Blow-up Behavior of Solutions of Semilinear Elliptic and Parabolic Inequalities

Steven D. Taliaferro

Mathematics Department
Texas A&M University
College Station, TX 77843-3368
stalia@math.tamu.edu

1. Introduction

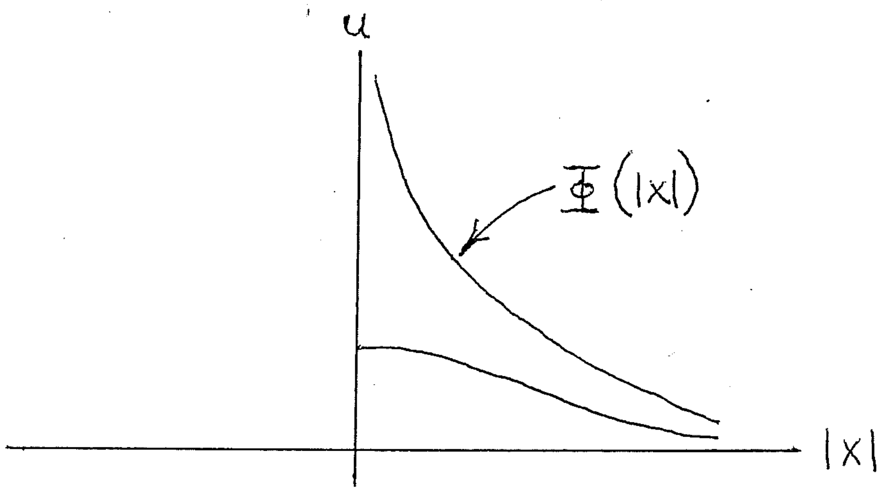
It is well-known that if u is positive and harmonic in a punctured neighborhood of the origin in $\mathbf{R}^n$ $(n \geq 2)$ then either u has a C^2 positive extension to the origin, or for some finite positive number m,

$$\lim_{x \to 0} \frac{u(x)}{\Phi(|x|)} = m,$$

where

$$\Phi(|x|) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x|}, & \text{if } n = 2 \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & \text{if } n \geq 3 \end{cases}$$

is the fundamental solution of $-\Delta$.



In particular, u satisfies the following two weaker conditions:

(i) u is *asymtotically radial* as $x \to 0$, i.e.

$$\lim_{x \to 0} \frac{u(x)}{\bar{u}(|x|)} = 1,$$

where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$, and

(ii) $u(x) = O(\Phi(|x|)) \quad \text{as} \quad x \to 0.$

Do similar results hold for C^2 positive solutions u of the differential inequalities

$$0 \le -\Delta u \le f(u) \qquad \text{in} \qquad \mathbf{B}^n \backslash \{0\}, \tag{1.1}$$

where $f\colon (0,\infty) \to (0,\infty)$ is a given continuous function? Here $\mathbf{B}^n$ is a ball in $\mathbf{R}^n$ centered at the origin whose radius depends on the solution u. Special case is Laplace equation.

Specifically, under what conditions on the function f does every C^2 positive solution u of (1.1) satisfy some (or all) of the following three conditions?

(i) u is asymptotically radial as $x \to 0$,
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$,
(iii) u is *asymptotically harmonic* as $x \to 0$, i.e.

$$\lim_{x \to 0} \frac{u(x)}{h(x)} = 1$$

for some function $h(x)$ which is positive and harmonic in a punctured neighborhood of the origin in $\mathbf{R}^n$.

Since (iii) implies (i) and (ii), the conditions on f for (iii) to hold will have to be at least as strong as the conditions on f for (i) or (ii) to hold.

2. Two dimensional results

Theorem 2.1. *Let $u(x)$ be a C^2 positive solution of*

$$0 \le -\Delta u \le f(u) \qquad \text{in} \qquad \mathbf{B}^2 \backslash \{0\},$$

where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\log f(t) = O(t) \quad \text{as} \quad t \to \infty.$$

Note that $f(t) = e^t$ satisfies this equation.

Then u is asymptotically harmonic as $x \to 0$.

The conformal Gauss curvature equation

$$-\Delta u = e^u$$

has been extensively studied. A corollary of the previous theorem is the following.

Corollary. *If $u(x)$ is a C^2 positive solution of*

$$0 \le -\Delta u \le e^u \qquad \text{in} \qquad \mathbf{B}^2 \backslash \{0\}$$

Not true if positivity is omitted.

then u is asymptotically harmonic as $x \to 0$.

The condition on f in the previous theorem was

$$\log f(t) = O(t) \quad \text{as} \quad t \to \infty. \tag{2.1}$$

The following theorem shows that this condition on f is essentially optimal.

