{r_j})^{\frac{(n-2)\sigma+2}{n}}}{A^\sigma B\varepsilon_j^{\lambda+\sigma-1}}\left(\frac{B\varepsilon_j^\lambda r_j^{-\alpha}}{\log\frac{1}{r_j}}\right)\left(\frac{A\varepsilon_j}{r_j^{n-2}(\log\frac{1}{r_j})^{\frac{n-2}{n}}}\right)^\sigma \\ &\leq (|x|^{-\alpha} * u^\lambda)u^\sigma. \end{aligned}$$

Thus u satisfies (1.2) in $B_2(0)\backslash\{0\}$. Finally, by (2.11) and (4.33) we have

$$u(x_j)\geq j\varphi(|x_j|)$$

and thus (4.32) holds. □

5 The case $\lambda\geq\frac{n}{n-2}$

In this section we prove Theorems 1.1–1.3 when $\lambda\geq\frac{n}{n-2}$. For these values of λ, our next result implies Theorem 1.1.

Theorem 5.1. *Suppose u is a nonnegative solution of (1.1,1.2) for some constants $\alpha\in(0,n)$,*

$$\lambda\geq\frac{n}{n-2} \quad \textit{and} \quad 0\leq\sigma<1-\frac{\alpha-2}{n}\lambda. \tag{5.1}$$

Then

$$u(x)=O(1) \quad \textit{as } x\to 0 \tag{5.2}$$

and u has a C^1 extension to the origin.

Proof. Let $v = u + 1$. Then by Lemma 2.1 we have that (2.1)–(2.5) hold. To prove (5.2) it clearly suffices to prove

$$v(x) = O(1) \quad \text{as } x \to 0. \tag{5.3}$$

By (2.1) and (5.1), the constant m in (2.5) is zero and thus by (2.5)

$$v(x) \le C\left[1 + \int_{|y|<1} \frac{-\Delta v(y)}{|x-y|^{n-2}} dy\right] \quad \text{for } 0 < |x| < 1 \tag{5.4}$$

for some positive constant C.

Since increasing σ increases the right side of the second inequality in $(2.2)_1$, we can assume instead of (5.1) that

$$\lambda \ge \frac{n}{n-2} \quad \text{and} \quad 0 < \sigma < 1 - \frac{\alpha - 2}{n}\lambda \tag{5.5}$$

which implies

$$\frac{\sigma}{\lambda} < \frac{2-\alpha}{n} + \frac{1}{\lambda} \le \frac{2-\alpha}{n} + \frac{n-2}{n} = \frac{n-\alpha}{n}. \tag{5.6}$$

By (5.5) there exists $\varepsilon = \varepsilon(n, \lambda, \sigma, \alpha) \in (0, 1)$ such that

$$\alpha + \varepsilon < n \quad \text{and} \quad \sigma < 1 - \frac{\alpha + \varepsilon - 2}{n}\lambda$$

which implies

$$\frac{\sigma - 1}{\lambda} < \frac{2 - \alpha - \varepsilon}{n}. \tag{5.7}$$

For the proof of Theorem 5.1 we will need the following lemma.

Lemma 5.1. *Suppose*

$$v \in L^p(B_1(0)) \tag{5.8}$$

for some constant

$$p \in \left[\lambda, \frac{n\lambda}{n-\alpha-\varepsilon}\right). \tag{5.9}$$

Then either

$$v \in L^{\frac{n\lambda}{n-\alpha-\varepsilon}}(B_1(0)) \tag{5.10}$$

or there exists a positive constant $C_0 = C_0(n, \lambda, \sigma, \alpha)$ *such that*

$$v \in L^q(B_1(0)) \tag{5.11}$$

for some $q \in (p, \infty)$ *satisfying*

$$\frac{1}{p} - \frac{1}{q} \ge C_0. \tag{5.12}$$

Proof. Define p_2 by

$$\frac{\lambda}{p} - \frac{1}{p_2} = \frac{n - \alpha - \varepsilon}{n}. \tag{5.13}$$

Then by (5.9)

$$1 \le \frac{p}{\lambda} < p_2 < \infty$$

and thus by Riesz potential estimates and (5.8) we have

$$\|I_{n-\alpha} v^\lambda\|_{p_2} \le C\|v^\lambda\|_{\frac{p}{\lambda}} = C\|v\|_p^\lambda < \infty \tag{5.14}$$

where I_β is defined in (2.3).

