INITIAL POINTWISE BOUNDS AND BLOW-UP FOR PARABOLIC CHOQUARD-PEKAR INEQUALITIES

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ABSTRACT. We study the behavior as $t \to 0^+$ of nonnegative functions

$$u \in C^{2,1}(\mathbb{R}^n \times (0,1)) \cap L^\lambda(\mathbb{R}^n \times (0,1)), \quad n \geq 1, \tag{0.1}$$

satisfying the parabolic Choquard-Pekar type inequalities

$$0 \leq u_t - \Delta u \leq (\Phi^{\alpha/n} * u^\lambda) u^\sigma \quad \text{in } B_1(0) \times (0,1) \tag{0.2}$$

where $\alpha \in (0, n+2)$, $\lambda > 0$, and $\sigma \geq 0$ are constants, Φ is the heat kernel, and $*$ is the convolution operation in $\mathbb{R}^n \times (0,1)$. We provide optimal conditions on α, λ, and σ such that nonnegative solutions u of (0.1),(0.2) satisfy pointwise bounds in compact subsets of $B_1(0)$ as $t \to 0^+$. We obtain similar results for nonnegative solutions of (0.1),(0.2) when $\Phi^{\alpha/n}$ in (0.2) is replaced with the fundamental solution Φ_α of the fractional heat operator $(\frac{\partial}{\partial t} - \Delta)^{\alpha/2}$.

1. Introduction

In this paper we study the behavior as $t \to 0^+$ of nonnegative functions

$$u \in C^{2,1}(\mathbb{R}^n \times (0,T)) \cap L^\lambda(\mathbb{R}^n \times (0,T)), \quad n \geq 1, \tag{1.1}$$

satisfying the nonlocal parabolic Choquard-Pekar type inequalities

$$0 \leq Hu \leq (\Phi^{\alpha/n} * u^\lambda) u^\sigma \quad \text{in } \Omega \times (0,T) \tag{1.2}$$

where $\alpha \in (0, n+2)$, $\lambda > 0$, $\sigma \geq 0$, and $T > 0$ are constants, Ω is an open subset of $\mathbb{R}^n$, $Hu = u_t - \Delta u$ is the heat operator,

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ 0 & \text{for } (x,t) \in \mathbb{R}^n \times (-\infty, 0] \end{cases} \tag{1.3}$$

is the heat kernel, and $*$ is the convolution operation in $\mathbb{R}^n \times (0,T)$, that is,

$$(\Phi^{\alpha/n} * u^\lambda)(x,t) = \iint_{\mathbb{R}^n \times (0,T)} \Phi(x-y, t-s)^{\alpha/n} u(y,s)^\lambda \, dy \, ds.$$

The regularity condition $u \in L^\lambda(\mathbb{R}^n \times (0,T))$ in (1.1) and the upper bound of $n+2$ for α are natural because one does not want the nonlocal convolution operation on the right side of (1.2) to be infinite at every point in $\mathbb{R}^n \times (0,T)$.

We also obtain results on the behavior as $t \to 0^+$ of nonnegative solutions of (1.1),(1.2) when $\Phi^{\alpha/n}$ in (1.2) is replaced with the fundamental solution Φ_α of the fractional heat operator $(\frac{\partial}{\partial t} - \Delta)^{\alpha/2}$. (See Remark 1.2.)

A motivation for the study of (1.1),(1.2) comes from the nonlocal elliptic equation

$$-\Delta u = (\Gamma^{\alpha/(n-2)} * u^\lambda)|u|^{\lambda-2} u \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

where $\alpha \in (0,n)$, $\lambda > 1$ and $\Gamma(x) = C(n)/|x|^{n-2}$ is a fundamental solution of $-\Delta$. For $n = 3$, $\alpha = 1$, and $\lambda = 2$, equation (1.4) is known in the literature as the *Choquard-Pekar equation* and

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was introduced in [16] as a model in quantum theory of a polaron at rest (see also [2]). Later, the equation (1.4) appears as a model of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [6]. More recently, the same equation (1.4) was used in a model of self-gravitating matter (see, e.g., [5, 12]) and it is known in this context as the *Schrödinger-Newton equation*.

The Choquard-Pekar equation (1.4) has been investigated for a few decades by variational methods starting with the pioneering works of Lieb [6] and Lions [7, 8]. More recently, new and improved techniques have been devised to deal with various forms of (1.4) (see, e.g., [10, 11, 13, 14, 15, 20] and the references therein).

Using nonvariational methods, the authors in [14] obtained sharp conditions for the nonexistence of nonnegative solutions to

$$-\Delta u \geq (\Gamma^{\alpha/(n-2)} * u^\lambda)u^\sigma$$

in an exterior domain of $\mathbb{R}^n$, $n \geq 3$.

For some very recent results on positive solutions Choquard-Pekar equations and inequalities which have an isolated singularity at the origin see [1] and [4].

Other examples of nonlocal equations which have been studied extensively in recent years are equations containing the fractional Laplacian and some of these equations are equivalent to equations containing convolutions with powers of the fundamental solution Γ of $-\Delta u$. For example, see [21] and [9].

On the other hand, we know of no results for nonlocal equations or inequalities when the nonlocal feature of the problem is due to convolutions with powers of the fundamental solution (1.3) of the heat equation. Our results for (1.1),(1.2) are, in this regard, new.

In this paper we consider the following question.

Question 1.1. Suppose $\alpha \in (0, n+2)$ and $\lambda > 0$ are constants and Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$. For which nonnegative constants σ, if any, does there exist a continuous function $\varphi : (0,1) \to (0,\infty)$ such that for all compact subsets K of Ω and all nonnegative solutions u of (1.1),(1.2) we have

$$\max_{x\in K} u(x,t) = O(\varphi(t)) \quad \text{as } t \to 0^+ \tag{1.5}$$

and what is the optimal such φ when it exists?

We call the function φ in (1.5) a pointwise bound for u on compact subsets of Ω as $t \to 0^+$.

Remark 1.1. Suppose $0 < \lambda < (n+2)/n$. Then, since $u = \Phi$, where Φ is the heat kernel given by (1.3), is a solution of (1.1),(1.2) and $\Phi(0,t) = (4\pi t)^{-n/2}$, we see that any pointwise bound for nonnegative solutions u of (1.1),(1.2) on compact subsets of Ω as $t \to 0^+$ must be at least as large as $t^{-n/2}$ and whenever $t^{-n/2}$ is such a bound it is necessarily optimal.

In order to state our results for Question 1.1, we define for each $\alpha \in (0, n+2)$ the continuous, piecewise linear functions $g_\alpha, G_\alpha : (0,\infty) \to [0,\infty)$ by

$$g_\alpha(\lambda) = \begin{cases} \frac{n+2}{n} & \text{if } 0 < \lambda < \frac{n+2-\alpha}{n} \\ \frac{2(n+2)-\alpha}{n} - \lambda & \text{if } \frac{n+2-\alpha}{n} \leq \lambda < \frac{n+2}{n} \\ \max\{0, 1 - \frac{\alpha-2}{n+2}\lambda\} & \text{if } \lambda \geq \frac{n+2}{n} \end{cases} \tag{1.6}$$

and

$$G_\alpha(\lambda) = \begin{cases} \frac{2(n+2)-\alpha}{n} - \lambda & \text{if } 0 < \lambda < \frac{n+2}{n} \\ \max\{0, 1 - \frac{\alpha-2}{n+2}\lambda\} & \text{if } \lambda \geq \frac{n+2}{n}. \end{cases}$$

These functions are graphed in Figure 1 (resp. Figure 2) when $\alpha \in (2, n+2)$ (resp. $\alpha \in (0,2]$). Note that

$$g_\alpha(\lambda) = G_\alpha(\lambda) \quad \text{for } \frac{n+2-\alpha}{n} \leq \lambda < \infty$$

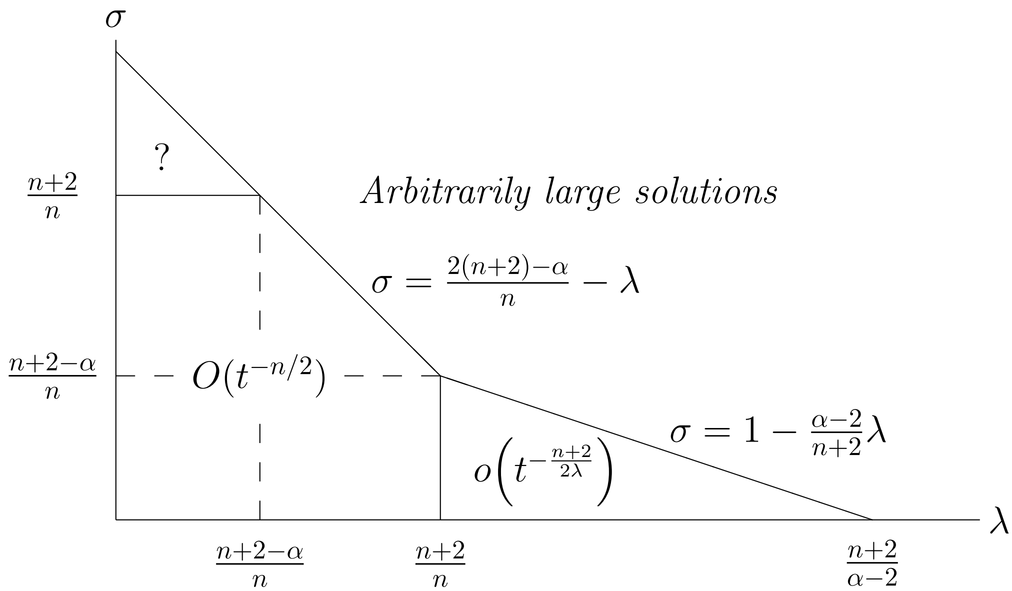

Figure 1. Case $\alpha \in (2, n+2)$.

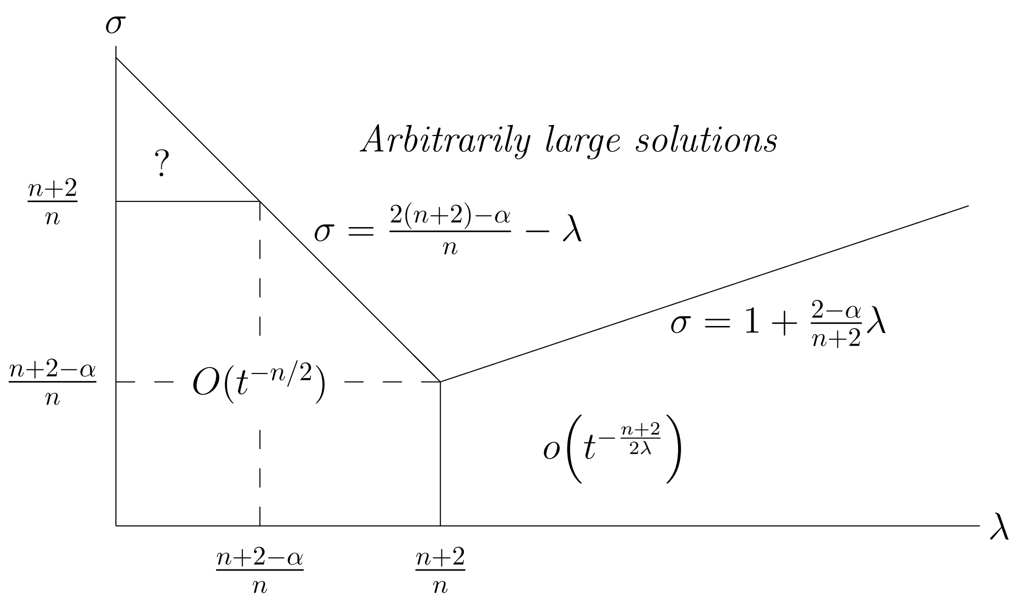

Figure 2. Case $\alpha \in (0, 2]$. When $\alpha = 2$ the graph on the interval $\lambda > (n+2)/n$ is the horizontal half line $\sigma = 1$.

and

$$g_\alpha(\lambda) < G_\alpha(\lambda) \quad \text{for } 0 < \lambda < \frac{n+2-\alpha}{n}.$$

According to the following theorem, if the point (λ, σ) lies below the graph of $\sigma = g_\alpha(\lambda)$ then there exists a pointwise bound for nonnegative solutions u of (1.1),(1.2) on compact subsets of Ω as $t \to 0^+$.

Theorem 1.1. *Suppose u is a nonnegative solution of* (1.1),(1.2) *where $\alpha \in (0, n+2)$, $\lambda > 0$, $T > 0$, and*

$$0 \le \sigma < g_\alpha(\lambda)$$

are constants and Ω *is an open subset of* $\mathbb{R}^n$*. Then for each compact subset* K *of* Ω *we have as* $t \to 0^+$ *that*

$$\max_{x\in K} u(x,t) = \begin{cases} O(t^{-n/2}) & \text{if } 0 < \lambda < \frac{n+2}{n} \quad (1.7)\\ o(t^{-(n+2)/(2\lambda)}) & \text{if } \lambda \geq \frac{n+2}{n}. \quad (1.8)\end{cases}$$

The estimate (1.7) is optimal by Remark 1.1. The exponent $-(n+2)/(2\lambda)$ in (1.8) is also optimal by the following result.

Theorem 1.2. *Suppose*

$$\lambda \geq \frac{n+2}{n} \quad \text{and} \quad \gamma = \frac{n+2-\varepsilon}{2\lambda}$$

for some $\varepsilon \in (0,1)$*. Then there exists a* C^∞ *positive solution* u *of*

$$Hu = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

such that

$$u \in L^\lambda(\mathbb{R}^n \times (0,T)) \quad \text{for all } T > 0$$

and

$$u(0,t) = t^{-\gamma} \quad \text{for all } t > 0.$$

By the next theorem, if the point (λ,σ) lies above the graph of $\sigma = G_\alpha(\lambda)$ then there does not exist a pointwise bound for nonnegative solutions u of (1.1),(1.2) on compact subsets of Ω as $t \to 0^+$.

Theorem 1.3. *Suppose* $\alpha, \lambda,$ *and* σ *are constants satisfying*

$$\alpha \in (0, n+2), \quad \lambda > 0, \quad \text{and} \quad \sigma > G_\alpha(\lambda).$$

Let $\varphi : (0,1) \to (0,\infty)$ *be a continuous function satisfying*

$$\lim_{t\to 0^+} \varphi(t) = \infty.$$

Then there exists a positive solution u *of* (1.1),(1.2) *with* $T = 1$ *and* $\Omega = \mathbb{R}^n$ *such that*

$$u(0,t) \neq O(\varphi(t)) \quad \text{as } t \to 0^+.$$

Theorems 1.1–1.3 completely answer Question 1.1 when the point (λ,σ) lies below the graph of g_α or above the graph of G_α. In particular, if u is a nonnegative solution of (1.1),(1.2) where (λ,σ) lies in the first quadrant of the $\lambda\sigma$-plane and either $\sigma < g_\alpha(\lambda)$ or $\sigma > G_\alpha(\lambda)$ then according to Theorems 1.1–1.3 either

(i) $\varphi(t) = t^{-n/2}$ is an optimal a priori pointwise bound for u on compact subsets of Ω as $t \to 0^+$; or

(ii) $\varphi(t) = t^{-(n+2)/(2\lambda)}$ is an optimal a priori pointwise bound for u on compact subsets of Ω as $t \to 0^+$; or

(iii) no pointwise a priori bound exists for u on compact subsets of Ω as $t \to 0^+$, that is solutions can be arbitrarily large as $t \to 0^+$.

The regions in which these three possibilities occur are shown in Figures 1 and 2. Also included in Figures 1 and 2 is an open triangular region marked with a question mark. For (λ,σ) in this region we have no results for Question 1.1.

Concerning the case that (λ,σ) lies on the graph of g_α we have the following result.

Theorem 1.4. *Suppose* $\alpha \in (0, n+2)$*.*

(i) If $0 < \lambda < \frac{n+2-\alpha}{n}$ *and* $\sigma = g_\alpha(\lambda)$ *then* $\varphi(t) = t^{-n/2}$ *is a poinwise bound for nonnegative solutions* u *of* (1.1),(1.2) *on compact subsets of* Ω *as* $t \to 0^+$*.*

(ii) If $\alpha \in (2, n+2)$*,* $\lambda > \frac{n+2}{\alpha-2}$*, and* $\sigma = g_\alpha(\lambda)$ *then there does not exist an a priori pointwise bound for nonnegative solutions* u *of* (1.1),(1.2) *on compact subsets of* Ω *as* $t \to 0^+$*.*

When a pointwise a priori bound as $t \to 0^+$ for nonnegative solutions u of (1.1),(1.2) on compact subsets of Ω does not exist, as in Theorems 1.3 and 1.4(ii), we prove this by constructing for any given continuous function $\varphi : (0,1) \to (0,\infty)$ a nonnegative solution u of (1.1),(1.2) consisting of a sequence of smoothly connected peaks centered at (x_j, t_j) where $t_j \to 0^+$ such that

$$u(x_j, t_j) \neq O(\varphi(t_j)) \quad \text{as } j \to \infty.$$

When such a pointwise a priori bound does exist, as in Theorems 1.1 and 1.4(i), we reduce the proof of this fact to ruling out the possibility of such peaked solutions.

If $\alpha \in (0, n+2)$ and $\lambda > 0$ then one of the following three conditions holds:

(i) $0 < \lambda < \frac{n+2-\alpha}{n}$;
(ii) $\frac{n+2-\alpha}{n} \le \lambda < \frac{n+2}{n}$;
(iii) $\frac{n+2}{n} \le \lambda < \infty$.

The proofs of Theorems 1.1–1.4 in case (i) (resp. (ii), (iii)) are given in Section 3 (resp. 4, 5). In Section 2 we provide some lemmas needed for these proofs. Our approach relies on an integral representation formula for nonnegative supertemperatures (see Appendix A), some integral estimates for heat potentials (see Appendix B), and Moser's iteration (see Lemmas 4.1 and 5.2).