Theorem 2.2. *Let $f\colon (0,\infty) \to (0,\infty)$ and $\varphi\colon (0,1) \to (0,\infty)$ be continuous functions such that*

$$\lim_{t\to\infty} \frac{\log f(t)}{t} = \infty \quad \textit{and} \quad \lim_{t\to 0+} \varphi(t) = \infty.$$

Then there exists a C^2 positive solution u of

$$0 \le -\Delta u \le f(u) \qquad \textit{in} \qquad \mathbf{B}^2\backslash\{0\}$$

such that

$$u(x) \ne O(\varphi(|x|) \quad \text{as} \quad x \to 0$$

and u is not asymptotically radial as $x \to 0$. (See picture)

Thus condition (2.1) is essentially optimal for any (or all) of the following conditions to hold:

(i) u is asymptotically radial as $x \to 0$,
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$,
(iii) u is asymptotically harmonic as $x \to 0$.

There is no analogous condition on f in three and higher dimensions.

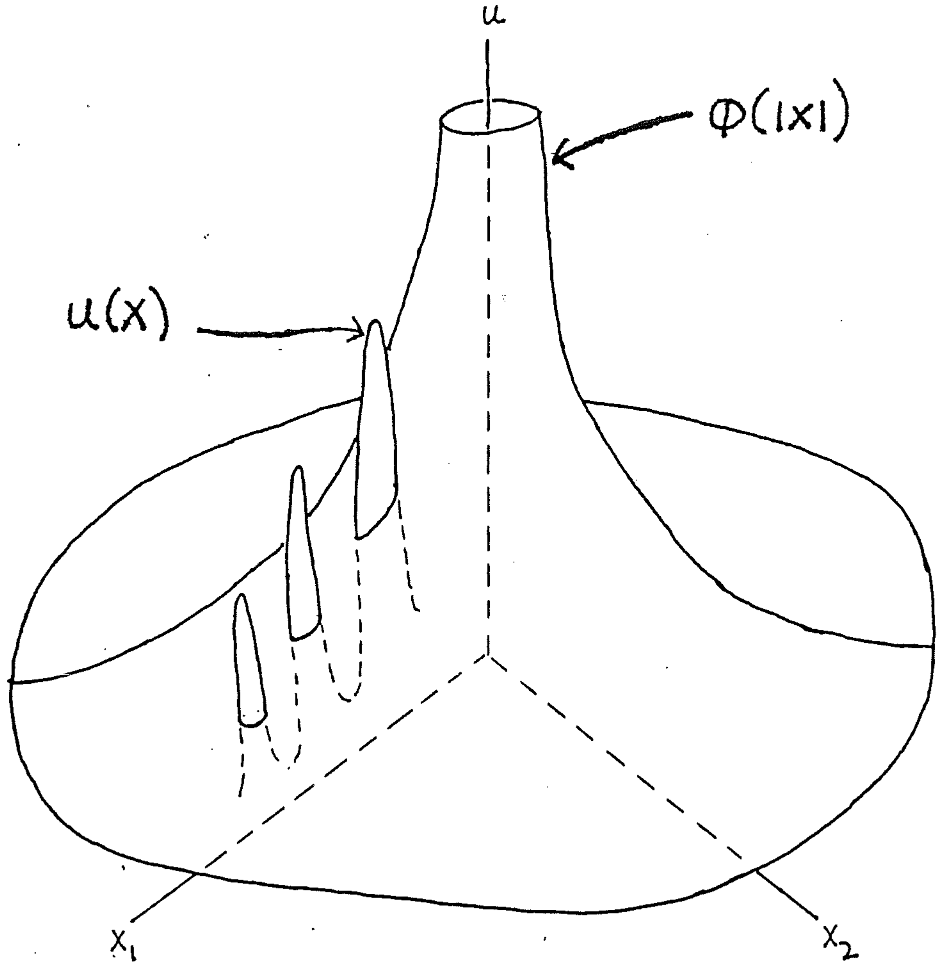


Figure 1

$$u(x) \neq O(\varphi(|x|)) \quad \text{as} \quad x \to 0$$

3. Three and higher dimensional results

Theorem 3.1. *Let $u(x)$ be a C^2 positive solution of*

$$0 \le -\Delta u \le f(u) \quad \text{in} \quad \mathbf{B}^n \setminus \{0\}, \quad n \ge 3,$$