Define $p_3 > 0$ by

$$\frac{1}{p_3} = \frac{1}{p_2} + \frac{\sigma}{p}. \tag{5.15}$$

Then by Hölder's inequality

$$\begin{aligned} \|((I_{n-\alpha}v^\lambda)v^\sigma)^{p_3}\|_1 &\le \|(I_{n-\alpha}v^\lambda)^{p_3}\|_{\frac{p_2}{p_3}} \|v^{\sigma p_3}\|_{\frac{p}{\sigma p_3}} \\ &= \|I_{n-\alpha}v^\lambda\|_{p_2}^{p_3} \|v\|_p^{\sigma p_3} < \infty \end{aligned}$$

by (5.8) and (5.14). Hence by (2.2)

$$-\Delta v \in L^{p_3}(B_1(0)). \tag{5.16}$$

Also by (5.15), (5.13), (5.9), and (5.7) we have

$$\begin{aligned} \frac{1}{p_3} &= \frac{\lambda+\sigma}{p} - \frac{n-\alpha-\varepsilon}{n} \le \frac{\lambda+\sigma}{\lambda} - \frac{n-\alpha-\varepsilon}{n} \\ &= \frac{\sigma}{\lambda} + \frac{\alpha+\varepsilon}{n} < \frac{1}{\lambda} - \frac{\alpha+\varepsilon-2}{n} + \frac{\alpha+\varepsilon}{n} \\ &= \frac{1}{\lambda} + \frac{2}{n}. \end{aligned}$$

Thus by (5.5) we see that

$$p_3 > 1. \tag{5.17}$$

Case I. Suppose $p_3 \ge \frac{n}{2}$. Then by (5.16), (5.4), and Riesz potential estimates we have $v \in L^q(B_1(0))$ for all $q > 1$ which implies (5.10).

Case II. Suppose $p_3 < \frac{n}{2}$. Define q by

$$\frac{1}{p_3} - \frac{1}{q} = \frac{2}{n}. \tag{5.18}$$

Then by (5.17)

$$1 < p_3 < q < \infty.$$

Hence by (5.16), (5.4) and Riesz potential estimates we have (5.11) holds.

Also by (5.18), (5.15), (5.13). (5.9), and (5.7) we get

$$\begin{aligned} \frac{1}{p} - \frac{1}{q} &= \frac{1}{p} + \frac{2}{n} - \frac{1}{p_3} = \frac{1}{p} + \frac{2}{n} - \frac{\sigma}{p} - \frac{\lambda}{p} + 1 - \frac{\alpha+\varepsilon}{n} \\ &= -\frac{\lambda+\sigma-1}{p} + 1 - \frac{\alpha+\varepsilon-2}{n} \\ &\ge \frac{1-(\lambda+\sigma)}{\lambda} + 1 + \frac{2-\alpha-\varepsilon}{n} > 0. \end{aligned}$$

Thus (5.12) holds. ∎

We now return to the proof of Theorem 5.1. By (2.1), $v \in L^\lambda(B_1(0))$. Starting with this fact and iterating Lemma 5.1 a finite number of times we see that (5.10) holds. In particular

$$v \in L^p(B_1(0)) \tag{5.19}$$

for some

$$p > \frac{n\lambda}{n-\alpha}. \tag{5.20}$$

Hence $v^\lambda \in L^{\frac{p}{\lambda}}(B_1(0))$ and $\frac{p}{\lambda} > \frac{n}{n-\alpha}$. Thus by Riesz potential estimates $I_{n-\alpha}(v^\lambda) \in L^\infty(B_1(0))$. So by (2.2)

$$0 \le -\Delta v < Cv^\sigma \text{ in } B_1(0)\backslash\{0\}. \tag{5.21}$$

Hence by (5.19), $-\Delta v \in L^{\frac{p}{\sigma}}(B_1(0))$ and by (5.20) and (5.6)

$$\frac{p}{\sigma} > \frac{n}{n-\alpha}\frac{\lambda}{\sigma} > \left(\frac{n}{n-\alpha}\right)^2 > 1.$$

Thus by (5.4) and Riesz potential estimates

$$v \in L^q(B_1(0)) \quad \text{ where } q = \begin{cases} \infty & \text{if } \frac{p}{\sigma} \ge \frac{n}{2-\varepsilon} \\ \frac{1}{\frac{\sigma}{p} - \frac{2-\varepsilon}{n}} & \text{if } \frac{p}{\sigma} < \frac{n}{2-\varepsilon}. \end{cases} \tag{5.22}$$