In this paper, we denote by $\mathcal{P}_r(x,t)$ the open circular cylinder in $\mathbb{R}^n \times \mathbb{R}$ of radius $\sqrt{r}$, height r, and top center point (x,t). Thus

$$\mathcal{P}_r(x,t) = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y-x| < \sqrt{r} \text{ and } t-r < s < t\}.$$

Remark 1.2. Note that

$$\Phi(x,t)^{\alpha/n} = \frac{1}{(4\pi)^{\alpha/2}} t^{-\alpha/2} e^{-\frac{\alpha}{4n}\frac{|x|^2}{t}} \chi_{(0,\infty)}(t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{1.9}$$

However, by checking the proofs of our results, we find that Theorems 1.1, 1.3, and 1.4 remain correct if $\Phi(x,t)^{\alpha/n}$ in (1.2) is replaced with any function of the form

$$C_1(n,\alpha) t^{-\alpha/2} e^{-C_2(n,\alpha)\frac{|x|^2}{t}} \chi_{(0,\infty)}(t) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{1.10}$$

where $C_1(n,\alpha)$ and $C_2(n,\alpha)$ are any given positive constants. In particular, since the fundamental solution Φ_α of the fractional heat operator $(\frac{\partial}{\partial t} - \Delta)^{\alpha/2}$, $\alpha \in (0, n+2)$, is given by

$$\Phi_\alpha(x,t) := \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)} \Phi(x,t),$$

where Φ is the heat kernel (1.3) (see [18, Chapter 9, Section 2]), we find for $0 < \alpha < n+2$ that

$$\Phi_{n+2-\alpha}(x,t) = \frac{1}{(4\pi)^{n/2}\Gamma((n+2-\alpha)/2)} t^{-\alpha/2} e^{-\frac{1}{4}\frac{|x|^2}{t}} \chi_{(0,\infty)}(t)$$

is of the form (1.10). Thus Theorems 1.1, 1.3, and 1.4 remain correct if $\Phi^{\alpha/n}$ in (1.2) is replaced with $\Phi_{n+2-\alpha}$.

2. Preliminary Lemmas

Lemma 2.1. *Suppose $\alpha \in (0, n+2)$, $\lambda > 0$, $\sigma \ge 0$, $T > 0$, and $\beta \ge 0$ are constants, Ω is an open subset of $\mathbb{R}^n$, and K is a compact subset of Ω, such that there exists a nonnegative solution u of (1.1),(1.2), where the convolution operation in (1.2) is in $\mathbb{R}^n \times (0,T)$, satisfying*

$$\max_{x\in K} u(x,t) \neq O(t^{-\beta}), \quad (\textit{resp. } o(t^{-\beta})) \quad \textit{as } t \to 0^+. \tag{2.1}$$

Then there exists a nonnegative function $v(\xi,\tau)$ such that

$$v \in C^{2,1}(\mathbb{R}^n \times (0,16)) \cap L^\lambda(\mathbb{R}^n \times (0,16)), \tag{2.2}$$

$$0 \le Hv \le (\Phi^{\alpha/n} * v^\lambda) v^\sigma \quad \textit{in } B_4(0) \times (0,16), \tag{2.3}$$

where $*$ is the convolution operation in $\mathbb{R}^n \times (0,16)$, and

$$\max_{|\xi|\le 1} v(\xi,\tau) \ne O(\tau^{-\beta}), \quad (\textit{resp. } o(\tau^{-\beta})) \quad \textit{as } \tau \to 0^+. \tag{2.4}$$

Proof. It follows from (2.1) and the compactness of K that there exist a sequence $\{(x_j,t_j)\} \subset K \times (0,T)$ and $x_0 \in K$ such that

$$(x_j,t_j) \to (x_0,0) \quad \text{as } j \to \infty \tag{2.5}$$

and

$$u(x_j,t_j) \ne O(t_j^{-\beta}) \quad (\text{resp. } o(t_j^{-\beta})) \quad \text{as } j \to \infty. \tag{2.6}$$

Choose $r \in (0,1)$ and $b > 0$ such that

$$\overline{B_{4r}(x_0)} \times (0,16r^2) \subset \Omega \times (0,T) \tag{2.7}$$

and

$$b^{\lambda+\sigma-1} < r^{-(n+4-\alpha)}. \tag{2.8}$$

Define $v(\xi,\tau)$ by $u(x,t) = bv(\xi,\tau)$ where $x = x_0 + r\xi$ and $t = r^2\tau$ and define (ξ_j,τ_j) by $x_j = x_0 + r\xi_j$ and $t_j = r^2\tau_j$. Then by (2.5)

$$(\xi_j,\tau_j) \to (0,0) \quad \text{as } j \to \infty. \tag{2.9}$$

Clearly $(x,t) \in \mathbb{R}^n \times (0,16r^2)$ if and only if $(\xi,\tau) \in \mathbb{R}^n \times (0,16)$. Also $16r^2 \le T$ by (2.7). It therefore follows from (1.1) that (2.2) holds.

For $(x,t) \in \mathcal{P}_{16r^2}(x_0,16r^2)$ (i.e. $(\xi,\tau) \in \mathcal{P}_{16}(0,16)$) we have under the change of variables $y = x_0 + r\eta$, $s = r^2\zeta$ that

$$\iint_{\mathbb{R}^n\times(0,T)} \Phi(x-y,t-s)^{\alpha/n} u(y,s)^\lambda \, dy\, ds = \frac{b^\lambda r^{n+2}}{r^\alpha} \iint_{\mathbb{R}^n\times(0,16)} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n} v(\eta,\zeta)^\lambda \, d\eta\, d\zeta$$

where in the last integral we were able to replace the region of integration $\mathbb{R}^n \times (0,T/r^2)$ with $\mathbb{R}^n \times (0,16)$ because $\tau < 16 \le T/r^2$ and $\Phi(x,t) = 0$ for $t < 0$. Thus by (1.2) and (2.8) we find that v satisfies (2.3).

Finally by (2.6) we have

$$\tau_j^\beta v(\xi_j,\tau_j) = \left(\frac{t_j}{r^2}\right)^\beta \frac{1}{b} u(x_j,t_j) \ne O(1) \quad (\text{resp. } o(1)) \quad \text{as } j \to \infty$$

which together with (2.9) implies (2.4). $\square$

Remark 2.1. Suppose α, λ, σ, T, β, Ω, and K are as in Lemma 2.1. Then in order to show that all nonnegative solutions u of (1.1),(1.2) satisfy

$$\max_{x\in K} u(x,t) = O(t^{-\beta}) \quad (\text{resp. } o(t^{-\beta})) \quad \text{as } t \to 0^+$$

it suffices by Lemma 2.1 to show that all nonnegative solutions $u(x,t)$ of

$$u \in C^{2,1}(\mathbb{R}^n \times (0,16)) \cap L^\lambda(\mathbb{R}^n \times (0,16)) \tag{2.10}$$

and

$$0 \le Hu \le (\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } B_4(0) \times (0,16), \tag{2.11}$$

where $*$ is the convolution operation in $\mathbb{R}^n \times (0,16)$, satisfy

$$\max_{|x|\le 1} u(x,t) = O(t^{-\beta}) \quad (\text{resp. } o(t^{-\beta})) \quad \text{as } t \to 0^+.$$

Throughout this paper we will repeatedly use the following simple lemma.

Lemma 2.2. *If* $\gamma > 0$ *and* $x \in \mathbb{R}^n$, $n \geq 1$, *then*

$$\int_{|x-y|<r} e^{-\gamma|x-y|^2} dy = \gamma^{-n/2} \int_{|z|<\sqrt{\gamma}r} e^{-|z|^2} dz$$

and

$$\int_{|x-y|>r} e^{-\gamma|x-y|^2} dy = \gamma^{-n/2} \int_{|z|>\sqrt{\gamma}r} e^{-|z|^2} dz.$$

In particular

$$\int_{\mathbb{R}^n} e^{-\gamma|x-y|^2} dy = C(n)\gamma^{-n/2}.$$

Proof. Make the change of variables $z = \sqrt{\gamma}(x-y)$. □

Lemma 2.3. *Suppose for some constants* $\alpha \in (0, n+2)$, $\lambda > 0$, *and* $\sigma \geq 0$, *the function* u *is a nonnegative solution of* (2.10),(2.11) *where* $*$ *is the convolution operation in* $\mathbb{R}^n \times (0,16)$. *Set* $v = u + 1$. *Then*

$$v \in C^{2,1}(\mathbb{R}^n \times (0,16)) \cap L^\lambda(B_{\sqrt{8}}(0) \times (0,8)) \tag{2.12}$$

and for some positive constant C, v *satisfies*

$$\left.\begin{aligned} &0 \leq Hv \leq C(\Phi^{\alpha/n} * v^\lambda)v^\sigma \\ &v \geq 1 \end{aligned}\right\} \quad \text{in } B_2(0) \times (0,8) \tag{2.13}$$

where $*$ *is the convolution operation in* $B_{\sqrt{8}}(0) \times (0,8)$. *Also*

$$Hv,\ v^\beta \in L^1(B_{\sqrt{8}}(0) \times (0,8)) \quad \text{for all } \beta \in \left[1, \frac{n+2}{n}\right) \tag{2.14}$$

and there exists a positive finite Borel measure μ *on* $B_{\sqrt{8}}(0)$ *and a bounded function* $h \in C^{2,1}(B_2(0) \times (-4,4))$ *satisfying*

$$\begin{aligned} Hh &= 0 \quad \text{in } B_2(0) \times (-4,4) \\ h &= 0 \quad \text{in } B_2(0) \times (-4,0] \end{aligned}$$

such that

$$v(x,t) = h(x,t) + \int_0^8 \int_{|y|<\sqrt{8}} \Phi(x-y,t-s)Hv(y,s)\,dy\,ds + \int_{|y|<\sqrt{8}} \Phi(x-y,t)\,d\mu(y) \tag{2.15}$$

for $(x,t) \in B_2(0) \times (0,4)$.

Remark 2.2. Under the assumptions of Lemma 2.3 we have

$$(4\pi t)^{n/2} \int_{|y|<\sqrt{8}} \Phi(x-y,t)\,d\mu(y) \leq \int_{|y|<\sqrt{8}} d\mu(y) < \infty \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Thus by (2.15) we see that

$$v(x,t) \leq C\left(\left(\frac{1}{\sqrt{t}}\right)^n + \int_0^8 \int_{|y|<\sqrt{8}} \Phi(x-y,t-s)Hv(y,s)\,dy\,ds\right) \quad \text{for } (x,t) \in B_2(0) \times (0,4).$$

Proof of Lemma 2.3. Clearly (2.10) implies (2.12). For

$$(x,t) \in B_2(0) \times (0,8), \quad (y,s) \in (\mathbb{R}^n \times (0,16)) \backslash (B_{\sqrt{8}}(0) \times (0,8)),$$

and $s < t$ we have $|x-y| > \sqrt{8}-2 > 1/\sqrt{2}$ and thus

$$\begin{aligned}\Phi(x-y,t-s) &\le \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{1}{8(t-s)}} \\ &\le \sup_{0<\tau<t} \frac{e^{-\frac{1}{8\tau}}}{(4\pi\tau)^{n/2}} \le C\frac{e^{-\frac{1}{8t}}}{t^{n/2}}.\end{aligned}$$

Hence for $(x,t) \in B_2(0)\times(0,8)$ we have

$$\begin{aligned}&\iint_{\mathbb{R}^n\times(0,16)\setminus B_{\sqrt{8}}(0)\times(0,8)} \Phi(x-y,t-s)^{\alpha/n} u(y,s)^\lambda \, dy\, ds \\ &\qquad \le C\left(\frac{e^{-\frac{1}{8t}}}{t^{n/2}}\right)^{\alpha/n} \iint_{\mathbb{R}^n\times(0,16)} u(y,s)^\lambda \, dy\, ds \le C\left(\frac{e^{-\frac{1}{8t}}}{t^{n/2}}\right)^{\alpha/n} \qquad (2.16)\end{aligned}$$

by (2.10).

On the other hand, for $(x,t)\in B_2(0)\times(0,8)$ we have $B_R(x)\subset B_{\sqrt{8}}(0)$ where $R=\sqrt{8}-2$ and thus by Lemma 2.2 we find that

$$\begin{aligned}\iint_{B_{\sqrt{8}}(0)\times(0,8)} \Phi(x-y,t-s)^{\alpha/n}\, dy\, ds &\ge \int_0^t \left(\int_{B_R(x)} \Phi(y-x,t-s)^{\alpha/n} dy\right) ds \\ &= \int_0^t \int_{B_R(x)} \Phi(y-x,\tau)^{\alpha/n} dy\, d\tau \\ &= \int_0^t \frac{1}{(4\pi\tau)^{\alpha/2}} \left(\int_{|y-x|<R} e^{-\frac{\alpha}{4n\tau}|y-x|^2} dy\right) d\tau \\ &= C\int_0^t \tau^{\frac{n-\alpha}{2}} \left(\int_{|z|<R\sqrt{\frac{\alpha}{4n\tau}}} e^{-|z|^2} dz\right) d\tau \\ &\ge C\int_0^t \tau^{\frac{n-\alpha}{2}} d\tau = Ct^{\frac{n+2-\alpha}{2}} \ge C\left(\frac{e^{-\frac{1}{8t}}}{t^{n/2}}\right)^{\alpha/n}. \qquad (2.17)\end{aligned}$$

Hence for $(x,t)\in B_2(0)\times(0,8)$ we obtain from (2.16) and (2.17) that

$$\begin{aligned}&\iint_{\mathbb{R}^n\times(0,16)} \Phi(x-y,t-s)^{\alpha/n} u(y,s)^\lambda \, dy\, ds \\ &\le \iint_{B_{\sqrt{8}}(0)\times(0,8)} \Phi(x-y,t-s)^{\alpha/n} u(y,s)^\lambda \, dy\, ds + C\iint_{B_{\sqrt{8}}(0)\times(0,8)} \Phi(x-y,t-s)^{\alpha/n} 1^\lambda \, dy\, ds \\ &\le C\iint_{B_{\sqrt{8}}(0)\times(0,8)} \Phi(x-y,t-s)^{\alpha/n} v(y,s)^\lambda \, dy\, ds.\end{aligned}$$

Thus, since u satisfies (2.11) we see that v satisfies (2.13). Finally, by (2.11), $Hv \ge 0$ in $B_4(0)\times(0,16)$. Hence Theorem A.1 and Remark A.1 with $R_1 = 4$, $R_2 = 8$, and $R_3 = 16$ imply (2.14) and (2.15). □

The following lemma will be needed to estimate the last integral in (2.15).

Lemma 2.4. *Suppose*

$$u \in L^p(\Omega\times(0,T)) \qquad (2.18)$$

for some open subset Ω *of* $\mathbb{R}^n$, $n\ge 1$, *and some constants* $p\in[1,\infty)$ *and* $T>0$. *Assume also that*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, d\mu(y)$$

for some finite positve Borel measure μ on $\mathbb{R}^n$. Then for each compact subset K of Ω we have

$$\max_{x\in K} u(x,t) = o\left(t^{-\frac{n+2}{2p}}\right) \quad \text{as } t \to 0^+. \tag{2.19}$$

Proof. The proof consists of two steps.

Step 1. In this step we prove Lemma 2.4 in the special case that

$$\Omega = B_{3r}(x_0) \quad \text{and} \quad K = \overline{B_r(x_0)} \tag{2.20}$$

for some $x_0 \in \mathbb{R}^n$ and some $r > 0$. Clearly we can assume $x_0 = 0$. Since $u = v + w$ where

$$v(x,t) = \int_{|y|<2r} \Phi(x-y,t)\,d\mu(y)$$

and

$$w(x,t) = \int_{|y|\ge 2r} \Phi(x-y,t)\,d\mu(y),$$

to complete step 1, it suffices to prove v and w satisfy (2.19) when Ω and K are given by (2.20).

Since for $|x-y| \ge r$ and $t > 0$

$$\Phi(x-y,t) \le \frac{1}{r^n}\left(\frac{r^2}{4\pi t}\right)^{n/2} e^{-\frac{r^2}{4t}} \le r^{-n}C(n)$$

we have

$$\max_{|x|\le r} w(x,t) \le r^{-n}C(n)\int_{\mathbb{R}^n} d\mu(y) < \infty \quad \text{for } t > 0.$$

Thus w satisfies (2.19) when Ω and K are given by (2.20).

For $|y| \le 2r$ and $\tau > 0$ it follows from Lemma 2.2 that

$$\begin{aligned}\int_{|x|\ge 3r} \Phi(x-y,\tau)^p dx &= \frac{1}{(4\pi\tau)^{np/2}}\int_{|x|\ge 3r} e^{-\frac{p|x-y|^2}{4\tau}} dx \\ &\le \frac{1}{(4\pi\tau)^{np/2}}\int_{|x-y|\ge r} e^{-\frac{p|x-y|^2}{4\tau}} dx \\ &= \frac{C(n,p)}{r^{n(p-1)}}\left[\left(\frac{r}{\sqrt{\tau}}\right)^{n(p-1)}\int_{|z|>\sqrt{\frac{p}{4}}\frac{r}{\sqrt{\tau}}} e^{-|z|^2}dz\right] \\ &\le C(n,p)/r^{n(p-1)}.\end{aligned}$$

We obtain therefore from Jensen's inequality and Fubini's theorem that

$$\begin{aligned}\|v\|^p_{L^p((\mathbb{R}^n\setminus B_{3r}(0))\times(0,t))} &= \int_0^t\int_{|x|\ge 3r}\left(\int_{|y|<2r}\Phi(x-y,\tau)\,d\mu(y)\right)^p dx\,d\tau \\ &\le |\mu|^{p-1}\int_{|y|<2r}\int_0^t\left(\int_{|x|\ge 3r}\Phi(x-y,\tau)^p dx\right)d\tau\,d\mu(y) \\ &\le |\mu|^p C(n,p)t/r^{n(p-1)} \quad \text{for all } t > 0.\end{aligned} \tag{2.21}$$

We now use (2.21) to show v satisfies (2.19).

For $0 < \tau < t$ and $x \in \mathbb{R}^n$ it follows from standard L^p-L^q estimates with $q = \infty$ (see [17, Prop. 48.4] that

$$v(x,t) \le (4\pi)^{\frac{-n}{2p}}(t-\tau)^{\frac{-n}{2p}}\|v(\cdot,\tau)\|_{L^p(\mathbb{R}^n)}.$$

Hence

$$v(x,t)^p\int_0^t (t-\tau)^{n/2}d\tau \le (4\pi)^{-n/2}\|v\|^p_{L^p(\mathbb{R}^n\times(0,t))}$$

which implies

$$\begin{aligned}\max_{x\in\mathbb{R}^n} v(x,t)t^{\frac{n+2}{2p}} &\le C(n,p)\|v\|_{L^p(\mathbb{R}^n\times(0,t))}\\ &\le C(n,p)\left[\|u\|_{L^p(B_{3r}(0)\times(0,t))}+\|v\|_{L^p(\mathbb{R}^n\setminus B_{3r}(0)\times(0,t))}\right]\\ &\to 0 \quad \text{as } t\to 0^+\end{aligned}$$

by (2.18) and (2.21). Thus v satisfies (2.19) when Ω and K are given by (2.20).