where $f\colon (0,\infty) \to (0,\infty)$ is a continuous function satisfying

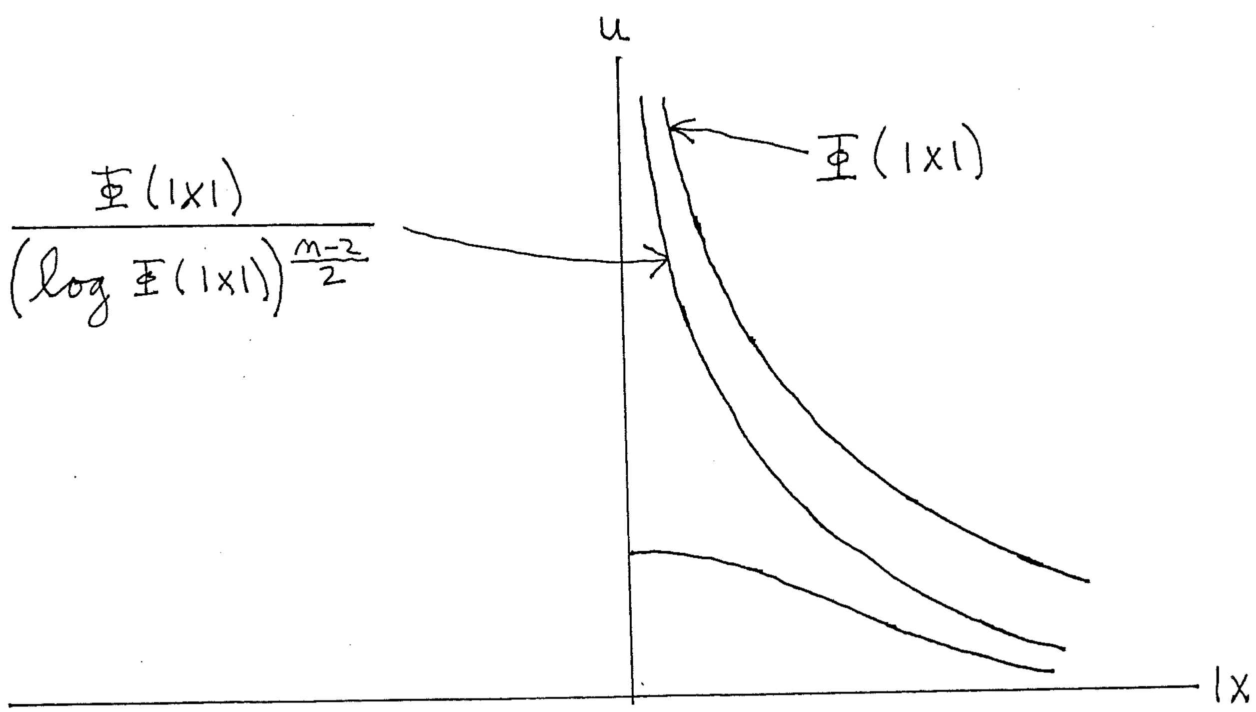
$$f(t) = O(t^{\frac{n}{n-2}}) \quad \text{as} \quad t \to \infty.$$

Then u is asymptotically radial as $x \to 0$. Moreover, either u is asymptotically harmonic as $x \to 0$ or u satisfies the following two conditions:

$$\lim_{x\to 0} \frac{u(x)}{\Phi(|x|)} = 0$$

and

$$\liminf_{x\to 0} \frac{u(x)}{\left(\frac{\Phi(|x|)}{(\log \Phi(|x|))^{(n-2)/2}}\right)} > 0.$$



The condition on f in the previous theorem was

$$f(t) = O(t^{\frac{n}{n-2}}) \quad \text{as} \quad t \to \infty. \tag{3.1}$$

The following theorem shows that this condition on f is essentially optimal.

Theorem 3.2. *Let $f\colon (0,\infty) \to (0,\infty)$ and $\varphi\colon (0,1) \to (0,\infty)$ be continuous functions such that*

$$\lim_{t\to\infty} \frac{f(t)}{t^{n/(n-2)}} = \infty \quad \textit{and} \quad \lim_{t\to 0+} \varphi(t) = \infty.$$

Then there exists a C^2 positive solution u of

$$0 \le -\Delta u \le f(u) \qquad \textit{in} \qquad \mathbf{B}^n\backslash\{0\}, \quad n \ge 3,$$

such that

$$u(x) \neq O(\varphi(|x|) \quad \text{as} \quad x \to 0$$

and u is not asymptotically radial as $x \to 0$.

Thus condition (3.1) is essentially optimal for either (or both) of the following conditions to hold:

(i) u is asymptotically radial as $x \to 0$,
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$;

but is too weak to imply
(iii) u is asymptotically harmonic as $x \to 0$,

because for $0 < \sigma < (n-2)/2$ the function

$$u_\sigma(x) := \frac{\Phi(|x|)}{(\log \Phi(|x|))^\sigma}$$

is a C^2 positive solution of $0 \le -\Delta u \le u^{\frac{n}{n-2}}$ in $\mathbf{B}^n\backslash\{0\}$ and $u_\sigma(x)$ is not asymptotically harmonic as $x \to 0$.

This is in contrast to the situation in two dimensions.

What condition on f is needed for u to be asy harmonic?

4. Asymptotically harmonic solutions in three and higher dimensions

By the last two theorems, the essentially optimal growth condition on f for u to be asymptotically radial as $x \to 0$ is

$$f(t) = O(t^{n/(n-2)}) \quad \text{as} \quad t \to \infty.$$

In the following theorem, we strenghen this growth condition on f in such a way as to conclude that u is asymptotically harmonic as $x \to 0$.

First we need a definition:
$\log_1 := \log, \quad \log_2 := \log \circ \log, \quad \log_3 := \log \circ \log \circ \log, \quad$ etc.

Theorem 4.1. *Let u be a C^2 positive solution of*

$$0 \leq -\Delta u \leq \frac{u^{\frac{n}{n-2}}}{(\log_1 u) \cdots (\log_{q-1} u)(\log_q u)^\beta} \tag{4.1}$$

in $\mathbf{B}^n \backslash \{0\}$, $n \geq 3$, where $\beta \in (1, \infty)$ and q is a positive integer. Then u is asymptotically harmonic as $x \to 0$.