If $q = \infty$ then (5.3) holds. Hence we can assume $\frac{p}{\sigma} < \frac{n}{2-\varepsilon}$. Then by (5.22)

$$\frac{1}{p} - \frac{1}{q} = \frac{1-\sigma}{p} + \frac{2-\varepsilon}{n}.$$

Thus, if $\sigma \in (0,1]$ then

$$\frac{1}{p} - \frac{1}{q} > \frac{1}{n}.$$

On the other hand, if $\sigma > 1$ then by (5.20) and (5.7)

$$\begin{aligned} \frac{1}{p} - \frac{1}{q} &= \frac{2-\varepsilon}{n} - \frac{\sigma-1}{p} \\ &> \frac{2-\varepsilon}{n} - \frac{\sigma-1}{\lambda}\frac{n-\alpha}{n} \\ &> \frac{2-\varepsilon}{n} - \frac{2-\alpha-\varepsilon}{n} = \frac{\alpha}{n}. \end{aligned}$$

Thus for $\sigma > 0$ we have

$$\frac{1}{p} - \frac{1}{q} > C(n,\alpha) > 0.$$

Hence, after a finite number of iterations of the procedure of going from (5.19) to (5.22) we get $v \in L^\infty(B_1(0))$ and hence we see again that (5.3) holds.

Finally by Lemma 2.4, u has a C^1 extension to the origin. □

The result below implies Theorems 1.2 and 1.3 when $\lambda \ge \frac{n}{n-2}$.

Theorem 5.2. *Suppose α, λ, and σ are constants satisfying $\alpha \in (0,n)$,*

$$\lambda \ge \frac{n}{n-2}, \quad \sigma \ge 0, \quad \text{and} \quad \sigma > 1 - \frac{\alpha-2}{n}\lambda.$$

Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying

$$\lim_{t\to 0^+} \varphi(t) = \infty.$$

Then there exists a nonnegative solution u of (1.1,1.2) such that

$$u(x) \ne O(\varphi(|x|)) \quad \text{as } x \to 0. \tag{5.23}$$

Proof. If $\lambda = \frac{n}{n-2}$ then Theorem 5.2 follows from Theorem 4.2. Hence we can assume $\lambda > \frac{n}{n-2}$.

Let $\{x_j\} \subset \mathbb{R}^n$ and $\{r_j\}, \{e_j\} \subset (0,1)$ be sequences satisfying (2.21) and (2.22). Holding x_j and ε_j fixed and decreasing r_j to a sufficiently small positive number we can assume

$$\frac{A\varepsilon_j}{r_j^{n/\lambda}} > j\varphi(|x_j|) \quad \text{for } j = 1, 2, ... \tag{5.24}$$

and

$$r_j^{\frac{n}{\lambda}(\sigma - (1 - \frac{\alpha-2}{n}\lambda))} < A^\sigma B \varepsilon_j^{\lambda+\sigma-1} \quad \text{for } j = 1, 2, ... \tag{5.25}$$

where A and B are as in Lemma 2.3. Let u be as in Lemma 2.3. By (2.25) u satisfies (1.2) in $B_2(0)\backslash(\{0\} \cup \cup_{j=1}^\infty B_{r_j}(x_j))$. Also, for $x \in B_{r_j}(x_j)$, it follows from (2.24), (5.25), (2.27), and (2.26) that

$$\begin{aligned} 0 \le -\Delta u \le \frac{\varepsilon_j}{r_j^{2+n/\lambda}} &= \frac{r_j^{\frac{n}{\lambda}(\sigma - (1 - \frac{\alpha-2}{n}\lambda))}}{A^\sigma B \varepsilon_j^{\lambda+\sigma-1}} \left(\frac{B\varepsilon_j^\lambda}{r_j^\alpha}\right) \left(\frac{A\varepsilon_j}{r_j^{n/\lambda}}\right)^\sigma \\ &\le (|x|^\alpha * u^\lambda) u^\sigma. \end{aligned}$$

Thus u satisfies (1.2) in $B_2(0)\backslash\{0\}$.

Finally by (2.26) and (5.24) we have

$$u(x_j) \ge \frac{A\varepsilon_j}{r_j^{n/\lambda}} > j\varphi(|x_j|)$$

and thus (5.23) holds. □

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