Step 2. We now use Step 1 to complete the proof. For each $x\in K$ choose $r_x>0$ such that $B_{3r_x}(x)\subset\Omega$. Since K is compact there exists finitely many points $x_1,...,x_m$ in K such that

$$K\subset\bigcup_{j=1}^m B_{r_j}(x_i) \quad \text{where } r_j=r_{x_j}. \tag{2.22}$$

For $j=1,2,...,m$ we have by Step 1 that

$$\max_{|x-x_j|\le r_j} u(x,t)=o\left(t^{-\frac{n+2}{2p}}\right) \quad \text{as } t\to 0^+.$$

Hence (2.19) follows from (2.22). □

Lemma 2.5. *Suppose $r>0$ and $\beta>n+2$ are constants and $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$. Then for $(x,t)\in\overline{\mathcal{P}_r(x_0,t_0)}$ we have*

$$\iint_{\mathbb{R}^n\times\mathbb{R}\setminus\mathcal{P}_{2r}(x_0,t_0)}\Phi(x-y,t-s)^{\beta/n}\,dy\,ds\le\frac{C}{\sqrt{r}^{\beta-(n+2)}}$$

where $C=C(n,\beta)>0$.

Proof. Throughout this proof $(x,t)\in\overline{\mathcal{P}_r(x_0,t_0)}$ and $C=C(n,\beta)$ is a positive constant whose value may change from line to line. Let

$$A=\mathbb{R}^n\times(-\infty,t_0-2r] \quad\text{and}\quad B=(\mathbb{R}^n\setminus B_{\sqrt{2r}}(x_0))\times(t_0-2r,\,t_0).$$

For $(y,s)\in B$ we have

$$\frac{|y-x|}{|y-x_0|}\ge\frac{|y-x_0|-|x-x_0|}{|y-x_0|}=1-\frac{|x-x_0|}{|y-x_0|}\ge 1-\frac{\sqrt{r}}{\sqrt{2r}}=1-\frac{1}{\sqrt{2}}>\frac{1}{4}.$$

It therefore follows from Lemma 2.2 that

$$\begin{aligned}\iint_B\Phi(x-y,t-s)^{\beta/n}\,dy\,ds &\le\int_{t_0-2r}^t\frac{1}{(4\pi(t-s))^{\beta/2}}\left(\int_{|y-x_0|>\sqrt{2r}}e^{-\frac{\beta|y-x_0|^2}{64n(t-s)}}dy\right)ds\\ &=Cr^{\frac{n-\beta}{2}}\int_{t_0-2r}^t\left(\frac{r}{t-s}\right)^{\frac{\beta-n}{2}}\int_{|z|>\sqrt{\frac{2\beta}{64n}}\sqrt{\frac{r}{t-s}}}e^{-|z|^2}dz\,ds\\ &\le Cr^{\frac{n+2-\beta}{2}}.\end{aligned}$$

Also, by Lemma 2.2, we obtain

$$\begin{aligned}\iint_A\Phi(x-y,t-s)^{\beta/n}\,dy\,ds &=\int_{-\infty}^{t_0-2r}\frac{1}{(4\pi(t-s))^{\beta/2}}\int_{\mathbb{R}^n}e^{-\frac{\beta|x-y|^2}{4n(t-s)}}\,dy\,ds\\ &=C\int_{-\infty}^{t_0-2r}(t-s)^{\frac{n-\beta}{2}}ds\\ &=C(t-t_0+2r)^{\frac{n+2-\beta}{2}}\le Cr^{\frac{n+2-\beta}{2}}.\end{aligned}$$

Thus Lemma 2.5 follows from the fact that

$$\iint_{\mathbb{R}^n\times\mathbb{R}\setminus\mathcal{P}_{2r}(x_0,t_0)} \Phi(x-y,t-s)^{\beta/n}\,dy\,ds = \iint_{A\cup B} \Phi(x-y,t-s)^{\beta/n}\,dy\,ds.$$

□

Lemma 2.6. *Suppose $r>0$ and $0<\beta<n+2$ are constants and $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$. Then*

$$\iint_{(y,s)\in\mathcal{P}_r(x_0,t_0)} \Phi(x-y,t-s)^{\beta/n}\,dy\,ds \le C\sqrt{r}^{\,n+2-\beta} \quad \text{for } (x,t)\in\mathcal{P}_r(x_0,t_0)$$

where $C=C(n,\beta)>0$.

Proof. By Lemma 2.2, we have for $(x,t)\in\mathcal{P}_r(x_0,t_0)$ that

$$\begin{aligned}\iint_{\mathcal{P}_r(x_0,t_0)} \Phi(x-y,t-s)^{\beta/n}\,dy\,ds &\le \int_{t_0-r}^{t} \frac{1}{(4\pi(t-s))^{\beta/2}}\left(\int_{\mathbb{R}^n} e^{-\frac{\beta|x-y|^2}{4n(t-s)}}\,dy\right)ds\\ &= C\int_{t_0-r}^{t}(t-s)^{\frac{n-\beta}{2}}\,ds\\ &= C(t-t_0+r)^{\frac{n+2-\beta}{2}} \le C\sqrt{r}^{\,n+2-\beta}.\end{aligned}$$

□

Lemma 2.7. *Suppose $\alpha\in(0,n+2)$ and $\beta\in[0,n+2)$ are constants. Then*

$$\iint_{\mathbb{R}^n\times(0,t)} \Phi(x-y,t-s)^{\alpha/n}\Phi(y-z,s)^{\beta/n}\,dy\,ds \le \frac{C}{\sqrt{t}^{\,\alpha+\beta-(n+2)}} \tag{2.23}$$

for all $x,z\in\mathbb{R}^n$ and $t>0$ where $C=C(n,\alpha,\beta)>0$.

Proof. When $\beta=0$, Lemma 2.7 follows directly from Lemma 2.2. Hence we can assume $\beta\in(0,n+2)$. Under the change of variables

$$x-z=\sqrt{t}\xi,\quad y-z=\sqrt{t}\eta,\quad s=t\zeta$$

we see that the left side of (2.23) equals

$$\begin{aligned}&\iint_{\mathbb{R}^n\times(0,1)} \Phi(\sqrt{t}(\xi-\eta),t(1-\zeta))^{\alpha/n}\Phi(\sqrt{t}\eta,t\zeta)^{\beta/n}\sqrt{t}^{\,n+2}\,d\eta\,d\zeta\\ &=\iint_{\mathbb{R}^n\times(0,1)}\left(\frac{1}{(4\pi t(1-\zeta))^{n/2}}\right)^{\alpha/n}\left(\frac{1}{(4\pi t\zeta)^{n/2}}\right)^{\beta/n} e^{-\frac{\alpha}{n}\frac{|\xi-\eta|^2}{4(1-\zeta)}-\frac{\beta}{n}\frac{|\eta|^2}{4\zeta}}\sqrt{t}^{\,n+2}\,d\eta\,d\zeta\\ &=\frac{C(n,\alpha,\beta)}{\sqrt{t}^{\,\alpha+\beta-(n+2)}}\int_0^1\frac{1}{(1-\zeta)^{\alpha/2}\zeta^{\beta/2}}\left(\int_{\mathbb{R}^n} e^{-\frac{\alpha}{n}\frac{|\xi-\eta|^2}{4(1-\zeta)}-\frac{\beta}{n}\frac{|\eta|^2}{4\zeta}}\,d\eta\right)d\zeta\\ &\le\frac{C(n,\alpha,\beta)}{\sqrt{t}^{\,\alpha+\beta-(n+2)}}\left[\int_0^{1/2}\frac{d\zeta}{\zeta^{\beta/2-n/2}}+\int_{1/2}^1\frac{d\zeta}{(1-\zeta)^{\alpha/2-n/2}}\right]\end{aligned}$$

by Lemma 2.2. □

Lemma 2.8. *Suppose $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$ and $r>0$. If*

$$(x,t)\in\overline{\mathcal{P}_r(x_0,t_0)} \quad \text{and} \quad (y,s)\in(\mathbb{R}^n\times\mathbb{R})\setminus\overline{\mathcal{P}_{2r}(x_0,t_0)}$$

then

$$\Phi(x-y,t-s)\le\frac{C(n)}{r^{n/2}}.$$

Proof. We consider three cases.
Case I. Suppose $t_0 - 2r \leq s < t$. Then $|x-y| \geq (\sqrt{2}-1)\sqrt{r}$ and hence

$$\begin{aligned}\Phi(x-y,t-s) &\leq \frac{e^{-\frac{(\sqrt{2}-1)^2 r}{4(t-s)}}}{(4\pi(t-s))^{n/2}} \leq \sup_{\tau>0} \frac{e^{-\frac{(\sqrt{2}-1)^2 r}{4\tau}}}{(4\pi\tau)^{n/2}} \\ &= \sup_{\zeta>0} \frac{e^{-(\sqrt{2}-1)^2\zeta}}{(\pi r/\zeta)^{n/2}} = \frac{C(n)}{r^{n/2}}.\end{aligned}$$

Case II. Suppose $s < t_0 - 2r$. Then $t - s \geq r$ and hence

$$\Phi(x-y,t-s) \leq \frac{1}{(4\pi r)^{n/2}} = \frac{C(n)}{r^{n/2}}.$$

Case III. Suppose $s \geq t$. Then $\Phi(x-y,t-s) = 0$. □

Lemma 2.9. *Suppose $\alpha > 0$ and T are constants. Then for $s < t \leq T$ and $|x| \leq \sqrt{T-t}$ we have*

$$\int_{|y|<\sqrt{T-s}} \Phi(x-y,t-s)^{\alpha/n} dy \geq \frac{C}{(t-s)^{(\alpha-n)/2}}$$

where $C = C(n,\alpha)$ is a positive constant.

Proof. Making the change of variables $z = \frac{x-y}{\sqrt{t-s}}$ and letting $e_1 = (1,0,...,0)$ we get

$$\begin{aligned}\int_{|y|<\sqrt{T-s}} \Phi(x-y,t-s)^{\alpha/n} dy &= \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{(t-s)^{\alpha/2}} \int_{|y|<\sqrt{T-s}} e^{-\frac{\alpha|x-y|^2}{4n(t-s)}} dy \\ &= \frac{1}{(4\pi)^{\alpha/2}(t-s)^{(\alpha-n)/2}} \int_{|z-\frac{x}{\sqrt{t-s}}|<\frac{\sqrt{T-s}}{\sqrt{t-s}}} e^{-\frac{\alpha}{4n}|z|^2} dz \qquad (2.24) \\ &\geq \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{(t-s)^{(\alpha-n)/2}} \int_{|z-\frac{\sqrt{T-s}}{\sqrt{t-s}}e_1|<\frac{\sqrt{T-s}}{\sqrt{t-s}}} e^{-\frac{\alpha}{4n}|z|^2} dz \qquad (2.25) \\ &\geq \frac{1}{(4\pi)^{\alpha/2}} \frac{1}{(t-s)^{(\alpha-n)/2}} \int_{|z-e_1|<1} e^{-\frac{\alpha}{4n}|z|^2} dz, \qquad (2.26)\end{aligned}$$

where the last two inequalities need some explanation. Since $|x| \leq \sqrt{T-t} < \sqrt{T-s}$, the center of the ball of integration in (2.24) is closer to the origin than the center of the ball of integration in (2.25). Thus, since the integrand is a decreasing function of $|z|$, we obtain (2.25). Since $\sqrt{T-s} \geq \sqrt{t-s}$, the ball of integration in (2.25) contains the ball of integration in (2.26) and hence (2.26) holds. □

3. The case $0 < \lambda < \frac{n+2-\alpha}{n}$

In this section we prove Theorems 1.1, 1.3, and 1.4 when $0 < \lambda < (n+2-\alpha)/n$. For these values of λ, Remark 2.1 and the following theorem imply Theorems 1.1 and 1.4.

Theorem 3.1. *Suppose u is a nonnegative solution of* (2.10),(2.11) *for some constants $\alpha \in (0, n+2)$,*

$$0 < \lambda < \frac{n+2-\alpha}{n} \quad \text{and} \quad 0 \leq \sigma \leq \frac{n+2}{n}. \qquad (3.1)$$

Then

$$\max_{|x|\leq 1} u(x,t) = O(t^{-n/2}) \quad \text{as } t \to 0^+. \qquad (3.2)$$

Proof. Let $v = u + 1$. Then by Lemma 2.3 we have (2.12)–(2.15) hold. To prove (3.2), it clearly suffices to prove

$$\max_{|x|\le 1} v(x,t) = O(t^{-n/2}) \quad \text{as } t \to 0^+. \tag{3.3}$$

Choose $\varepsilon \in (0,1)$ such that

$$\lambda < \frac{n+2-\alpha}{n+\varepsilon}. \tag{3.4}$$

By (2.14),

$$v^\lambda \in L^{\frac{n+2}{(n+\varepsilon)\lambda}}(\mathcal{P}_8(0,8)).$$

Thus, since (3.4) implies

$$\frac{\lambda(n+\varepsilon)}{n+2} < \frac{n+2-\alpha}{n+2}$$

we have by Theorem B.2 (with α replaced with $n+2-\alpha$) that

$$\Phi^{\alpha/n} * v^\lambda \in L^\infty(\mathcal{P}_8(0,8))$$

where the convolution operation is in $\mathcal{P}_8(0,8)$. Hence by (2.12) and (2.13), v is a $C^{2,1}$ positive solution of

$$0 \le Hv \le Cv^\sigma \quad \text{in } B_2(0) \times (0,8).$$

Thus by $(3.1)_2$ and [19, Theorem 1.1], v satisfies (3.3). □

The following theorem implies Theorem 1.3 when $0 < \lambda < \frac{n+2}{n}$.

Theorem 3.2. *Suppose α, λ, and σ are constants satisfying*

$$\alpha \in (0, n+2), \quad 0 < \lambda < \frac{n+2}{n}, \quad \text{and} \quad \sigma > \frac{2(n+2)-\alpha}{n} - \lambda. \tag{3.5}$$

Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying

$$\lim_{t\to 0^+} \varphi(t) = \infty.$$

Then there exists a positive function

$$u \in C^\infty(\mathbb{R}^n \times (0,1)) \cap L^\lambda(\mathbb{R}^n \times (0,1)) \tag{3.6}$$

satisfying

$$0 \le Hu \le (\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } \mathbb{R}^n \times (0,1), \tag{3.7}$$

where $*$ is the convolution operation in $\mathbb{R}^n \times (0,1)$, such that

$$u(0,t) \ne O(\varphi(t)) \quad \text{as } t \to 0^+. \tag{3.8}$$

Proof. By scaling u and noting by (3.5) that $\sigma + \lambda \ne 1$ we see that it suffices to prove Theorem 3.2 with (3.7) replaced with the weaker statement that there exists a positive constant $C = C(n,\lambda,\sigma,\alpha)$ such that u satisfies

$$0 \le Hu \le C(\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } \mathbb{R}^n \times (0,1) \tag{3.9}$$

where $*$ is the convolution operation in $\mathbb{R}^n \times (0,1)$.

By (3.5) there exists $\varepsilon = \varepsilon(n,\lambda,\sigma,\alpha) \in (0,1)$ such that

$$\sigma > \frac{2(n+2)-\alpha}{n-\varepsilon} - \lambda. \tag{3.10}$$

Let

$$p = \frac{n-\varepsilon}{2} \tag{3.11}$$

and let $\{T_j\} \subset (0,1)$ be a sequence that $T_j \to 0$ as $j \to \infty$. Define $w_j : (-\infty, T_j) \to (0,\infty)$ by

$$w_j(t) = (T_j - t)^{-p} \tag{3.12}$$

and define $t_j \in (0, T_j)$ by

$$w_j(t_j) = t_j^{-n/2}. \tag{3.13}$$

Then

$$\frac{T_j - t_j}{t_j} = \frac{w_j(t_j)^{-1/p}}{t_j} = t_j^{n/(2p)-1} \to 0 \quad \text{as } j \to \infty \tag{3.14}$$

by (3.11).

Choose $a_j \in ((t_j + T_j)/2, T_j)$ such that $w_j(a_j) > j\varphi(a_j)$. Then

$$\frac{w_j(a_j)}{\varphi(a_j)} \to \infty \quad \text{as } j \to \infty. \tag{3.15}$$

Let $h_j(s) = \sqrt{a_j - s}$ and $H_j(s) = \sqrt{a_j + \varepsilon_j - s}$ where $\varepsilon_j > 0$ satisfies

$$a_j + 2\varepsilon_j < T_j, \quad t_j - \varepsilon_j > t_j/2, \quad \varepsilon_j < T_j^2, \quad \text{and} \quad w_j(t_j - \varepsilon_j) > \frac{w_j(t_j)}{2}. \tag{3.16}$$

Define

$$\omega_j = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y| < h_j(s) \quad \text{and} \quad t_j < s < a_j\}$$
$$\Omega_j = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y| < H_j(s) \quad \text{and} \quad t_j - \varepsilon_j < s < a_j + \varepsilon_j\}.$$

By taking a subsequence we can assume the sets Ω_j are pairwise disjoint.

Let $\chi_j : \mathbb{R}^n \times \mathbb{R} \to [0,1]$ be a C^∞ function such that $\chi_j \equiv 1$ in w_j and $\chi_j \equiv 0$ in $\mathbb{R}^n \times \mathbb{R} \backslash \Omega_j$. Define f_j, $u_j : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ by

$$f_j(y,s) = \chi_j(y,s) w_j'(s) \tag{3.17}$$

and

$$u_j(x,t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x-y, t-s) f_j(y,s)\, dy\, ds. \tag{3.18}$$

Then f_j and u_j are C^∞ and

$$Hu_j = f_j \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{3.19}$$

By Theorem B.2 with $p = n+2$ and $q = \infty$ we see that

$$\begin{aligned}
&\|\iint_{\Omega_j \backslash \omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds\|_{L^\infty(\mathbb{R}^n \times (0,1))} \\
&\qquad = \|\iint_{\mathbb{R}^n \times (0,1)} \Phi(x-y, t-s) \chi_{\Omega_j \backslash \omega_j}(y,s) w_j'(s)\, dy\, ds\|_{L^\infty(\mathbb{R}^n \times (0,1))} \\
&\qquad \le C_n \|w_j'(s)\|_{L^{n+2}(\Omega_j \backslash \omega_j)} \\
&\qquad \le w_j(t_j)
\end{aligned} \tag{3.20}$$

provided we decrease ε_j if necessary because $|\Omega_j \backslash \omega_j| \to 0$ as $\varepsilon_j \to 0$.