This theorem is essentially optimal because a solution of (4.1) when $\beta = 1$ is

$$u(|x|) = \frac{\Phi(|x|)}{\log_{q+2} \Phi(|x|)}$$

which is not asymptotically harmonic as $x \to 0$.

5. Summary of results

In summary, the essentially optimal condition on a continuous function $f\colon (0,\infty) \to (0,\infty)$ for every C^2 positive solution u of

$$0 \le -\Delta u \le f(u) \qquad \text{in} \qquad \mathbf{B}^n \backslash \{0\}$$

to satisfy

(i) u is asymptotically radial as $x \to 0$, and/or
(ii) $u(x) = O(\Phi(|x|))$ as $x \to 0$

is

$$\log f(t) = O(t) \quad \text{when} \quad n = 2,$$

$$f(t) = O(t^{n/(n-2)}) \quad \text{when} \quad n \ge 3.$$

Moreover, the essentially optimal condition on f for u to be asymptotically harmonic as $x \to 0$ is

$$\log f(t) = O(t) \quad \text{when} \quad n = 2,$$

$$f(t) = O\left(\frac{t^{n/(n-2)}}{(\log_1 t)\cdots(\log_{q-1} t)(\log_q t)^\beta}\right) \quad \text{when} \quad n \ge 3$$

for some $\beta \in (1,\infty)$ and some positive integer q.

6. Further results

Recall that C^2 solutions u of

$$\begin{array}{l} 0 \le -\Delta u \le u^{n/(n-2)} \\ u > 0 \end{array} \quad \text{in} \quad \mathbf{B}^n \setminus \{0\}, \quad n \ge 3$$

satisfy $\quad u(x) = O(|x|^{2-n}) \quad$ as $\quad x \to 0.$

However the problem

$$\begin{array}{l} 0 \le -\Delta u \le u^{\lambda} \\ u > 0 \end{array} \quad \text{in} \quad \mathbf{B}^n \setminus \{0\}, \quad \frac{n}{n-2} < \lambda$$

has arbitrarily large solutions near the origin. (That is given a continuous function $\varphi \colon (0,1) \to (0,\infty)$ there exists a C^2 solution u such that $u(x) \ne O(\varphi(|x|))$ as $x \to 0$.)

Consider instead the more restricted problem

$$\begin{array}{l} au^{\lambda} \le -\Delta u \le u^{\lambda} \\ u > 0 \end{array} \quad \text{in} \quad \mathbf{B}^n \setminus \{0\}, \quad \frac{n}{n-2} < \lambda < \frac{n+2}{n-2}$$

where $0 < a < 1$.
Arbitrarily large solutions near the origin?
Answer depends on a.
Thus this is the correct problem to study for λ as above.

More precisely, consider the differential inequalities

$$au^\lambda \le -\Delta u \le u^\lambda \quad \text{in} \quad \mathbf{B}^n \backslash \{0\}, \quad n \ge 3 \tag{1}$$

where

$$\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}. \tag{2}$$

Theorem 1. *Suppose λ satisfies (2). Then there exists $a = a(n,\lambda) \in (0,1)$ such that (1) has C^2 positive solutions which are arbitrarily large near the origin.*

Theorem 2. *Suppose λ satisfies (2). Then there exists $a = a(n,\lambda) \in (0,1)$ such that if u is a C^2 positive solution of (1) then*

$$u(x) = O(|x|^{-2/(\lambda-1)}) \quad \textit{as} \quad x \to 0$$

and

$$0 < C_1 \le \frac{u(x)}{\bar{u}(|x|)} \le C_2 < \infty \quad \textit{for } |x| \textit{ small and positive}$$

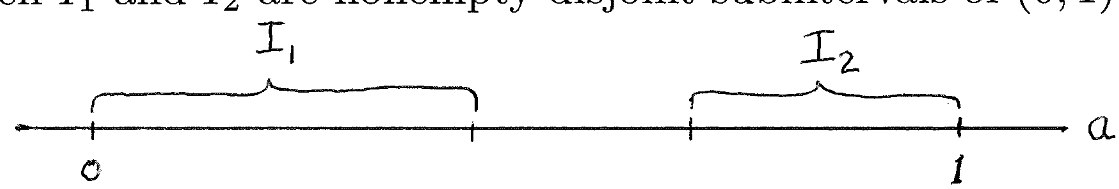
where $\bar{u}(r)$ is the average of u on the sphere $|x| = r$.

Let λ satisfy (2) and let

$$I_1 = I_1(n,\lambda) = \{a \in (0,1)\colon \text{Theorem 1 is true}\}$$

$$I_2 = I_2(n,\lambda) = \{a \in (0,1)\colon \text{Theorem 2 is true}\}.$$

Then I_1 and I_2 are nonempty disjoint subintervals of $(0,1)$.



Open Question. *Does $I_1 \cup I_2 = (0,1)$? If not, what is the behavior of C^2 positive solutions of (1) when*

$$a \in (0,1) \backslash (I_1 \cup I_2)?$$

The proofs of some of these results for elliptic inequalities use a representation formula of Brezis and Lions for nonnegative solutions of $-\Delta u \geq 0$ in a punctured neighborhood of the origin in $\mathbf{R}^n$.

Brezis-Lions Lemma. *Let u be a C^2 solution of*

$$\left.\begin{aligned} -\Delta u &\geq 0 \\ u &\geq 0 \end{aligned}\right. \qquad \textit{in} \quad B_2(0)\backslash\{0\}.$$

Then $-\Delta u \in L^1(B_1(0)$ and for some nonnegative constant m and some solution of h of

$$-\Delta h = 0 \quad \textit{in} \quad B_1(0)$$

we have

$$u = m\Phi + N + h \quad \textit{in} \quad B_1(0)\backslash\{0\}$$

where Φ is the fundamental solution of $-\Delta$ and

$$N(x) = \int_{|y|<1} \Phi(x-y)(-\Delta u(y))\, dy.$$

7. Polyharmonic inequalities

Next consider C^{2m} nonnegative solutions of

$$-\Delta^m u \geq 0 \quad \text{in} \quad \mathbf{B}^n \backslash \{0\} \tag{7.1}$$

where $n \geq 2$ and $m \geq 1$ are integers. If $m = 1$ then this inequality has C^2 positive solutions which are pointwise arbitrarily large near the origin. What about other choices for m? Results in this section are due to Ghergu, Moradifam, T.

Theorem 7.1. *A necessary and sufficient condition on integers $n \geq 2$ and $m \geq 1$ such that C^{2m} nonnegative solutions $u(x)$ of (7.1) satisfy a pointwise a priori bound as $x \to 0$ is that*

$$\textit{either } m \textit{ is even or } n < 2m. \tag{7.2}$$

In this case, u is harmonically bounded at 0, that is

$$u(x) = O(\Phi(x)) \quad \text{as} \quad x \to 0.$$

This bound is optimal.

By this theorem, in order to get a pointwise bound for nonnegative solutions $u(x)$ of (7.1) when (7.2) does not hold we have to impose additional conditions on u. To this end, we consider

$$0 \leq -\Delta^m u \leq f(u) \quad \text{in} \quad \mathbf{B}^n \backslash \{0\} \tag{7.3}$$

and ask the following question.

Question. *For which continuous fuctions* $f : [0,\infty) \to [0,\infty)$ *does there exist a continuous function* $\varphi : (0,1) \to (0,\infty)$ *such that every* C^{2m} *nonnegative solution* $u(x)$ *of*

$$0 \leq -\Delta^m u \leq f(u) \quad \text{in} \quad \mathbf{B}^n \backslash \{0\} \tag{7.3}$$

satisfies

$$u(x) = O(\varphi(|x|)) \quad \text{as} \quad x \to 0$$

and what is the optimal such φ *when one exists?*

The last theorem completely answers this question when

$$\text{either } m \text{ is even or } n < 2m. \tag{7.2}$$

(In this case all nonnegative solutions of (7.3) are harmonically bounded, regardless of f.)

As an example of our results for this question when (7.2) does not hold, we have the following result which deals with the case

$$m \geq 2 \text{ is odd and } n = 2m,$$

which is the most interesting case when (7.2) does not hold.

Theorem 7.2. *Let $u(x)$ be a C^{2m} nonnegative solution of*

$$0 \le -\Delta^m u \le f(u) \quad \text{in} \quad \mathbf{B}^n\backslash\{0\},$$

where $m \ge 2$ is odd, $n = 2m$, and either

$$\begin{array}{lll} (i) & f(t) = t^\lambda, & 0 \le \lambda \le \dfrac{2n-2}{n-2}; \\ (ii) & f(t) = t^\lambda, & \lambda > \dfrac{2n-2}{n-2}; \\ (iii) & f(t) = e^{t^\lambda}, & 0 \le \lambda < 1; \text{ or} \\ (iv) & f(t) = e^{t^\lambda}, & \lambda \ge 1. \end{array}$$

Then, as $x \to 0$, $u(x)$ respectively satisfies

$$\begin{array}{ll} (i) & u(x) = O\left(|x|^{-(n-2)}\right) \quad \text{that is } u \text{ is harmonically bounded;} \\ (ii) & u(x) = o\left(|x|^{-(n-2)} \log \dfrac{1}{|x|}\right); \\ (iii) & u(x) = o\left(|x|^{\frac{-(n-2)}{1-\lambda}}\right); \text{ or} \\ (iv) & \text{no pointwise bound.} \end{array}$$

These bounds are all optimal.

This theorem is "non-radial" in the sense that all *radial* solutions of

$$0 \le -\Delta^m u \quad \text{in} \quad \mathbf{B}^n\backslash\{0\},\ m \ge 1,\ n \ge 2,$$

are harmonically bounded at 0.