Also, for $(x,t) \in \Omega_j$ we have $|x| < \sqrt{T_j - t_j}$ by $(3.16)_1$, and thus using $(3.16)_2$ we obtain

$$\sup_{(x,t) \in \Omega_j} \frac{|x|^2}{t} \le \frac{T_j - t_j}{t_j - \varepsilon_j} \le \frac{2(T_j - t_j)}{t_j} \to 0 \quad \text{as } j \to \infty$$

by (3.14). Hence by (3.14), $(3.16)_2$, and (3.13) there exists a positive number M, independent of j, such that

$$M\Phi(x,t) \ge 2/t_j^{n/2} = 2w_j(t_j) \quad \text{for } (x,t) \in \Omega_j. \tag{3.21}$$

In order to obtain a lower bound for u_j in Ω_j, note first that for $s < t \le a_j + \varepsilon_j$ and $|x| \le H_j(t)$ we have by Lemma 2.9 that

$$\int_{|y| < H_j(s)} \Phi(x-y, t-s)\, dy \ge \beta \tag{3.22}$$

for some constant

$$\beta = \beta(n) \in (0,1). \tag{3.23}$$

Next using (3.22) and (3.23), we find for $(x,t) \in \Omega_j$ that

$$\begin{aligned}\iint_{\Omega_j} \Phi(x-y,t-s)w_j'(s)\,dy\,ds &= \int_{t_j-\varepsilon_j}^{t} w_j'(s)\left(\int_{|y|<H_j(s)} \Phi(x-y,t-s)dy\right)ds \\ &\geq \beta(w_j(t) - w_j(t_j-\varepsilon_j)) \\ &\geq \beta w_j(t) - w_j(t_j).\end{aligned}$$

It therefore follows from (3.17), (3.18), and (3.20) that for $(x,t) \in \Omega_j$ we have

$$\begin{aligned}u_j(x,t) &\geq \iint_{\omega_j} \Phi(x-y,t-s)w_j'(s)\,dy\,ds \\ &= \iint_{\Omega_j} \Phi(x-y,t-s)w_j'(s)\,dy\,ds - \iint_{\Omega_j\setminus\omega_j} \Phi(x-y,t-s)w_j'(s)\,dy\,ds \\ &\geq \beta w_j(t) - 2w_j(t_j).\end{aligned} \tag{3.24}$$

Also by (3.17), (3.12), and (3.16) we obtain

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}} f_j(y,s)\,dy\,ds &\leq \iint_{\Omega_j} w_j'(s)\,dy\,ds \\ &\leq p\int_0^{T_j} (T_j-s)^{-(p+1)}\left(\int_{|y|<\sqrt{T_j-s}} dy\right)ds \\ &= p|B_1(0)|\int_0^{T_j} (T_j-s)^{n/2-p-1}ds \\ &= p|B_1(0)|\int_0^{T_j} \tau^{n/2-p-1}d\tau \to 0 \quad \text{as } j\to\infty\end{aligned} \tag{3.25}$$

by (3.11). Hence for $1 \leq \lambda < (n+2)/n$ it follows from (3.18) and Theorem B.2 that

$$\|u_j\|_{L^\lambda(\mathbb{R}^n\times(0,1))} \to 0 \quad \text{as } j\to\infty. \tag{3.26}$$

We next prove (3.26) when

$$0 < \lambda < 1. \tag{3.27}$$

(Theorem B.2 cannot be directly used in this case.) Choose $z_0 > 1$ such that the expression $z^{n/2}e^{-z/4}$ is decreasing on the interval $z_0 \leq z < \infty$. Let $r_0 = \sqrt{z_0}+1$. Then $r_0 > 2$ and by (3.17) and (3.18) we have

$$\begin{aligned}\iint_{\mathbb{R}^n\times(0,1)} u_j(x,t)^\lambda\,dx\,dt &= \iint_{\mathbb{R}^n\times(0,1)}\left(\iint_{\mathbb{R}^n\times(0,1)} \Phi(x-y,t-s)f_j(y,s)\,dy\,ds\right)^\lambda dx\,dt \\ &= I_j + J_j\end{aligned} \tag{3.28}$$

where

$$I_j := \iint_{B_{r_0}(0)\times(0,1)}\left(\iint_{\mathbb{R}^n\times(0,1)} \Phi(x-y,t-s)f_j(y,s)\,dy\,ds\right)^\lambda dx\,dt$$

and

$$J_j := \iint_{(\mathbb{R}^n\setminus B_{r_0}(0))\times(0,1)}\left(\iint_{\Omega_j} \Phi(x-y,t-s)f_j(y,s)\,dy\,ds\right)^\lambda dx\,dt.$$

By (3.27) and Hölder's inequality

$$I_j \le \left(\iint_{B_{r_0}(0)\times(0,1)} dx\,dt\right)^{1-\lambda}\left(\iint_{B_{r_0}(0)\times(0,1)}\left(\iint_{\mathbb{R}^n\times(0,1)}\Phi(x-y,t-s)f_j(y,s)\,dy\,ds\right)dx\,dt\right)^{\lambda}$$
$$\to 0 \quad \text{as } j\to\infty \tag{3.29}$$

by (3.25) and Theorem B.2 with $p=q=1$. Also

$$J_j \le \iint_{(\mathbb{R}^n\setminus B_{r_0}(0))\times(0,1)} A_j(x,t)^\lambda\,dx\,dt\|f_j\|^\lambda_{L^1(\Omega_j)} \tag{3.30}$$

where

$$A_j(x,t) := \max_{(y,s)\in\Omega_j,\, s<t}\Phi(x-y,t-s).$$

For $s<t$, $(y,s)\in\Omega_j$ and $(x,t)\in(\mathbb{R}^n\setminus B_{r_0}(0))\times(0,1)$ we have $0<s<t<1$ and

$$|x-y|>|x|-|y|>|x|-1.$$

Thus

$$(4\pi)^{n/2}\Phi(x-y,t-s)\le \frac{1}{(t-s)^{n/2}}e^{-\frac{(|x|-1)^2}{4(t-s)}}$$
$$= \frac{1}{(|x|-1)^n}\left(\frac{(|x|-1)^2}{t-s}\right)^{n/2}e^{-\frac{(|x|-1)^2}{4(t-s)}}. \tag{3.31}$$

Since $|x|\ge r_0$ and $0<s<t<1$ we have

$$\frac{(|x|-1)^2}{t-s} > (|x|-1)^2\ge z_0$$

and thus by the definition of z_0 we obtain from (3.31) that

$$(4\pi)^{n/2}\Phi(x-y,t-s)\le \frac{1}{(|x|-1)^n}((|x|-1)^2)^{n/2}e^{-(|x|-1)^2/4}$$
$$= e^{-(|x|-1)^2/4}.$$

Hence

$$A_j(x,t)^\lambda \le e^{-\lambda(|x|-1)^2/4} \quad \text{for } (x,t)\in(\mathbb{R}^n\setminus B_{r_0}(0))\times(0,1).$$

It therefore follows from (3.30) and (3.25) that $J_j\to 0$ as $j\to\infty$ which together with (3.29) and (3.28) yields (3.26) when λ satisfies (3.27).

By (3.25) we find that

$$\iint_{\mathbb{R}^n\times\mathbb{R}}\sum_{j=1}^\infty f_j(y,s)\,dy\,ds<\infty$$

provided we take a subsequence if necessary. Hence, since the C^∞ functions f_j have disjoint supports, we see that the function $u:(\mathbb{R}^n\times\mathbb{R})\setminus\{(0,0)\}\to[0,\infty)$ defined by

$$u(x,t)=(M+1)\Phi(x,t)+\sum_{j=1}^\infty u_j(x,t) \tag{3.32}$$

is C^∞ and by (3.18) we have

$$Hu=\sum_{j=1}^\infty f_j \quad \text{in } (\mathbb{R}^n\times\mathbb{R})\setminus\{(0,0)\} \tag{3.33}$$
$$u\equiv 0 \quad \text{in } \mathbb{R}^n\times(-\infty,0).$$

From (3.26) we have

$$u \in L^\lambda(\mathbb{R}^n \times (0,1))$$

provided we take a subsequence of u_j if necessary. Thus (3.6) holds.

We now prove (3.9). By (3.33) and (3.17) we have $Hu \equiv 0$ in $(\mathbb{R}^n \times (0,1)) \backslash \bigcup_{j=1}^\infty \Omega_j$. Hence to prove (3.9), it suffice to prove there exists a positve constant $C = C(n,\lambda,\sigma,\alpha)$ such that

$$0 \le Hu \le C(\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } \Omega_j \tag{3.34}$$

for $j = 1, 2, ...$

By (3.32), (3.24), and (3.21) we have for $(x,t) \in \Omega_j$ that

$$\begin{aligned} u(x,t) &\ge (M+1)\Phi(x,t) + \beta w_j(t) - 2w_j(t_j) \\ &\ge \Phi(x,t) + \beta w_j(t). \end{aligned} \tag{3.35}$$

Thus for $(x,t) \in \Omega_j$ we see by (3.33), (3.17), and (3.12) that

$$\begin{aligned} Hu(x,t) &= f_j(x,t) \le w_j'(t) = p w_j(t)^{1+1/p} \\ &= p w_j(t)^{1+1/p-\sigma} w_j(t)^\sigma \le \frac{p}{\beta^\sigma} w_j(t)^{1+1/p-\sigma} u(x,t)^\sigma. \end{aligned}$$

Hence to prove (3.34) it suffices to show

$$w_j(t)^{1+1/p-\sigma} < C \iint_{\mathbb{R}^n \times (0,1)} \Phi(x-y,t-s)^{\alpha/n} u(y,s)^\lambda \, dy \, ds \quad \text{for } (x,t) \in \Omega_j. \tag{3.36}$$

Our proof of (3.36) consists of two cases.

Case I. Suppose

$$(x,t) \in \Omega_j \quad \text{and} \quad t \le \frac{T_j + t_j}{2}. \tag{3.37}$$

Then using (3.16), (3.11), and the fact that w_j is an increasing function we have

$$\frac{1}{2} \le \frac{w_j(t)}{2w_j(t_j - \varepsilon_j)} \le \frac{w_j(t)}{w_j(t_j)} \le \left(\frac{T_j - \frac{T_j+t_j}{2}}{T_j - t_j} \right)^{-p} = 2^p < 2^{n/2}.$$

Also by (3.13) and (3.14)

$$\frac{w_j(t_j)}{T_j^{-n/2}} = \left(\frac{T_j}{t_j} \right)^{n/2} \in (1,2)$$

provided we take a subsequence if necessary. Thus (3.37) implies

$$\frac{1}{2} < \frac{w_j(t)}{T_j^{-n/2}} < 2^{(n+2)/2}. \tag{3.38}$$

Next, making the change of variables

$$x = \sqrt{T_j}\xi, \quad t = T_j\tau, \quad \text{and} \quad y = \sqrt{T_j}\eta, \quad s = T_j\zeta,$$

we get

$$\begin{aligned} &\iint_{\mathbb{R}^n \times (0,1)} \Phi(x-y,t-s)^{\alpha/n} \Phi(y,s)^\lambda \, dy \, ds \\ &= \iint_{\mathbb{R}^n \times (0,\tau)} \frac{1}{T_j^{\frac{n}{2}\frac{\alpha}{n}}} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n} \frac{1}{T_j^{n\lambda/2}} \Phi(\eta,\zeta)^\lambda \sqrt{T_j}^{n+2} \, d\eta \, d\zeta \\ &\ge \frac{G(\xi,\tau)}{\sqrt{T_j}^{\alpha+n\lambda-(n+2)}} \end{aligned} \tag{3.39}$$

where

$$G(\xi,\tau) := \iint_{B_1(0)\times(1/2,\tau)} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n}\Phi(\eta,\zeta)^\lambda\,d\eta\,d\zeta.$$

Since by $(3.37)_1$, $(3.16)_1$, (3.14), and $(3.16)_3$,

$$1>\tau=\frac{t}{T_j}\geq\frac{t_j-\varepsilon_j}{T_j}\to 1 \quad \text{as } j\to\infty$$

we have by $(3.37)_1$ that

$$|\xi|=\frac{|x|}{\sqrt{T_j}}<\frac{\sqrt{T_j-t}}{\sqrt{T_j}}=\sqrt{1-\frac{t}{T_j}}\to 0 \quad \text{as } j\to\infty.$$

Thus, since G is clearly continuous at $(\xi,\tau)=(0,1)$ and $G(0,1)>0$ we have by (3.39) that

$$\iint_{\mathbb{R}^n\times(0,1)} \Phi(x-y,t-s)^{\alpha/n}\Phi(y,s)^\lambda\,dy\,ds \geq \frac{C}{\sqrt{T_j}^{\alpha+n\lambda-(n+2)}} \quad \text{for } (x,t)\in\Omega_j, \tag{3.40}$$

where $C:=G(0,1)/2>0$, provided we take a subsequence if necessary.

Since by (3.10) and (3.11),

$$\begin{aligned}\sigma &> \frac{n+2}{n-\varepsilon}+\frac{n+2-\alpha}{n-\varepsilon}-\lambda\\ &> \frac{n+2-\varepsilon}{n-\varepsilon}+\frac{n+2-\alpha}{n}-\lambda\\ &= \frac{1}{p}+1+\frac{n+2-\alpha}{n}-\lambda\end{aligned}$$

we have

$$\begin{aligned}\left(\frac{1}{p}+1-\sigma\right)\frac{n}{2} &< \frac{n}{2}\left(\lambda-\frac{n+2-\alpha}{n}\right)\\ &= \frac{\alpha+n\lambda-(n+2)}{2}.\end{aligned}$$

Thus (3.36) follows from (3.35), (3.38), and (3.40).

Case II. Suppose

$$(x,t)\in\Omega_j \quad \text{and} \quad t\geq\frac{T_j+t_j}{2}. \tag{3.41}$$

Then for $s<t$ we have by Lemma 2.9 with $T=a_j+\varepsilon_j$ that

$$\int_{|y|<H_j(s)} \Phi(x-y,t-s)^{\alpha/n}dy \geq \frac{C}{(t-s)^{(\alpha-n)/2}}$$

for some positive constant $C=C(n,\alpha)$. Thus for (x,t) satisfying (3.41) we get

$$\begin{aligned}\iint_{\Omega_j} \Phi(x-y,t-s)^{\alpha/n}w_j(s)^\lambda\,dy\,ds &\geq \int_{t_j}^t w_j(s)^\lambda\left(\int_{|y|<H_j(s)}\Phi(x-y,t-s)^{\alpha/n}dy\right)ds\\ &\geq C\int_{t_j}^t\frac{ds}{(t-s)^a(T_j-s)^b} \quad \text{where } a=(\alpha-n)/2 \text{ and } b=\lambda p\\ &= \frac{C}{(T_j-t)^{a+b-1}}\int_1^{\frac{T_j-t_j}{T_j-t}}\frac{dz}{(z-1)^a z^b} \text{ under the change of variables } T_j-s=(T_j-t)z\\ &\geq \frac{C}{(T_j-t)^{a+b-1}}\int_1^2\frac{dz}{(z-1)^a z^b}=\frac{C}{(T_j-t)^{(\alpha-n)/2+\lambda p-1}}.\end{aligned} \tag{3.42}$$

Since by (3.10) and (3.11)

$$p(\sigma+\lambda-1) > \frac{n-\varepsilon}{2}\left(\frac{2(n+2)-\alpha}{n-\varepsilon}-1\right)$$
$$= n+2-\frac{\alpha}{2}-\frac{n}{2}+\frac{\varepsilon}{2}$$
$$> \frac{n-\alpha}{2}+2,$$

we have

$$p(1+\frac{1}{p}-\sigma) < \frac{\alpha-n}{2}+\lambda p-1.$$

Thus (3.36) follows from (3.12), (3.35), and (3.42). This completes the proof of (3.36) in all cases. Hence (3.34) and (3.9) hold.

Finally (3.8) follows from (3.15) and (3.35) with $(x,t)=(0,a_j)$. □

4. The case $\frac{n+2-\alpha}{n} \le \lambda < \frac{n+2}{n}$

In this section we prove Theorem 1.1 when

$$\frac{n+2-\alpha}{n} \le \lambda < \frac{n+2}{n}. \tag{4.1}$$

(For these values of λ, Theorem 1.3 follows from Theorem 3.2 in the last section and Theorems 1.2 and 1.4 are vacuously true.)

For λ satisfying (4.1), Remark 2.1 and the following theorem imply Theorem 1.1.

Theorem 4.1. *Suppose u is a nonnegative solution of* (2.10),(2.11) *for some constants $\alpha\in(0,n+2)$,*

$$\frac{n+2-\alpha}{n} \le \lambda < \frac{n+2}{n} \quad \textit{and} \quad 0\le\sigma<\frac{2(n+2)-\alpha}{n}-\lambda. \tag{4.2}$$

Then

$$\max_{|x|\le 1} u(x,t) = O(t^{-n/2}) \quad \textit{as } t\to 0^+. \tag{4.3}$$

Proof. Let $v=u+1$. Then by Lemma 2.3 we have that (2.12)–(2.15) hold. To prove (4.3), it clearly suffices to prove

$$\max_{|x|\le 1} v(x,t) = O(t^{-n/2}) \quad \text{as } t\to 0^+. \tag{4.4}$$

Since increasing λ or σ increases the right side of the second inequality in $(2.13)_1$, we can assume instead of (4.2) that

$$\frac{n+2-\alpha}{n} < \lambda < \frac{n+2}{n}, \quad \sigma>0, \quad \text{and} \quad 1<\lambda+\sigma<\frac{2(n+2)-\alpha}{n}. \tag{4.5}$$

Since the increased value of λ is less than $\frac{n+2}{n}$, it follows from (2.14) that (2.12) still holds.