To prove the results in this section, we need a representation formula for C^{2m} nonnegative solutions $u(x)$ of

$$-\Delta^m u \geq 0 \quad \text{in} \quad B_2(0)\backslash\{0\} \subset \mathbf{R}^n, \tag{7.3}$$

which extends to $m \geq 2$ the Brezis-Lions representation formula for (7.3) when $m = 1$. We now discuss this extension.

A fundamental solution of Δ^m in $\mathbf{R}^n$ is given by

$$\Gamma(x) = C|x|^{2m-n} \quad or \quad C|x|^{2m-n} \log |x|.$$

If $u(x)$ is a C^{2m} nonnegative solution of (7.3), where $m \geq 1$ and $n \geq 2$ are integers, then

$$\int_{|y|<1} |y|^{2m-2}(-\Delta^m u(y))\, dy < \infty \tag{7.4}$$

but, when $m \geq 2$,

$$\int_{|y|<1} (-\Delta^m u(y))\, dy, \quad \text{and hence} \quad \int_{|y|<1} \Gamma(x-y)\Delta^m u(y)\, dy,$$

may be infinite. So the straight-forward generalization of the Brezis-Lions formula does not work, because the last integral is the natural extension to $m \geq 2$ of the Newtonian potential term in the Brezis-Lions formula. To overcome this difficulty, let $\Psi(x, y)$ be the difference between $\Gamma(x - y)$ and the partial sum of degree $2m - 3$ of the Taylor series of Γ at x. Then for all $x \neq 0$, we have by Taylor's theorem that

$$\Psi(x, y) = O(|y|^{2m-2}) \quad \text{as} \quad y \to 0.$$

Thus by (7.4),

$$N(x) := \int_{|y|<1} \Psi(x, y)\Delta^m u(y)\, dy < \infty \quad \text{for} \quad x \neq 0 \tag{7.5}$$

and one can check that $\Delta^m N = \Delta^m u$. Thus (7.5) is the correct extension to $m \geq 2$ of the Newtonian potential term in the Brezis-Lions formula.

Our polyharmonic extension of the Brezis-Lions representation formula is the following theorem. A similar result was obtained by Futamura and Mizuta.

For $x \neq 0$ and $y \neq x$, let

$$\Psi(x,y) = \Gamma(x-y) - \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Gamma(x)$$

be the difference between $\Gamma(x-y)$ and the partial sum of degree $2m-3$ of the Taylor series of Γ at x.

Theorem 7.3. *Let $u(x)$ be a C^{2m} nonnegative solution of*

$$-\Delta^m u \geq 0 \quad in \quad B_2(0)\backslash\{0\} \subset \mathbf{R}^n$$

where $m \geq 1$ and $n \geq 2$ are integers. Then

$$\int_{|y|<1} |y|^{2m-2}(-\Delta^m u(y))\, dy < \infty$$

and

$$u = N + h + \sum_{|\alpha| \leq 2m-2} a_\alpha D^\alpha \Gamma \quad in \quad B_1(0)\backslash\{0\}$$

where a_α are constants, $h \in C^\infty(B_1(0))$ is a solution of

$$\Delta^m h = 0 \quad in \quad B_1(0),$$

and

$$N(x) = \int_{|y|<1} \Psi(x,y) \Delta^m u(y)\, dy \quad for \quad x \neq 0.$$

8. Systems

We study the behavior near the origin of C^2 positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{aligned} 0 \leq -\Delta u \leq f(v) \\ 0 \leq -\Delta v \leq g(u) \end{aligned} \quad \text{in} \quad \mathbf{B}^2 \backslash \{0\},$$

where $f, g : (0, \infty) \to (0, \infty)$ are continuous functions.

We say such a function f is *exponential bounded* at ∞ if

$$\log^+ f(t) = O(t) \quad \text{as} \quad t \to \infty.$$

There are three possiblities to consider:

(i) f and g are both exponentially bounded at ∞;
(ii) neither f nor g is exponentially bounded at ∞;
(iii) one and only one of the functions f and g is exponentially bounded at ∞.

The following three results [Ghergu, T, Verbitsky] deal with these three possiblities.

By the following theorem, if the functions f and g are both exponentially bounded at ∞ then all positive solutions u and v are harmonically bounded at 0.

Theorem 8.1. *Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system*

$$\begin{aligned} 0 \le -\Delta u \le f(v) \\ 0 \le -\Delta v \le g(u) \end{aligned} \quad \textit{in} \quad \mathbf{B}^2\backslash\{0\},$$

where $f, g : (0,\infty) \to (0,\infty)$ are continuous and exponentially bounded at ∞. Then both u and v are harmonically bounded, that is

$$u(x), v(x) = O\left(\log\frac{1}{|x|}\right) \quad \textit{as} \quad x \to 0.$$

This bound for u and v is optimal

By the following theorem, it is *essentially* the case that if neither of the functions f and g is exponentially bounded at ∞ then neither of the positive solutions u and v satisfies an apriori pointwise bound at 0.

Theorem 8.2. *Suppose $f, g : (0,\infty) \to (0,\infty)$ are continuous functions satisfying*

$$\lim_{t\to\infty} \frac{\log f(t)}{t} = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\log g(t)}{t} = \infty.$$

Let $h : (0,1) \to (0,\infty)$ be a continuous function satisfying $\lim_{r\to 0^+} h(r) = \infty$. Then there exist C^2 positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{aligned} 0 \le -\Delta u \le f(v) \\ 0 \le -\Delta v \le g(u) \end{aligned} \quad \text{in} \quad \mathbf{B}^2\backslash\{0\},$$

such that

$$u(x) \ne O(h(|x|)) \quad \text{as } x \to 0$$

and

$$v(x) \ne O(h(|x|)) \quad \text{as } x \to 0.$$

By the following theorem, if at least one of the functions f and g is exponentially bounded at ∞ then at least one of the positive solutions u and v is harmonically bounded at 0.

Theorem 8.3. *Suppose $u(x)$ and $v(x)$ are C^2 positive solutions of the system*

$$\begin{aligned} 0 &\le -\Delta u \\ 0 &\le -\Delta v \le g(u) \end{aligned} \qquad \text{in} \quad \mathbf{B}^2 \backslash \{0\},$$

where $g : (0,\infty) \to (0,\infty)$ is continuous and exponentially bounded at ∞. Then v is harmonically bounded, that is

$$v(x) = O\left(\log \frac{1}{|x|}\right) \qquad \text{as } x \to 0.$$

If, in addition,

$$-\Delta u \le f(v) \quad \text{in} \quad \mathbf{B}^2 \backslash \{0\},$$

where $f : (0,\infty) \to (0,\infty)$ is a continuous function satisfying

$$\log^+ f(t) = O(t^\lambda) \quad \text{as } t \to \infty$$

for some $\lambda > 1$ then

$$u(x) = o\left(\left(\log \frac{1}{|x|}\right)^\lambda\right) \qquad \text{as } x \to 0.$$

Note that in these theorems we impose no conditions on the growth of $f(t)$ (or $g(t)$) as $t \to 0^+$. Also, all bounds given in these three theorems are optimal. Similar results hold in three and higher dimensions.

9. Parabolic inequalities

It is not hard to prove that if u is a nonnegative solution of the heat equation

$$u_t - \Delta u = 0 \qquad \text{in} \quad \Omega \times (0,1), \tag{1}$$

where Ω is an open subset of $\mathbf{R}^n$, $n \geq 1$, then for each compact subset K of Ω, we have

$$\max_{x \in K} u(x,t) = O(t^{-n/2}) \qquad \text{as} \quad t \to 0^+. \tag{2}$$

The exponent $-n/2$ in (2) is optimal because the Gaussian

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

is a nonnegative solution of the heat equation in $\mathbf{R}^n \times \mathbf{R} - (0,0)$ and

$$G(0,t) = (4\pi t)^{-n/2} \qquad \text{for} \quad t > 0.$$

Do similar results hold for nonnegative solutions u of the differential inequatities

$$0 \leq u_t - \Delta u \leq f(u) \qquad \text{in} \quad \Omega \times (0,1), \tag{3}$$

where $f : [0,\infty) \to (0,\infty)$ is a given continuous function? Note that solutions of the heat equation satisfy (3). By the following theorem, the estimate (2) remains true provided

$$f(s) = O(s^{(n+2)/n}) \qquad \text{as} \quad s \to \infty.$$

Theorem 9.1. *Suppose $u(x,t)$ is a $C^{2,1}$ nonnegative solution of*

$$0 \le u_t - \Delta u \le u^{\frac{n+2}{n}} + 1 \qquad \textit{in} \quad \Omega \times (0,1), \tag{4}$$

where Ω is an open subset of $\mathbf{R}^n$, $n \ge 1$. Then, for each compact subset K of Ω, we again have

$$\max_{x \in K} u(x,t) = O(t^{-n/2}) \qquad \textit{as} \quad t \to 0^+. \tag{5}$$

One of the main accomplishments of this paper is the proof of Theorem 9.1 when the nonlinear term on the right side of (4) is $u^{\frac{n+2}{n}}$. When the nonlinear term is u^λ, $\lambda < \frac{n+2}{n}$, the proof of Theorem 9.1 is much easier.

Theorem 1 is optimal in two ways. First, as before, the exponent $-n/2$ in (5) is optimal because the Gaussian $G(x,t)$ is a solution of (4) and

$$G(0,t) = (4\pi t)^{-n/2} \qquad \text{for} \quad t > 0.$$

And second, the exponent $\frac{n+2}{n}$ on u in (4) cannot be increased by the following theorem.