By (4.5) there exists $\varepsilon=\varepsilon(n,\lambda,\sigma,\alpha)\in(0,1)$ such that

$$\left(\frac{n+4-\alpha}{n+4-\alpha-\varepsilon}\right)\frac{n+2-\alpha}{n} < \lambda < \frac{n+2}{n+\varepsilon} \quad \text{and} \quad \lambda+\sigma<\frac{2(n+2)-\alpha}{n+\varepsilon} \tag{4.6}$$

which implies

$$\sigma < \frac{2(n+2)-\alpha}{n+\varepsilon}-\lambda < \frac{2(n+2)-\alpha}{n+\varepsilon}-\frac{n+2-\alpha}{n} < \frac{n+2}{n+\varepsilon}. \tag{4.7}$$

Suppose for contradition that (4.4) is false. Then there is a sequence $\{(x_j,t_j)\}\subset\overline{B_1(0)}\times(0,1)$ and $x_0\in\overline{B_1(0)}$ such that $(x_j,t_j)\to(x_0,0)$ as $j\to\infty$ and

$$\lim_{j\to\infty} t_j^{n/2} v(x_j,t_j) = \infty. \tag{4.8}$$

By Lemma 2.8 we have for $(x,t) \in \mathcal{P}_{t_j/4}(x_j,t_j)$ that

$$\iint_{\mathcal{P}_8(0,8)\setminus\mathcal{P}_{t_j/2}(x_j,t_j)} \Phi(x-y,t-s)Hv(y,s)\,dy\,ds \le \frac{C(n)}{t_j^{n/2}} \iint_{\mathcal{P}_8(0,8)} Hv(y,s)\,dy\,ds.$$

It therefore follows from (2.14) and Remark 2.2 that

$$v(x,t) \le C\left[\left(\frac{1}{\sqrt{t}}\right)^n + \iint_{\mathcal{P}_{t_j/2}(x_j,t_j))} \Phi(x-y,t-s)Hv(y,s)\,dy\,ds\right] \quad \text{for } (x,t)\in\mathcal{P}_{t_j/4}(x_j,t_j). \tag{4.9}$$

Substituting $x = x_j$ and $t = t_j$ in (4.9) and using (4.8) we find that

$$t_j^{n/2}\iint_{\mathcal{P}_{t_j/2}(x_j,t_j)} \Phi(x_j-y,t_j-s)Hv(y,s)\,dy\,ds \to \infty \quad \text{as } j\to\infty. \tag{4.10}$$

Also, by (2.14) we have

$$\iint_{\mathcal{P}_{t_j/2}(x_j,t_j)} Hv(y,s)\,dy\,ds \to 0 \quad \text{as } j\to\infty. \tag{4.11}$$

Defining

$$f_j(\eta,\zeta) = r_j^{\frac{n+2}{2}} Hv(x_j+\sqrt{r_j}\eta, t_j + r_j\zeta) \quad \text{where } r_j = t_j/8 \tag{4.12}$$

and making the change of variables

$$y = x_j + \sqrt{r_j}\eta, \quad s = t_j + r_j\zeta \tag{4.13}$$

in (4.11) and (4.10) we get

$$\iint_{\mathcal{P}_4(0,0)} f_j(\eta,\zeta)\,d\eta\,d\zeta \to 0 \quad \text{as } j\to\infty \tag{4.14}$$

and

$$\iint_{\mathcal{P}_4(0,0)} \Phi(-\eta,-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta \to \infty \quad \text{as } j\to\infty. \tag{4.15}$$

Let

$$N(y,s) = \iint_{\mathcal{P}_8(0,8)} \Phi(y-\bar{y},s-\bar{s})Hv(\bar{y},\bar{s})\,d\bar{y}\,d\bar{s}.$$

By (2.14) and Theorem B.2 we find that $N \in L^{\frac{n+2}{n+\varepsilon}}(\mathcal{P}_8(0,8))$ and thus $N^\lambda \in L^{\frac{n+2}{\lambda(n+\varepsilon)}}(\mathcal{P}_8(0,8))$. Hence by Hölder's inequality, (4.6), and Lemma 2.5 we have for $R\in(0,1]$ and $(x,t)\in\mathcal{P}_{Rt_j/8}(x_j,t_j)$ that

$$\iint_{\mathcal{P}_8(0,8)\setminus\mathcal{P}_{Rt_j/4}(x_j,t_j)} \Phi(x-y,t-s)^{\alpha/n}N(y,s)^\lambda\,dy\,ds$$

$$\le \|N^\lambda\|_{L^{\frac{n+2}{\lambda(n+\varepsilon)}}(\mathcal{P}_8(0,8))}\left(\iint_{\mathbb{R}^n\times\mathbb{R}\setminus\mathcal{P}_{Rt_j/4}(x_j,t_j)} \Phi(x-y,t-s)^{\frac{\alpha q}{n}}\,dy\,ds\right)^{1/q} \quad \text{where } \frac{\lambda(n+\varepsilon)}{n+2}+\frac{1}{q} = 1$$

$$\le C\left(\frac{1}{\sqrt{Rt_j}^{\alpha q-(n+2)}}\right)^{1/q} \tag{4.16}$$

$$= C\frac{1}{\sqrt{t_j}^{(n+\varepsilon)\lambda-(n+2-\alpha)}} \tag{4.17}$$

where $C > 0$ depends on R but not on j.

Since by (2.14) and Lemma 2.8 we have

$$N(y,s) \le C\left[\frac{1}{\sqrt{t_j}^{\,n}} + \iint_{(\bar y,\bar s)\in\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(y-\bar y, s-\bar s) Hv(\bar y,\bar s)\, d\bar y\, d\bar s\right] \quad \text{for } (y,s)\in P_{Rt_j/4}(x_j,t_j)$$

it follows from Lemma 2.6 that for $(x,t)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)$ we have

$$\begin{aligned}
&\iint_{(y,s)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)} \Phi(x-y,t-s)^{\alpha/n} N(y,s)^{\lambda}\, dy\, ds \\
&\qquad \le C\left[\frac{1}{\sqrt{t_j}^{\,n\lambda-(n+2-\alpha)}} + \iint_{(y,s)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)} \Phi(x-y,t-s)^{\alpha/n}\right. \\
&\qquad\quad \left.\times\left(\iint_{(\bar y,\bar s)\in\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(y-\bar y,s-\bar s) Hv(\bar y,\bar s)\, d\bar y\, d\bar s\right)^{\lambda} dy\, ds\right].
\end{aligned} \tag{4.18}$$

Also by Jensen's inequality, (4.5) and Lemma 2.7 we have for $x\in\mathbb{R}^n$, $t>0$, and $\lambda\ge 1$ that

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}} \Phi(x-y,t-s)^{\alpha/n}\left(\int_{|z|<\sqrt{8}} \Phi(y-z,s)\, d\mu(z)\right)^{\lambda} dy\, ds \\
&\qquad \le C\int_{|z|<\sqrt{8}}\left(\iint_{\mathbb{R}^n\times(0,t)} \Phi(x-y,t-s)^{\alpha/n}\Phi(y-z,s)^{\lambda n/n}\, dy\, ds\right) d\mu(z) \\
&\qquad \le \frac{C}{\sqrt{t}^{\,\alpha+\lambda n-(n+2)}}.
\end{aligned} \tag{4.19}$$

We claim that (4.19) also holds for $0<\lambda<1$. To see this, let $x\in\mathbb{R}^n$ and $t>0$ be fixed and define

$$f(y,s) = \Phi(x-y,t-s)^{\alpha/n} \quad \text{and} \quad g(y,s) = \int_{|z|<\sqrt{8}} \Phi(y-z,s)\, d\mu(z).$$

Then by Lemma 2.7 with $\beta=0$ and $\beta=n$ we have

$$\|f\|_1 := \iint_{\mathbb{R}^n\times(0,t)} \Phi(x-y,t-s)^{\alpha/n} dy\, ds \le C\sqrt{t}^{\,n+2-\alpha}$$

and

$$\iint_{\mathbb{R}^n\times(0,t)} fg\, dy\, ds = \int_{|z|<\sqrt{8}}\iint_{\mathbb{R}^n\times(0,t)} \Phi(x-y,t-s)^{\alpha/n}\Phi(y-z,s)\, dy\, ds\, d\mu(z) \le C\sqrt{t}^{\,2-\alpha},$$

respectively, where C depends on neither x nor t. Thus by Jensen's inequality we find for $(x,t)\in\mathbb{R}^n\times(0,\infty)$ and $0<\lambda<1$ that

$$\begin{aligned}
\iint_{\mathbb{R}^n\times\mathbb{R}} fg^{\lambda} dy\, ds &= \iint_{\mathbb{R}^n\times(0,t)} \left(g\|f\|_1^{1/\lambda}\right)^{\lambda} \frac{f}{\|f\|_1}\, dy\, ds \\
&\le \left(\iint_{\mathbb{R}^n\times(0,t)} g\|f\|_1^{1/\lambda}\frac{f}{\|f\|_1}\, dy\, ds\right)^{\lambda} \\
&= \|f\|_1^{1-\lambda}\left(\iint_{\mathbb{R}^n\times(0,t)} fg\, dy\, ds\right)^{\lambda} \le C\sqrt{t}^{\,n+2-\alpha-\lambda n}.
\end{aligned}$$

That is (4.19) also holds for $0<\lambda<1$.

It therefore follows from (2.15), (4.17), (4.18), and Lemma 2.7 that for $(x,t)\in\mathcal{P}_{Rt_j/8}(x_j,t_j)$ we have

$$\begin{aligned}
&\int_{(y,s)\in\mathcal{P}_8(0,8)}\Phi(x-y,t-s)^{\alpha/n}v(y,s)^\lambda\,dy\,ds\\
&\qquad\le C\left[\iint_{(y,s)\in\mathcal{P}_8(0,8)}\Phi(x-y,t-s)^{\alpha/n}\left(\int_{|z|<\sqrt{8}}\Phi(y-z,s)\,d\mu(z)\right)^\lambda dy\,ds\right.\\
&\qquad\quad\left.+\iint_{(y,s)\in\mathcal{P}_8(0,8)}\Phi(x-y,t-s)^{\alpha/n}(N(y,s)+1)^\lambda\,dy\,ds\right]\\
&\qquad\le C\left[\frac{1}{\sqrt{t_j}^{(n+\varepsilon)\lambda-(n+2-\alpha)}}+\iint_{(y,s)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)}\Phi(x-y,t-s)^{\alpha/n}\right.\\
&\qquad\quad\left.\times\left(\iint_{(\bar{y},\bar{s})\in\mathcal{P}_{Rt_j/2}(x_j,t_j)}\Phi(y-\bar{y},s-\bar{s})Hv(\bar{y},\bar{s})\,d\bar{y}\,d\bar{s}\right)^\lambda dy\,ds\right]
\end{aligned}$$

where $C>0$ depends on R but not on j.

Also, similar to the way (4.9) was derived, we obtain

$$v(x,t)\le C\left[\frac{1}{\sqrt{t_j}^n}+\iint_{\mathcal{P}_{Rt_j/4}(x_j,t_j)}\Phi(x-y,t-s)Hv(y,s)\,dy\,ds\right]\quad\text{for }(x,t)\in P_{Rt_j/8}(x_j,t_j).$$

We see therefore from (2.13) that for $(x,t)\in\mathcal{P}_{Rt_j/8}(x_j,t_j)$ and $R\in(0,1]$ we have

$$\begin{aligned}
Hv(x,t)\le C&\left[\frac{1}{\sqrt{t_j}^{(n+\varepsilon)\lambda-(n+2-\alpha)}}+\iint_{(y,s)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)}\Phi(x-y,t-s)^{\alpha/n}\right.\\
&\left.\times\left(\iint_{(\bar{y},\bar{s})\in\mathcal{P}_{Rt_j/2}(x_j,t_j)}\Phi(y-\bar{y},s-\bar{s})Hv(\bar{y},\bar{s})\,d\bar{y}\,d\bar{s}\right)^\lambda dy\,ds\right]\\
&\times\left[\frac{1}{\sqrt{t_j}^{n\sigma}}+\left(\iint_{\mathcal{P}_{Rt_j/2}(x_j,t_j)}\Phi(x-y,t-s)Hv(y,s)\,dy\,ds\right)^\sigma\right].
\end{aligned}$$

Hence under the change of variables (4.13),

$$x=x_j+\sqrt{r_j}\xi,\quad t=t_j+r_j\tau,$$

and

$$\bar{y}=x_j+\sqrt{r_j}\bar{\eta},\quad \bar{s}=t_j+r_j\bar{\zeta},$$

we obtain from (4.12) and (4.6) that

$$\begin{aligned}
&f_j(\xi,\tau)=r_j^{\frac{n+2}{2}}Hv(x,t)\le r_j^{\frac{(n+\varepsilon)(\lambda+\sigma)-(n+2-\alpha)}{2}}Hv(x,t)\\
&\le C\left[1+\iint_{(\eta,\zeta)\in\mathcal{P}_{4R}(0,0)}\Phi(\xi-\eta,\tau-\zeta)^{\alpha/n}\left(\iint_{(\bar{\eta},\bar{\zeta})\in\mathcal{P}_{4r}(0,0)}\Phi(\eta-\bar{\eta},\zeta-\bar{\zeta})f_j(\bar{\eta},\bar{\zeta})\,d\bar{\eta}\,d\bar{\zeta}\right)^\lambda d\eta\,d\zeta\right]\\
&\quad\times\left[1+\left(\iint_{\mathcal{P}_{4R}(0,0)}\Phi(\xi-\eta,\tau-\zeta)f_j(\eta,\zeta)\,d\eta\,d\zeta\right)^\sigma\right] \qquad (4.20)
\end{aligned}$$

for $(\xi,\tau)\in\mathcal{P}_R(0,0)$ and $R\in(0,1]$ where $C>0$ depends on R but not on j.

To complete the proof of Theorem 4.1 we will need the following lemma.

Lemma 4.1. *Suppose the sequence*

$$\{f_j\} \text{ is bounded in } L^p(\mathcal{P}_{4R}(0,0)) \tag{4.21}$$

for some constants $p \in [1, \frac{n+2}{2}]$ *and* $R \in (0,1]$. *Then there exists a positive constant* $C_0 = C_0(n,\lambda,\sigma,\alpha)$ *such that the sequence*

$$\{f_j\} \text{ is bounded in } L^q(\mathcal{P}_R(0,0)) \tag{4.22}$$

for some $q \in (p,\infty)$ *satisfying*

$$\frac{1}{p} - \frac{1}{q} \geq C_0. \tag{4.23}$$

Proof. For $R \in (0,1]$ we formally define operators N_R and I_R by

$$(N_R f)(\xi,\tau) = \iint_{\mathcal{P}_{4R}(0,0)} \Phi(\xi-\eta,\tau-\zeta) f(\eta,\zeta)\, d\eta\, d\zeta$$

and

$$(I_R f)(\xi,\tau) = \iint_{\mathcal{P}_{4R}(0,0)} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n} f(\eta,\zeta)\, d\eta\, d\zeta.$$

Define p_2 by

$$\frac{1}{p} - \frac{1}{p_2} = \frac{2-\varepsilon}{n+2} \tag{4.24}$$

where ε is as in (4.6). Then $p_2 \in (p,\infty)$ and thus by Theorem B.2 we have

$$\|(N_R f_j)^\lambda\|_{p_2/\lambda} = \|N_R f_j\|_{p_2}^\lambda \leq C\|f_j\|_p^\lambda \tag{4.25}$$

and

$$\|(N_R f_j)^\sigma\|_{p_2/\sigma} = \|N_R f_j\|_{p_2}^\sigma \leq C\|f_j\|_p^\sigma \tag{4.26}$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{P}_{4R}(0,0))}$. Since

$$\frac{1}{p_2} = \frac{1}{p} - \frac{2-\varepsilon}{n+2} \leq 1 - \frac{2-\varepsilon}{n+2} = \frac{n+\varepsilon}{n+2}$$

we see by (4.6) that

$$\frac{p_2}{\lambda} > 1. \tag{4.27}$$

Now there are two cases to consider.

Case I. Suppose

$$\frac{p_2}{\lambda} < \frac{n+2}{n+2-\alpha}. \tag{4.28}$$

Define p_3 and q by

$$\frac{\lambda}{p_2} - \frac{1}{p_3} = \frac{n+2-\alpha}{n+2} \tag{4.29}$$

and

$$\frac{1}{q} := \frac{1}{p_3} + \frac{\sigma}{p_2} = \frac{\lambda+\sigma}{p_2} - \frac{n+2-\alpha}{n+2}. \tag{4.30}$$

It follows from (4.27)–(4.30), (4.24), and (4.5) that

$$1 < \frac{p_2}{\lambda} < p_3 < \infty, \quad q > 0, \tag{4.31}$$

and

$$\begin{aligned}\frac{1}{p}-\frac{1}{q}&=\frac{1}{p}-\left((\lambda+\sigma)\left(\frac{1}{p}-\frac{2-\varepsilon}{n+2}\right)-\frac{n+2-\alpha}{n+2}\right)\\&=\frac{(2-\varepsilon)(\lambda+\sigma)+(n+2-\alpha)}{n+2}-\frac{\lambda+\sigma-1}{p}\\&\geq\frac{(2-\varepsilon)(\lambda+\sigma)+(n+2-\alpha)-(n+2)(\lambda+\sigma-1)}{n+2}\\&=\frac{2(n+2)-\alpha-(n+\varepsilon)(\lambda+\sigma)}{n+2}.\end{aligned}$$

Thus (4.23) holds by (4.6).

By (4.29), (4.31), (4.25), and Theorem B.1 we find that

$$\begin{aligned}\|(I_R((N_Rf_j)^\lambda))^q\|_{p_3/q}&=\|I_R((N_Rf_j)^\lambda)\|_{p_3}^q\\&\leq C\|(N_Rf_j)^\lambda\|_{p_2/\lambda}^q\\&\leq C\|f_j\|_p^{\lambda q}.\end{aligned}$$

Also by (4.26) we get

$$\|(N_Rf_j)^{\sigma q}\|_{\frac{p_2}{\sigma q}}=\|(N_Rf_j)^\sigma\|_{p_2/\sigma}^q\leq C\|f_j\|_p^{\sigma q}.$$

It therefore follows from (4.20), (4.30), Hölder's inequality, and (4.21) that (4.22) holds.

Case II. Suppose

$$\frac{p_2}{\lambda}\geq\frac{n+2}{n+2-\alpha}. \tag{4.32}$$

Then by Theorem B.2, (4.21), and (4.25) we find that the sequence

$$\{I_R((N_Rf_j)^\lambda)\}\text{ is bounded in }L^\gamma(\mathcal{P}_{4R}(0,0))\quad\text{for all }\gamma\in(1,\infty). \tag{4.33}$$

Let $\hat{q}=p_2/\sigma$. Then by (4.24),

$$\frac{1}{p}-\frac{1}{\hat{q}}=\frac{1}{p}-\frac{\sigma}{p_2}=\frac{2-\varepsilon}{n+2}+\frac{1-\sigma}{p_2}.$$

Thus for $\sigma\leq 1$ we have

$$\frac{1}{p}-\frac{1}{\hat{q}}\geq\frac{2-\varepsilon}{n+2}>0$$

and for $\sigma>1$ it follows from (4.32) and (4.6) that

$$\begin{aligned}\frac{1}{p}-\frac{1}{\hat{q}}&\geq\frac{2-\varepsilon}{n+2}-\frac{\sigma-1}{\frac{(n+2)\lambda}{n+2-\alpha}}\\&\geq\frac{2-\varepsilon}{n+2}-\frac{\frac{2(n+2)-\alpha}{n}-\lambda-1}{\frac{(n+2)\lambda}{n+2-\alpha}}\\&=\frac{n+4-\alpha-\varepsilon}{(n+2)\lambda}\left(\lambda-\left(\frac{n+4-\alpha}{n+4-\alpha-\varepsilon}\right)\frac{n+2-\alpha}{n}\right)>0.\end{aligned}$$

Thus defining $q\in(p,\hat{q})$ by

$$\frac{1}{q}=\frac{\frac{1}{p}+\frac{1}{\hat{q}}}{2}$$

we have for $\sigma>0$ that

$$\frac{1}{p}-\frac{1}{q}=\frac{1}{2}\left(\frac{1}{p}-\frac{1}{\hat{q}}\right)\geq C_0(n,\lambda,\sigma,\alpha)>0.$$

That is (4.23) holds.

Since $q\sigma/p_2 < \hat{q}\sigma/p_2 = 1$ there exists $\gamma \in (q, \infty)$ such that

$$\frac{q}{\gamma} + \frac{q\sigma}{p_2} = 1. \tag{4.34}$$

Also

$$\|(I_R((N_R f_j)^\lambda))^q\|_{\gamma/q} = \|I_R((N_R f_j)^\lambda)\|_\gamma^q$$

and by (4.26)

$$\|(N_R f_j)^{\sigma q}\|_{\frac{p_2}{\sigma q}} = \|(N_R f_j)^\sigma\|_{p_2/\sigma}^q \le C\|f_j\|_p^{\sigma q}.$$

It therefore follows from (4.20), (4.34), Hölder's inequality, (4.33), and (4.21) that (4.22) holds. □

We return now to the proof of Theorem 4.1. By (4.14) the sequence

$$\{f_j\} \text{ is bounded in } L^1(\mathcal{P}_4(0,0)). \tag{4.35}$$

Starting with this fact and iterating Lemma 4.1 a finite number of times (m times is enough if $m > 1/C_0$) we see that there exists $R_0 \in (0,1)$ such that the sequence

$$\{f_j\} \text{ is bounded in } L^p(\mathcal{P}_{4R_0}(0,0))$$

for some $p > (n+2)/2$. Hence by Theorem B.2 the sequence $\{N_{R_0} f_j\}$ is bounded in $L^\infty(\mathcal{P}_{4R_0}(0,0))$. Thus (4.20) implies the sequence

$$\{f_j\} \text{ is bounded in } L^\infty(\mathcal{P}_{R_0}(0,0)). \tag{4.36}$$

Since by Lemma 2.8,

$$\iint_{\mathcal{P}_4(0,0)} \Phi(-\eta,-\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta$$
$$\le \iint_{\mathcal{P}_{R_0}(0,0)} \Phi(-\eta,-\zeta) f_j(\eta,\zeta)\, d\eta\, d\zeta + \frac{C(n)}{R_0^{n/2}} \iint_{\mathcal{P}_4(0,0)\setminus\mathcal{P}_{R_0}(0,0)} f_j(\eta,\zeta)\, d\eta\, d\zeta$$

we see that (4.35) and (4.36) contradict (4.15). This contradiction completes the proof of Theorem 4.1. □

5. The case $\lambda \ge \frac{n+2}{n}$

In this section we prove Theorems 1.1–1.4 when $\lambda \ge \frac{n+2}{n}$. For these values of λ, Remark 2.1 and the following theorem imply Theorem 1.1.

Theorem 5.1. *Suppose u is a nonnegative solution of* (2.10),(2.11) *for some constants $\alpha \in (0, n+2)$,*

$$\lambda \ge \frac{n+2}{n} \quad \textit{and} \quad 0 \le \sigma < 1 - \frac{\alpha-2}{n+2}\lambda. \tag{5.1}$$

Then

$$\max_{|x|\le 1} u(x,t) = o(t^{-\frac{n+2}{2\lambda}}) \quad \textit{as } t \to 0^+. \tag{5.2}$$

Proof. Let $v = u+1$. Then by Lemma 2.3 we have that (2.12)–(2.15) hold. To prove (5.2) it clearly suffices to prove

$$\max_{|x|\le 1} v(x,t) = o(t^{-\frac{n+2}{2\lambda}}) \quad \text{as } t \to 0^+. \tag{5.3}$$

Since increasing σ increases the right side of the second inequality in $(2.13)_1$, we can assume instead of (5.1) that

$$\lambda \ge \frac{n+2}{n} \quad \text{and} \quad 0 < \sigma < 1 - \frac{\alpha-2}{n+2}\lambda \tag{5.4}$$

which implies

$$\frac{\sigma}{\lambda} < \frac{2-\alpha}{n+2} + \frac{1}{\lambda} \le \frac{2-\alpha}{n+2} + \frac{n}{n+2} = \frac{n+2-\alpha}{n+2}. \tag{5.5}$$

By (5.4) there exists $\varepsilon = \varepsilon(n, \lambda, \sigma, \alpha) \in (0, 1)$ such that

$$\alpha + \varepsilon < n + 2 \quad \text{and} \quad \sigma < 1 - \frac{\alpha + \varepsilon - 2}{n + 2}\lambda \tag{5.6}$$

which implies

$$\frac{\sigma - 1}{\lambda} < \frac{2 - \alpha - \varepsilon}{n + 2}. \tag{5.7}$$

Part of the proof of Theorem 5.1 will consist of two lemmas, the first of which is the following.

Lemma 5.1. *Suppose Ω is a bounded open subset of $\mathbb{R}^n \times \mathbb{R}$ and*

$$p \in \left[\lambda, \frac{(n+2)\lambda}{n + 2 - \alpha - \varepsilon}\right). \tag{5.8}$$

Then for all $w \in L^p(\Omega)$ we have

$$\left\| \left(\iint_\Omega \Phi(\cdot - y, \cdot - s)^{\alpha/n} w(y,s)^\lambda \, dy \, ds \right) w^\sigma \right\|_{L^{p_3}(\Omega)} \leq C \|w\|^{\lambda+\sigma}_{L^p(\Omega)} \tag{5.9}$$

where

$$\frac{1}{p_3} = \frac{\lambda + \sigma}{p} - \frac{n + 2 - \alpha - \varepsilon}{n + 2} \tag{5.10}$$

and $C = C(n, \lambda, \sigma, \alpha, \Omega, p)$ is a positive constant. Moreover,

$$p_3 > 1. \tag{5.11}$$

Proof. Define p_2 by

$$\frac{\lambda}{p} - \frac{1}{p_2} = \frac{n + 2 - \alpha - \varepsilon}{n + 2}.$$

Then by (5.8) and $(5.6)_1$, $1 \leq p/\lambda < p_2 < \infty$ and thus by Theorem B.2 we have, letting

$$I(f) = \iint_\Omega \Phi(\cdot - y, \cdot - s)^{\alpha/n} f(y, s) \, dy \, ds,$$

that

$$\|I(w^\lambda)\|_{L^{p_2}(\Omega)} \leq C \|w^\lambda\|_{L^{p/\lambda}(\Omega)} = C \|w\|^\lambda_{L^p(\Omega)}. \tag{5.12}$$

Since $\frac{1}{p_3} = \frac{1}{p_2} + \frac{\sigma}{p}$ we have by Hölder's inequality that

$$\begin{aligned}
\|I(w^\lambda) w^\sigma\|^{p_3}_{L^{p_3}(\Omega)} &= \|(I(w^\lambda) w^\sigma)^{p_3}\|_{L^1(\Omega)} \\
&\leq \|I(w^\lambda)^{p_3}\|_{L^{p_2/p_3}(\Omega)} \|w^{\sigma p_3}\|_{L^{\frac{p}{\sigma p_3}}(\Omega)} \\
&= \|I(w^\lambda)\|^{p_3}_{L^{p_2}(\Omega)} \|w\|^{\sigma p_3}_{L^p(\Omega)}.
\end{aligned}$$

Thus (5.9) follows from (5.12).

Also from (5.8) and (5.7) we find that

$$\frac{1}{p_3} \leq \frac{\lambda + \sigma}{\lambda} - \frac{n + 2 - \alpha - \varepsilon}{n + 2} = \frac{\sigma}{\lambda} + \frac{\alpha + \varepsilon}{n + 2} < \frac{1}{\lambda} + \frac{2}{n + 2}.$$

Thus (5.11) follows from $(5.4)_1$. □

We now continue with the proof of Theorem 5.1. Suppose for contradiction that (5.3) is false. Then there exists a sequence $\{(x_j, t_j)\} \subset \overline{B_1(0)} \times (0, 1/2)$ and $x_0 \in \overline{B_1(0)}$ such that

$$(x_j, t_j) \to (x_0, 0) \quad \text{as } j \to \infty \tag{5.13}$$

and

$$\liminf_{j \to \infty} t_j^{\frac{n+2}{2\lambda}} v(x_j, t_j) > 0. \tag{5.14}$$

Define $p_3 > 0$ by $\frac{1}{p_3} = \frac{\alpha+\varepsilon}{n+2} + \frac{\sigma}{\lambda}$. Then by (2.12), (2.13) and Lemma 5.1 with $\Omega = \mathcal{P}_8(0,8)$, $p = \lambda$, and $w = v$ we have $p_3 > 1$ and $Hv \in L^{p_3}(\mathcal{P}_4(0,4)$. Hence defining p_4 by $\frac{1}{p_3} + \frac{1}{p_4} = 1$, using Hölder's inequality, and making the change of variables

$$\begin{aligned} x &= x_j + \sqrt{Rt_j}\xi, \quad t = t_j + Rt_j\tau \\ y &= x_j + \sqrt{Rt_j}\eta, \quad s = t_j + Rt_j\zeta \end{aligned}$$

we see for $R \in (0,1]$ that

$$\begin{aligned} &\sup_{(x,t)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)} \iint_{\mathcal{P}_4(0,4)\setminus\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(x-y,t-s)Hv(y,s)\,ds \\ &\le \sup_{(x,t)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)} \left(\iint_{\mathcal{P}_4\setminus\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(x-y,t-s)^{p_4}\,dy\,ds\right)^{1/p_4} \|Hv\|_{L^{p_3}(\mathcal{P}_4(0,4))} \\ &\le C \sup_{(\xi,\tau)\in\mathcal{P}_{1/4}(0,0)} \left(\iint_{\mathbb{R}^n\times\mathbb{R}\setminus\mathcal{P}_{1/2}(0,0)} \left(\frac{1}{(Rt_j)^{n/2}}\right)^{p_4} \Phi(\xi-\eta,\tau-\zeta)^{p_4}(Rt_j)^{\frac{n+2}{2}}\,d\eta\,d\zeta\right)^{1/p_4} \\ &= C\left(\frac{1}{Rt_j}\right)^{(\frac{np_4}{2}-\frac{n+2}{2})\frac{1}{p_4}} \sup_{(\xi,\tau)\in\mathcal{P}_{1/4}(0,0)} \left(\iint_{\mathbb{R}^n\times\mathbb{R}\setminus\mathcal{P}_{1/2}(0,0)} \Phi(\xi-\eta,\tau-\zeta)^{p_4}\,d\eta\,d\zeta\right)^{1/p_4} \\ &= C\left(\frac{1}{Rt_j}\right)^{\frac{n+2}{2\lambda}(\sigma-\frac{2-\alpha-\varepsilon}{n+2}\lambda)} \end{aligned}$$

where C depends on neither R nor j and

$$\sigma - \frac{2-\alpha-\varepsilon}{n+2}\lambda < 1$$

by $(5.6)_2$.

Also, using (2.14), Lemma 2.8, and the fact that $\mathcal{P}_{t_j/4}(x_j,t_j) \subset \mathcal{P}_2(0,2)$ we see for $R \in (0,1]$ that

$$\begin{aligned} \sup_{(x,t)\in\mathcal{P}_{Rt_j/4}(x_j,t_j)} &\iint_{\mathcal{P}_8(0,8)\setminus\mathcal{P}_4(0,4)} \Phi(x-y,t-s)Hv(y,s)\,dy\,ds \\ &\le C(n)\iint_{\mathcal{P}_8(0,8)} Hv(y,s)\,dy\,ds < \infty. \end{aligned}$$

Thus by (2.12), (2.15) and Lemma 2.4 with $p = \lambda$, $\Omega \times (0,T) = B_2(0) \times (0,4)$, and $K = \overline{B_{3/2}(0)}$ we have for $(x,t) \in \mathcal{P}_{Rt_j/4}(x_j,t_j)$ and $R \in (0,1]$ that

$$v(x,t) \le \iint_{\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(x-y,t-s)Hv(y,s)\,dy\,ds + \frac{\varepsilon_j}{(Rt_j)^{\frac{n+2}{2\lambda}}} \tag{5.15}$$

for some sequence $\{\varepsilon_j\} \subset (0,1)$ which tends to zero as $j \to \infty$ and which depends in neither (x,t) nor R.

Also, for $(x,t) \in \mathcal{P}_{Rt_j/4}(x_j,t_j)$ and $R \in (0,1]$ we have by (2.12) and Lemma 2.8 that

$$\begin{aligned} \iint_{\mathcal{P}_8(0,8)\setminus\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(x-y,t-s)^{\alpha/n}v(y,s)^\lambda\,dy\,ds &\le \left(C(n)\left(\frac{4}{Rt_j}\right)^{n/2}\right)^{\alpha/n} \|v^\lambda\|_{L^1(\mathcal{P}_8(0,8))} \\ &= \frac{C}{(Rt_j)^{\alpha/2}} \end{aligned}$$

where C depends on neither (x,t), R, nor j. Thus for $(x,t) \in \mathcal{P}_{Rt_j/4}(x_j,t_j)$ and $R \in (0,1]$ we have by (2.13) that

$$0 \le Hv(x,t) \le C\left(\frac{1}{(Rt_j)^{\alpha/2}} + \iint_{\mathcal{P}_{Rt_j/2}(x_j,t_j)} \Phi(x-y,t-s)^{\alpha/n} v(y,s)^\lambda \, dy\, ds\right) v(x,t)^\sigma \tag{5.16}$$

where C depends on neither (x,t), R, nor j.

Next, making the change of variables

$$v(y,s) = t_j^{-\frac{n+2}{2\lambda}} v_j(\eta,\zeta),$$

$$x = x_j + \sqrt{t_j}\xi, \quad t = t_j + t_j\tau; \qquad y = x_j + \sqrt{t_j}\eta, \quad s = t_j + t_j\zeta,$$

we obtain

$$\iint_{\mathcal{P}_{1/2}(0,0)} v_j(\eta,\zeta)^\lambda \, d\eta\, d\zeta = \iint_{\mathcal{P}_{t_j/2}(x_j,t_j)} v(y,s)^\lambda \, dy\, ds \tag{5.17}$$

and from (5.15) and (5.16) we find for $(\xi,\tau) \in \mathcal{P}_{R/4}(0,0)$ and $R \in (0,1]$ that

$$\begin{aligned} v_j(\xi,\tau) &\le \iint_{\mathcal{P}_{R/2}(0,0)} \frac{1}{t_j^{n/2}} \Phi(\xi-\eta,\tau-\zeta) t_j^{-1} Hv_j(\eta,\zeta) t_j^{\frac{n+2}{2}} \, d\eta\, d\zeta + \frac{\varepsilon_j}{R^{\frac{n+2}{2\lambda}}} \\ &= \iint_{\mathcal{P}_{R/2}(0,0)} \Phi(\xi-\eta,\tau-\zeta) Hv_j(\eta,\zeta)\, d\eta\, d\zeta + \frac{\varepsilon_j}{R^{\frac{n+2}{2\lambda}}}, \end{aligned} \tag{5.18}$$

where $\varepsilon_j \to 0$ as $j \to \infty$ and ε_j depends on neither (ξ,τ) nor R, and

$$\begin{aligned} 0 \le Hv_j(\xi,\tau) &\le Ct_j^{\frac{n+2}{2\lambda}+1-\frac{\alpha}{2}} \left(\frac{1}{R^{\alpha/2}} + \iint_{\mathcal{P}_{R/2}(0,0)} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n} v_j(\eta,\zeta)^\lambda \, d\eta\, d\zeta\right) \\ &\quad \times \left(t_j^{-\frac{(n+2)\sigma}{2\lambda}} v_j(\xi,\eta)^\sigma\right) \\ &= C\hat{\varepsilon}_j \left(\frac{1}{R^{\alpha/2}} + \iint_{\mathcal{P}_{R/2}(0,0)} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n} v_j(\eta,\zeta)^\lambda \, d\eta\, d\zeta\right) v_j(\xi,\eta)^\sigma \end{aligned} \tag{5.19}$$

where C depends on neither (ξ,τ), R, nor j and

$$\hat{\varepsilon}_j := t_j^{-\frac{n+2}{2\lambda}(\sigma - 1 + \frac{\alpha-2}{n+2}\lambda)} \to 0 \quad \text{as } j \to \infty$$

by (5.1) and (5.13).

Also by (5.14) we have

$$\liminf_{j\to\infty} v_j(0,0) > 0. \tag{5.20}$$

To complete the proof of Theorem 5.1 we will require the following lemma.

Lemma 5.2. *Suppose the sequence*

$$\{v_j\} \text{ is bounded in } L^p(P_{R/2}(0,0)) \tag{5.21}$$

for some constants $R \in (0,1]$ and

$$p \in \left[\lambda, \frac{(n+2)\lambda}{n+2-\alpha-\varepsilon}\right). \tag{5.22}$$

Then either the sequence

$$\{v_j\} \text{ tends to zero in } L^{\frac{(n+2)\lambda}{n+2-\alpha-\varepsilon}}(P_{R/8}(0,0)) \tag{5.23}$$

or there exists a positive constant $C_0 = C_0(n,\lambda,\sigma,\alpha)$ such that the sequence

$$\{v_j\} \text{ tends to zero in } L^q(P_{R/8}(0,0)) \tag{5.24}$$

for some $q \in (p, \infty)$ *satisfying*

$$\frac{1}{p} - \frac{1}{q} \geq C_0. \tag{5.25}$$

Proof. It follows from (5.19), (5.21), (5.22), and Lemma 5.1 that the sequence

$$\{Hv_j\} \text{ tends to 0 in } L^{p_3}(P_{R/4}(0,0)) \tag{5.26}$$

where p_3, defined by (5.10), satisfies (5.11).

Case I. Suppose $p_3 \geq \frac{n+2}{2}$. Then by (5.26), (5.18), and Theorem B.2 we have the sequence

$$\{v_j\} \text{ tends to zero in } L^q(P_{R/8}(0,0)) \quad \text{for all } q > 1$$

which implies (5.23).

Case II. Suppose $p_3 < \frac{n+2}{2}$. Define q by

$$\frac{1}{p_3} - \frac{1}{q} = \frac{2}{n+2}. \tag{5.27}$$

Then by (5.11)

$$1 < p_3 < q < \infty.$$

Hence by (5.26), (5.18) and Theorem B.1 we have (5.24) holds.

Also by (5.27), (5.10), (5.22), and (5.7) we get

$$\begin{aligned}
\frac{1}{p} - \frac{1}{q} = \frac{1}{p} + \frac{2}{n+2} - \frac{1}{p_3} &= \frac{1}{p} + \frac{2}{n+2} - \frac{\sigma}{p} - \frac{\lambda}{p} + 1 - \frac{\alpha+\varepsilon}{n+2} \\
&= -\frac{\lambda+\sigma-1}{p} + 1 - \frac{\alpha+\varepsilon-2}{n+2} \\
&\geq \frac{1-(\lambda+\sigma)}{\lambda} + 1 + \frac{2-\alpha-\varepsilon}{n+2} > 0.
\end{aligned}$$

Thus (5.25) holds. □

We now return to the proof of Theorem 5.1. By (2.12) and (5.17), the sequence $\{v_j\}$ tends to zero in $L^\lambda(P_{1/2}(0,0))$. Starting with this fact on iterating Lemma 5.2 a finite number of times we see that the sequence

$$\{v_j\} \text{ tends to zero in } L^p(\mathcal{P}_{R/2}(0,0)) \tag{5.28}$$

for some $R \in (0,1)$ and for some

$$p > \frac{(n+2)\lambda}{n+2-\alpha}. \tag{5.29}$$

Hence the sequence $\{v_j^\lambda\}$ tends to zero in $L^{p/\lambda}(\mathcal{P}_{R/2}(0,0))$ and $\frac{p}{\lambda} > \frac{n+2}{n+2-\alpha}$. Thus by Theorem B.2, the sequence whose jth term is the integral on the right side of (5.19), tends to zero in $L^\infty(\mathcal{P}_{R/2}(0,0))$. So by (5.19)

$$0 \leq Hv_j < Cv_j^\sigma \quad \text{in } \mathcal{P}_{R/4}(0,0) \tag{5.30}$$

where C does not depend on j. Hence by (5.28) the sequence $\{Hv_j\}$ tends to zero in $L^{p/\sigma}(\mathcal{P}_{R/4}(0,0))$ and by (5.29) and (5.5)

$$\frac{p}{\sigma} > \frac{(n+2)\lambda}{(n+2-\alpha)\sigma} > \left(\frac{n+2}{n+2-\alpha}\right)^2 > 1.$$

Thus by (5.18) and Theorem B.2 the sequence

$$\{v_j\} \text{ tends to zero in } L^q(\mathcal{P}_{R/8}(0,0)) \text{ where } q = \begin{cases} \infty, & \text{if } \frac{p}{\sigma} \geq \frac{n+2}{2-\varepsilon} \\ \frac{1}{\frac{\sigma}{p} - \frac{2-\varepsilon}{n+2}}, & \text{if } \frac{p}{\sigma} < \frac{n+2}{2-\varepsilon}. \end{cases} \tag{5.31}$$

However the possibility that $q=\infty$ is ruled out by (5.20). Hence we can assume $\frac{p}{\sigma}<\frac{n+2}{2-\varepsilon}$. Then by (5.31),

$$\frac{1}{p}-\frac{1}{q}=\frac{1-\sigma}{p}+\frac{2-\varepsilon}{n+2}.$$

Thus, if $\sigma\in(0,1]$ then

$$\frac{1}{p}-\frac{1}{q}>\frac{1}{n+2}.$$

On the other hand, if $\sigma>1$ then by (5.29) and (5.7)

$$\begin{aligned}\frac{1}{p}-\frac{1}{q}&=\frac{2-\varepsilon}{n+2}-\frac{\sigma-1}{p}\\&>\frac{2-\varepsilon}{n+2}-\frac{\sigma-1}{\lambda}\frac{n+2-\alpha}{n+2}\\&>\frac{2-\varepsilon}{n+2}-\frac{2-\alpha-\varepsilon}{n+2}=\frac{\alpha}{n+2}.\end{aligned}$$

Thus for $\sigma>0$ we have

$$\frac{1}{p}-\frac{1}{q}>C(n,\alpha)>0.$$

Hence, after a finite number of iterations of the procedure of going from (5.28) to (5.31) we see that the sequence $\{v_j\}$ tends to zero in $L^\infty(\mathcal{P}_{\hat{R}}(0,0))$ for some $\hat{R}\in(0,R)$ which again contrdicts (5.20). This completes the proof of Theorem 5.1. □

The following theorem implies Theorem 1.2.

Theorem 5.2. *Suppose*

$$\lambda\geq\frac{n+2}{n}\quad\text{and}\quad\gamma=\frac{n+2-\varepsilon}{2\lambda}\tag{5.32}$$

for some $\varepsilon\in(0,1)$. *Then the function*

$$u(x,t)=\Psi(x,t):=\int_{\mathbb{R}^n}\Phi(x-y,t)|y|^{-2\gamma}dy\tag{5.33}$$

is a C^∞ *positive solution of*

$$Hu=0\quad\text{in }\mathbb{R}^n\times(0,\infty)\tag{5.34}$$

such that

$$u\in L^\lambda(\mathbb{R}^n\times(0,T))\quad\text{for all }T>0,\tag{5.35}$$

$$t^\gamma u(0,t)=u(0,1)\quad\text{for }0<t<\infty,\tag{5.36}$$

and

$$t^\gamma u(x,t)\text{ is bounded between positive constants}\tag{5.37}$$

on $\{(x,t)\in\mathbb{R}^n\times(0,t):|x|<\sqrt{t}\}$.

Proof. By (5.32) we have $2\gamma<n$. Thus (5.33) is a C^∞ positive solution of (5.34).

For $a>0$ and $(x,t)\in\mathbb{R}^n\times(0,\infty)$ we find making the change of variables $y=az$ that

$$\begin{aligned}u(ax,a^2t)&=\int_{\mathbb{R}^n}\Phi(ax-y,a^2t)|y|^{-2\gamma}dy\\&=\int_{\mathbb{R}^n}\Phi(ax-az,a^2t)a^{-2\gamma}|z|^{-2\gamma}a^ndz\\&=a^{-2\gamma}\int_{\mathbb{R}^n}\Phi(x-z,t)|z|^{-2\gamma}dz\\&=a^{-2\gamma}u(x,t).\end{aligned}\tag{5.38}$$

Taking $x = 0$ and $t = 1$ in (5.38) we get

$$u(0, a^2) = a^{-2\gamma} u(0,1) \quad \text{for all } a > 0. \tag{5.39}$$

Thus (5.36) holds.

Taking $x \neq 0$, $t > 0$, and $a = 1/|x|$ in (5.38) and using the fact that $u(x,t)$ is radially symmetric in x about the origin we get

$$u(x,t) = a^{2\gamma} u(ax, a^2 t) = |x|^{-2\gamma} u(e_1, \frac{t}{|x|^2}) = |x|^{-2\gamma} g\left(\frac{t}{|x|^2}\right) \tag{5.40}$$

where $g(\zeta) = u(e_1, \zeta)$ and $e_1 = (1, 0, ..., 0) \in \mathbb{R}^n$. By (5.33),

$$g(\zeta) \to 1 \quad \text{as } \zeta \to 0^+ \tag{5.41}$$

and using (5.40) and (5.36) we obtain for $t > 0$ that

$$1 = \lim_{x\to 0} \frac{u(x,t)}{u(0,t)} = \lim_{x\to 0} \frac{|x|^{-2\gamma} g\left(\frac{t}{|x|^2}\right)}{u(0,1)t^{-\gamma}} = \lim_{x\to 0} \frac{1}{u(0,1)} \frac{g\left(\frac{t}{|x|^2}\right)}{\left(\frac{t}{|x|^2}\right)^{-\gamma}}.$$

Thus

$$\frac{g(\zeta)}{\zeta^{-\gamma}} \to u(0,1) \quad \text{as } \zeta \to \infty. \tag{5.42}$$

For $t > 0$, it follows from (5.40)–(5.42) and (5.32) that

$$\begin{aligned}\int_{\mathbb{R}^n} u(x,t)^\lambda dx &= \int_{\mathbb{R}^n} |x|^{-2\lambda\gamma} g\left(\frac{t}{|x|^2}\right)^\lambda dx \\ &\le C\left[\int_{\sqrt{t}<|x|} |x|^{-2\lambda\gamma} dx + \int_{|x|<\sqrt{t}} |x|^{-2\lambda\gamma} \left(\frac{t}{|x|^2}\right)^{-\gamma\lambda} dx\right] \\ &\le C t^{-1+\varepsilon/2}\end{aligned}$$

which implies (5.35).

Making the change of variables

$$x = \sqrt{t}\xi \quad \text{and} \quad y = \sqrt{t}\eta$$

in (5.33) we get

$$u(x,t) = \frac{1}{t^\gamma} U\left(\frac{x}{\sqrt{t}}\right) \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty)$$

where

$$U(\xi) = \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|\xi-\eta|^2/4} |\eta|^{-2\gamma} d\eta.$$

Thus since $U(\xi)$ is bounded between positive constants for $|\xi| \le 1$, we find that (5.37) holds. □

The following theorem implies Theorems 1.3 and 1.4 when $\lambda \ge (n+2)/n$.

Theorem 5.3. *Suppose α, λ, and σ are constants satisfying*

$$\alpha \in (0, n+2), \quad \lambda \ge \frac{n+2}{n}, \quad \sigma \ge 0, \quad \textit{and} \quad \sigma > 1 + \frac{2-\alpha}{n+2}\lambda. \tag{5.43}$$

Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying

$$\lim_{t\to 0^+} \varphi(t) = \infty.$$

Then there exists a positive function

$$u \in C^\infty(\mathbb{R}^n \times (0,1)) \cap L^\lambda(\mathbb{R}^n \times (0,1)) \tag{5.44}$$

satisfying

$$0 \le Hu \le (\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } \mathbb{R}^n \times (0,1), \tag{5.45}$$

where $*$ *is the convolution operation in* $\mathbb{R}^n \times (0,1)$*, such that*

$$u(0,t) \neq O(\varphi(t)) \quad \text{as } t \to 0^+. \tag{5.46}$$

Proof. By scaling u and noting by (5.43) that $\sigma + \lambda \neq 1$ we see that it suffices to prove Theorem 5.3 with (5.45) replaced with the weaker statement that there exists a positive constant $C = C(n, \lambda, \sigma, \alpha)$ such that u satisfies

$$0 \le Hu \le C(\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } \mathbb{R}^n \times (0,1), \tag{5.47}$$

where $(*)$ is the convolution operation in $\mathbb{R}^n \times (0,1)$.

By (5.43) there exists $\varepsilon = \varepsilon(n, \lambda, \sigma, \alpha) \in (0,1)$ such that

$$2\varepsilon < \alpha \tag{5.48}$$

and

$$\sigma > 1 + \frac{2 - \alpha + 2\varepsilon}{n + 2 - 2\varepsilon}\lambda. \tag{5.49}$$

Let

$$\gamma = \frac{n+2-\varepsilon}{2\lambda} \quad \text{and} \quad p = \frac{2\lambda}{n+2-2\varepsilon}. \tag{5.50}$$

Then

$$\gamma p > 1. \tag{5.51}$$

Let $\{T_j\} \subset (0,1)$ be a sequence such that $T_j \to 0$ as $j \to \infty$. Define $w_j : (-\infty, T_j) \to (0, \infty)$ by

$$w_j(t) = (T_j - t)^{-1/p} \tag{5.52}$$

and define $t_j \in (0, T_j)$ by

$$w_j(t_j) = t_j^{-\gamma}. \tag{5.53}$$

Then

$$\frac{T_j - t_j}{t_j} = \frac{w_j(t_j)^{-p}}{t_j} = t_j^{\gamma p - 1} \to 0 \quad \text{as } j \to \infty \tag{5.54}$$

by (5.51).

Choose $a_j \in ((t_j + T_j)/2, T_j)$ such that $w_j(a_j) > j\varphi(a_j)$. Then

$$\frac{w_j(a_j)}{\varphi(a_j)} \to \infty \quad \text{as } j \to \infty. \tag{5.55}$$

Let $h_j(s) = \sqrt{a_j - s}$ and $H_j(s) = \sqrt{a_j + \varepsilon_j - s}$ where $\varepsilon_j > 0$ satisfies

$$a_j + 2\varepsilon_j < T_j, \quad t_j - \varepsilon_j > t_j/2, \quad \varepsilon_j < T_j^2, \quad \text{and} \quad w_j(t_j - \varepsilon_j) > \frac{w_j(t_j)}{2}. \tag{5.56}$$

Define

$$\omega_j = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y| < h_j(s) \quad \text{and} \quad t_j < s < a_j\},$$
$$\Omega_j = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |y| < H_j(s) \quad \text{and} \quad t_j - \varepsilon_j < s < a_j + \varepsilon_j\}.$$

By taking a subsequence we can assume the sets Ω_j are pairwise disjoint.

Let $\chi_j : \mathbb{R}^n \times \mathbb{R} \to [0,1]$ be a C^∞ function such that $\chi_j \equiv 1$ in ω_j and $\chi_j \equiv 0$ in $\mathbb{R}^n \times \mathbb{R} \backslash \Omega_j$. Define f_j, $u_j : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ by

$$f_j(y,s) = \chi_j(y,s) w_j'(s) \tag{5.57}$$

and

$$u_j(x,t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x - y, t - s) f_j(y,s)\, dy\, ds. \tag{5.58}$$

Then f_j and u_j are C^∞ and

$$Hu_j = f_j \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{5.59}$$

By Theorem B.2 with $p = n+2$ and $q = \infty$ we see that

$$\begin{aligned}
&\left\| \iint_{\Omega_j \setminus \omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds \right\|_{L^\infty(\mathbb{R}^n \times (0,1))} \\
&\qquad = \left\| \iint_{\mathbb{R}^n \times (0,1)} \Phi(x-y, t-s) \chi_{\Omega_j \setminus \omega_j}(y,s) w_j'(s)\, dy\, ds \right\|_{L^\infty(\mathbb{R}^n \times (0,1))} \\
&\qquad \le C_n \|w_j'(s)\|_{L^{n+2}(\Omega_j \setminus \omega_j)} \\
&\qquad \le w_j(t_j)
\end{aligned} \tag{5.60}$$

provided we decrease ε_j if necessary because $|\Omega_j \setminus \omega_j| \to 0$ as $\varepsilon_j \to 0$.

Also, it follows from $(5.43)_2$, $(5.50)_1$, (5.37), $(5.56)_1$, (5.54), and (5.53) that there exists a positive constant M, independent of j, such that for $(x,t) \in \Omega_j$ we have

$$M\Psi(x,t) > \frac{2^{\gamma+1}}{t^\gamma} > \frac{2^{\gamma+1}}{T_j^\gamma} > \frac{2^{\gamma+1}}{(2t_j)^\gamma} = 2w_j(t_j), \tag{5.61}$$

provided we take a subsequence if necessary, where Ψ is defined by (5.33).

In order to obtain a lower bound for u_j in ω_j, note first that for $s < t \le a_j + \varepsilon_j$ and $|x| \le H_j(t)$ we have by Lemma 2.9 that

$$\int_{|y|<H_j(s)} \Phi(x-y, t-s)\, dy \ge b \tag{5.62}$$

for some constant

$$b = b(n) \in (0,1). \tag{5.63}$$

Next using (5.62) and (5.63), we find for $(x,t) \in \Omega_j$ that

$$\begin{aligned}
\iint_{\Omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds &= \int_{t_j - \varepsilon_j}^t w_j'(s) \left(\int_{|y|<H_j(s)} \Phi(x-y, t-s)\, dy \right) ds \\
&\ge b(w_j(t) - w_j(t_j - \varepsilon_j)) \\
&\ge bw_j(t) - w_j(t_j).
\end{aligned}$$

It therefore follows from (5.58), (5.57), and (5.60) that for $(x,t) \in \Omega_j$ we have

$$\begin{aligned}
u_j(x,t) &\ge \iint_{\omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds \\
&= \iint_{\Omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds - \iint_{\Omega_j \setminus \omega_j} \Phi(x-y, t-s) w_j'(s)\, dy\, ds \\
&\ge bw_j(t) - 2w_j(t_j).
\end{aligned} \tag{5.64}$$

Define $\beta > 0$ by

$$\frac{1}{\beta} - \frac{1}{\lambda} = \frac{2}{n+2}. \tag{5.65}$$

Then by (5.43)

$$\frac{2}{n+2} < \frac{1}{\beta} = \frac{1}{\lambda} + \frac{2}{n+2} \le \frac{n}{n+2} + \frac{2}{n+2} = 1 \tag{5.66}$$

and by (5.50)

$$p > \frac{2\lambda}{n+2} = \frac{2}{(n+2)/\lambda} = \frac{2}{\frac{n+2}{\beta} - 2} = \frac{2\beta}{n+2-2\beta}.$$

Thus

$$\frac{n}{2} - \frac{\beta(p+1)}{p} + 1 = \frac{(n+2-2\beta)p - 2\beta}{2p} > 0. \tag{5.67}$$

Next we slightly increase β in such a way that (5.67) and the first inequality in (5.66) still hold. Then instead of (5.65) and (5.66) we get

$$\frac{1}{\beta} - \frac{1}{\lambda} < \frac{2}{n+2} \tag{5.68}$$

and

$$\frac{2}{n+2} < \frac{1}{\beta} < 1 \tag{5.69}$$

respectively.

From (5.57), (5.52), $(5.56)_1$ and (5.67) we find that

$$\begin{aligned} p^\beta \iint_{\mathbb{R}^n\times\mathbb{R}} f_j(y,s)^\beta\, dy\, ds &\le p^\beta \iint_{\Omega_j} w_j'(s)^\beta\, dy\, ds \\ &\le p^\beta \int_0^{T_j} \int_{|y|<\sqrt{T_j-s}} w_j'(s)^\beta\, dy\, ds \\ &= |B_1(0)| \int_0^{T_j} (T_j - s)^{n/2-\beta(p+1)/p} ds \\ &= |B_1(0)| \int_0^{T_j} \tau^{n/2-\beta(p+1)/p}\, d\tau \to 0 \quad \text{as } j \to \infty. \end{aligned} \tag{5.70}$$

Hence by (5.58), (5.68), (5.69), and Theorem B.2 we obtain

$$\|u_j\|_{L^\lambda(\mathbb{R}^n\times(0,1))} \to 0 \quad \text{as } j \to \infty. \tag{5.71}$$

Repeating the derivation of (5.70) with β replaced with 1, we find that

$$\iint_{\mathbb{R}^n\times\mathbb{R}} f_j(y,s)\, dy\, ds \to 0 \quad \text{as } j \to \infty.$$

Thus

$$\iint_{\mathbb{R}^n\times\mathbb{R}} \sum_{j=1}^\infty f_j(y,s) d\eta ds < \infty$$

provided we take a subsequence if necessary. Hence, since the C^∞ functions f_j have disjoint supports, it follows from Theorem 5.2 that the function $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ defined by

$$u(x,t) = (M+1)\Psi(x,t) + \sum_{j=1}^\infty u_j(x,t) \tag{5.72}$$

is C^∞ and from (5.59) and Theorem 5.2 we have

$$Hu = \sum_{j=1}^\infty f_j \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{5.73}$$

By (5.71) and Theorem 5.2,

$$u \in L^\lambda(\mathbb{R}^n \times (0,1))$$

provided we take a subsequence of u_j if necessary. Thus (5.44) holds.

We now prove (5.47). By (5.73) and (5.57) we have

$$Hu \equiv 0 \quad \text{in } (\mathbb{R}^n \times (0,1)) \setminus \bigcup_{j=1}^\infty \Omega_j.$$

Hence to prove (5.47), it suffice to prove there exists a positive constant $C = C(n, \lambda, \sigma, \alpha)$ such that

$$0 \le Hu \le C(\Phi^{\alpha/n} * u^\lambda)u^\sigma \quad \text{in } \Omega_j \tag{5.74}$$

for $j = 1, 2, ...$

By (5.72), (5.64), and (5.61) we have for $(x, t) \in \Omega_j$ that

$$u(x, t) \ge (M + 1)\Psi(x, t) + bw_j(t) - 2w_j(t_j) \ge \Psi(x, t) + bw_j(t). \tag{5.75}$$

Thus for $(x, t) \in \Omega_j$ we see by (5.73), (5.57), and (5.52) that

$$\begin{aligned} Hu(x, t) = f_j(x, t) &\le w_j'(t) = \frac{1}{p}w_j(t)^{1+p} \\ &= \frac{1}{p}w_j(t)^{1+p-\sigma}w_j(t)^\sigma \le \frac{1}{pb^\sigma}w_j(t)^{1+p-\sigma}u(x, t)^\sigma. \end{aligned}$$

Hence to prove (5.74) it suffices to show

$$w_j(t)^{1+p-\sigma} < C \iint_{\mathbb{R}^n\times(0,1)} \Phi(x - y, t - s)^{\alpha/n}u(y, s)^\lambda \, dy \, ds \quad \text{for } (x, t) \in \Omega_j. \tag{5.76}$$

Our proof of (5.76) consists of two cases.

Case I. Suppose

$$(x, t) \in \Omega_j \quad \text{and} \quad t \le \frac{T_j + t_j}{2}. \tag{5.77}$$

Then using $(5.56)_4$, $(5.50)_2$, $(5.43)_2$ and the fact that w_j is an increasing function we have

$$\begin{aligned} \frac{1}{2} \le \frac{w_j(t)}{2w_j(t_j - \varepsilon_j)} &< \frac{w_j(t)}{w_j(t_j)} \\ &\le \left(\frac{T_j - \frac{T_j+t_j}{2}}{T_j - t_j}\right)^{-1/p} = 2^{1/p} < 2^{n/2}. \end{aligned}$$

Also by (5.53) and (5.54)

$$\frac{w_j(t_j)}{T_j^{-\gamma}} = \left(\frac{T_j}{t_j}\right)^\gamma \in (1, 2)$$

provided we take a subsequence if necessary. Thus (5.77) implies

$$\frac{1}{2} < \frac{w_j(t)}{T_j^{-\gamma}} < 2^{(n+2)/2}. \tag{5.78}$$

Next making the change of variables

$$x = \sqrt{T_j}\xi, \quad t = T_j\tau; \qquad y = \sqrt{T_j}\eta, \quad s = T_j\zeta; \qquad \text{and} \quad \hat{y} = \sqrt{T_j}\hat{\eta},$$

we get for $(y, s) \in \mathbb{R}^n \times (0, \infty)$ that

$$\begin{aligned} \Psi(y, s) &= \int_{\mathbb{R}^n} \Phi(y - \hat{y}, s)|\hat{y}|^{-2\gamma}d\hat{y} \\ &= \int_{\mathbb{R}^n} \frac{1}{T_j^{n/2}}\Phi(\eta - \hat{\eta}, \zeta)T_j^{-\gamma}|\hat{\eta}|^{-2\gamma}T_j^{n/2}d\hat{\eta} \\ &= T_j^{-\gamma}\Phi(\eta, \zeta) \end{aligned}$$

and thus for $(x,t) \in \Omega_j$ we obtain from $(5.50)_1$ that

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times(0,1)} \Phi(x-y,t-s)^{\alpha/n}\Psi(y,s)^\lambda \, dy\, ds \\
&\qquad = \iint_{\mathbb{R}^n\times(0,\tau)} \left(\frac{1}{T_j^{n/2}}\Phi(\xi-\eta,\tau-\zeta)\right)^{\alpha/n} (T_j^{-\gamma}\Psi(\eta,\zeta))^\lambda T_j^{\frac{n+2}{2}}\, d\eta\, d\zeta \\
&\qquad \geq \frac{G(\xi,\tau)}{\sqrt{T_j}^{\alpha+2\gamma\lambda-(n+2)}} = \frac{G(\xi,\tau)}{\sqrt{T_j}^{\alpha-\varepsilon}} \qquad (5.79)
\end{aligned}$$

where

$$G(\xi,\tau) := \iint_{B_1(0)\times(1/2,\tau)} \Phi(\xi-\eta,\tau-\zeta)^{\alpha/n}\Psi(\eta,\zeta)^\lambda \, d\eta\, d\zeta.$$

Since by $(5.77)_1$, $(5.56)_1$, (5.54), and $(5.56)_3$,

$$1 > \tau = \frac{t}{T_j} \geq \frac{t_j-\varepsilon_j}{T_j} \to 1 \quad \text{as } j\to\infty$$

we have by $(5.77)_1$ that

$$|\xi| = \frac{|x|}{\sqrt{T_j}} < \frac{\sqrt{T_j-t}}{\sqrt{T_j}} = \sqrt{1-\frac{t}{T_j}} \to 0 \quad \text{as } j\to\infty.$$

Thus, since G is clearly continuous at $(\xi,\tau)=(0,1)$ and $G(0,1)>0$ we have by (5.79) that

$$\iint_{\mathbb{R}^n\times(0,1)} \Phi(x-y,t-s)^{\alpha/n}\Psi(y,s)^\lambda \, dy\, ds \geq \frac{C}{\sqrt{T_j}^{\alpha-\varepsilon}} \quad \text{for } (x,t)\in\Omega_j \qquad (5.80)$$

provided we take a subsequence if necessary.

Since by (5.49) and $(5.50)_2$

$$\sigma-1 > \left(\frac{2-\alpha+2\varepsilon}{n+2-2\varepsilon}\right)\lambda = p - \frac{\alpha-2\varepsilon}{n+2-2\varepsilon}\lambda > p - \frac{\alpha-\varepsilon}{n+2-\varepsilon}\lambda$$

we have by $(5.50)_1$ that

$$\gamma(1+p-\sigma) < \gamma\left(\frac{(\alpha-\varepsilon)\lambda}{n+2-\varepsilon}\right) = \frac{\alpha-\varepsilon}{2}.$$

Thus (5.76) follows from (5.75), (5.78), and (5.80).

Case II. Suppose

$$(x,t)\in\Omega_j \quad \text{and} \quad t \geq \frac{T_j+t_j}{2}. \qquad (5.81)$$

Then for $s<t$ we have by Lemma 2.9 that

$$\int_{|y|<H_j(s)} \Phi(x-y,t-s)^{\alpha/n} dy \geq \frac{C}{(t-s)^{(\alpha-n)/2}}$$

for some positive constant $C = C(n, \alpha)$. Thus for (x, t) satisfying (5.81) we get

$$\begin{aligned}\iint_{\Omega_j} \Phi(x-y,t-s)^{\alpha/n} w_j(s)^\lambda \, dy\, ds &\geq \int_{t_j}^t w_j(s)^\lambda \left(\int_{|y|<H_j(s)} \Phi(x-y,t-s)^{\alpha/n} dy \right) ds \\ &\geq C \int_{t_j}^t \frac{ds}{(t-s)^{(\alpha-n)/2}(T_j-s)^{\lambda/p}} \\ &= \frac{C}{(T_j-t)^{(\alpha-n)/2+\lambda/p-1}} \int_1^{\frac{T_j-t_j}{T_j-t}} \frac{dz}{(z-1)^{(\alpha-n)/2} z^{\lambda/p}} \text{ where } T_j - s = (T_j-t)z \\ &\geq \frac{C}{(T_j-t)^{(\alpha-n)/2+\lambda/p-1}} \int_1^2 \frac{dz}{(z-1)^{(\alpha-n)/2} z^{\lambda/p}} \\ &= \frac{C}{(T_j-t)^{(\alpha-n)/2+\lambda/p-1}} = \frac{C}{(T_j-t)^{(\alpha-2\varepsilon)/2}} \end{aligned} \tag{5.82}$$

by $(5.50)_2$.

Since by (5.49) and $(5.50)_2$

$$\sigma - 1 > \frac{2-\alpha+2\varepsilon}{n+2-2\varepsilon}\lambda = p\frac{2-\alpha+2\varepsilon}{2}$$

we see that

$$\frac{1}{p}(1+p-\sigma) = 1 + \frac{1-\sigma}{p} < 1 + \frac{\alpha-2-2\varepsilon}{2} = \frac{\alpha-2\varepsilon}{2}.$$

Thus (5.76) follows from (5.75), (5.52), and (5.82).

Finally from (5.75) and (5.55) we get

$$\frac{u(0,a_j)}{\varphi(a_j)} \geq \frac{bw_j(a_j)}{\varphi(a_j)} \to \infty \quad \text{as } j \to \infty,$$

which gives (5.46). □

Appendix A. Representation formula

In this appendix we provide the following representation formula for nonnegative supertemperatures.

Theorem A.1. *Suppose $0 < R_1 < R_2 < R_3$ are constants and u is a $C^{2,1}$ nonnegative solution of*

$$Hu \geq 0 \quad \text{in } B_{\sqrt{R_3}}(0) \times (0, R_3) \subset \mathbb{R}^n \times \mathbb{R}, \ n \geq 1, \tag{A.1}$$

where $Hu = u_t - \Delta u$ is the heat operator. Then

$$Hu \in L^1(B_{\sqrt{R_2}}(0) \times (0, R_2)), \tag{A.2}$$

$$u^\beta \in L^1(B_{\sqrt{R_1}}(0) \times (0, R_1)) \quad \text{for } 1 \leq \beta < \frac{n+2}{n} \tag{A.3}$$

and there exist a finite positive Borel measure μ on $B_{\sqrt{R_2}}(0)$ and a bounded function $h \in C^{2,1}(B_{\sqrt{R_1}}(0) \times (-R_1, R_1))$ satisfying

$$Hh = 0 \quad \text{in } B_{\sqrt{R_1}}(0) \times (-R_1, R_1) \tag{A.4}$$

$$h = 0 \quad \text{in } \quad B_{\sqrt{R_1}}(0) \times (-R_1, 0] \tag{A.5}$$

such that

$$u = N + v + h \quad \text{in } B_{\sqrt{R_1}}(0) \times (0, R_1) \tag{A.6}$$

where

$$N(x,t) := \int_0^{R_2} \int_{|y|<\sqrt{R_2}} \Phi(x-y,t-s) Hu(y,s)\,dy\,ds, \tag{A.7}$$

$$v(x,t) := \int_{|y|<\sqrt{R_2}} \Phi(x-y,t)\,d\mu(y), \tag{A.8}$$

and Φ *is the heat kernel* (1.3).

Proof. When $\beta = 1$, $R_1 = 1$, $R_2 = 4$, and $R_3 = 16$, Theorem A.1 was proved in [19]. The proof of Theorem A.1 when $\beta = 1$ is obtained by making straighforward changes to the proof in [19]. It remains only to prove (A.3) for $1 < \beta < (n+2)/n$. To do this, it suffices by (A.6) to show

$$N^\beta \in L^1(\mathbb{R}^n \times (0,R_2)) \quad \text{for } 1 < \beta < (n+2)/n \tag{A.9}$$

and

$$v^\beta \in L^1(\mathbb{R}^n \times (0,R_2)) \quad \text{for } 1 < \beta < (n+2)/n. \tag{A.10}$$

Theorem B.2 and (A.2) imply (A.9).

Finally, for $t > 0$, $\beta > 1$, and β' the conjugate Hölder exponent of β we have

$$\begin{aligned}
\int_{\mathbb{R}^n} v(x,t)^\beta\,dx &= \int_{\mathbb{R}^n} \left(\int_{|y|<\sqrt{R_2}} \Phi(x-y,t)\,d\mu(y) \right)^\beta dx \\
&\le \int_{\mathbb{R}^n} \left(\int_{|y|<\sqrt{R_2}} 1^{\beta'}\,d\mu(y) \right)^{\beta/\beta'} \int_{|y|<\sqrt{R_2}} \Phi(x-y,t)^\beta\,d\mu(y)\,dx \\
&= C \int_{|y|<\sqrt{R_2}} \left(\int_{\mathbb{R}^n} \Phi(x-y,t)^\beta\,dx \right) d\mu(y) \\
&= C \int_{|y|<\sqrt{R_2}} t^{-n\beta/2} \int_{\mathbb{R}^n} e^{-\frac{\beta|x-y|^2}{4t}}\,dx\,d\mu(y) \\
&= C t^{n(1-\beta)/2} \quad \text{by Lemma 2.2}
\end{aligned}$$

which implies (A.10). ☐

Remark A.1. If u is a $C^{2,1}$ nonnegative solution of (A.1) where $R_3 > 0$ then by Theorem A.1,

$$u^\beta \in L^1(B_{\sqrt{R}}(0) \times (0,R)) \quad \text{for } 1 \le \beta < \frac{n+2}{n} \text{ and } 0 < R < R_3.$$

Thus the conclusion (A.3) in Theorem A.1 can be replaced with

$$u^\beta \in L^1(B_{\sqrt{R_2}}(0) \times (0,R_2)) \quad \text{for } 1 \le \beta < \frac{n+2}{n}.$$

Appendix B. Heat potential estimates

In this appendix we provide estimates for the heat potentials

$$(J_\alpha f)(x,t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x-y,t-s)^{\frac{n+2-\alpha}{n}} f(y,s)\,dy\,ds$$

and

$$(V_\alpha f)(x,t) = \iint_\Omega \Phi(x-y,t-s)^{\frac{n+2-\alpha}{n}} f(y,s)\,dy\,ds,$$

where Φ is given by (1.3), $\Omega = \mathbb{R}^n \times (a,b)$, and $\alpha \in (0, n+2)$. The proofs of these estimates are given in [3, Appendix B].

Theorem B.1. *Suppose* $0 < \alpha < n+2$ *and* $1 < p < \frac{n+2}{\alpha}$ *are constants and* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *is a nonnegative measurable function. Let*

$$q = \frac{(n+2)p}{n+2-\alpha p}.$$

Then

$$\|J_\alpha f\|_{L^q(\mathbb{R}^n \times \mathbb{R})} \le C\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R})}$$

where $C = C(n,p,\alpha)$ *is a positive constant.*

Theorem B.2. *Let* $p, q \in [1,\infty]$, α, *and* δ *satisfy*

$$0 \le \delta = \frac{1}{p} - \frac{1}{q} < \frac{\alpha}{n+2} < 1. \tag{B.1}$$

Then V_α *maps* $L^p(\Omega)$ *continuously into* $L^q(\Omega)$ *and for* $f \in L^p(\Omega)$ *we have*

$$\|V_\alpha f\|_{L^q(\Omega)} \le M\|f\|_{L^p(\Omega)},$$

where

$$M = C(b-a)^{(\alpha-(n+2)\delta)/2} \quad \text{for some constant } C = C(n,\alpha,\delta) > 0.$$

Theorem B.2 is weaker than Theorem B.1 in that the second inequality in (B.1) cannot be replaced with equality. However it is stronger in that the cases $p = 1$ and $q = \infty$ are allowed.

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