Theorem 9.2. *Let $\lambda > \frac{n+2}{n}$ and let $\varphi(t)$, $0 < t \leq 1$, be any large positive continuous function. Then there exists a $C^{2,1}$ nonnegative solution $u(x,t)$ of*

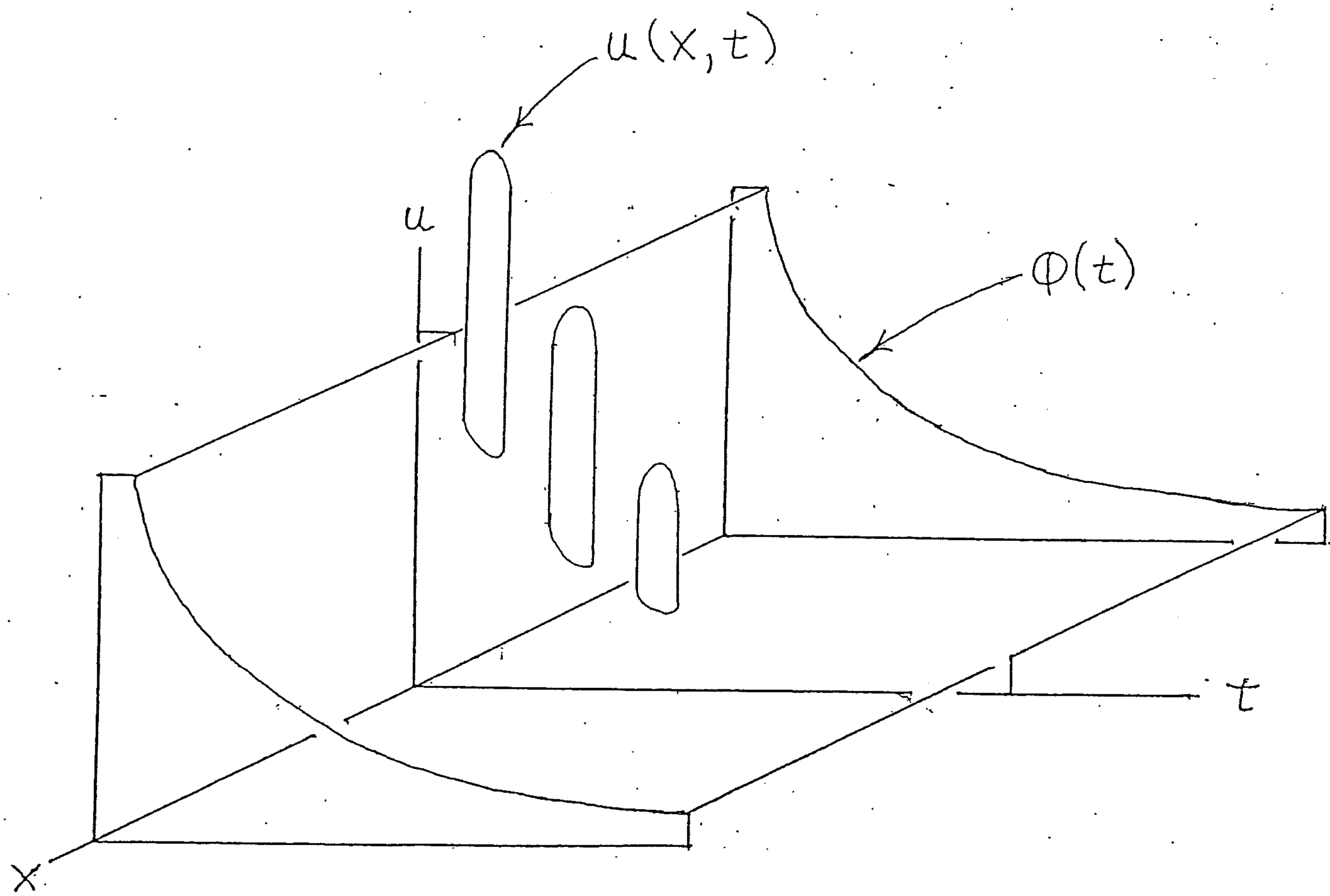
$$0 \leq u_t - \Delta u \leq u^\lambda \qquad \text{in} \quad (\mathbf{R}^n \times \mathbf{R}) - \{(0,0)\} \qquad (6)$$

satisfying $u \equiv 0$ in $\mathbf{R}^n \times (-\infty, 0)$ and

$$u(0,t) \neq O(\varphi(t)) \quad \text{as} \quad t \to 0^+.$$

Actually, Theorem 9.2 is true if the nonlinear term u^λ in (6) is replaced with the smaller term

$$u^{\frac{n+2}{n}}(\log(1+u))^\beta, \quad \beta > 2/n.$$



For the proofs of some of my results for parabolic inequalities, I prove and use a

Parabolic Brezis-Lions Lemma. *Let u be a $C^{2,1}$ solution of*

$$\begin{aligned} u_t - \Delta u &\geq 0 \\ u &\geq 0 \end{aligned} \qquad \text{in} \quad B_3(0) \times (0,3).$$

Then for some finite positive Borel measure μ on $B_2(0)$ and some solution of h of

$$h_t - \Delta h = 0 \quad \text{in} \quad B_1(0) \times (-1,1)$$

we have

$$u = N + v + h \quad \text{in} \quad B_1(0) \times (0,1)$$

where

$$N(x,t) = \int_0^2 \int_{|y|<2} G(x-y, t-s)(u_t - \Delta u)(y,s)\, dy\, ds,$$

$$v(x,t) = \int_{|y|<2} G(x-y,t)\, d\mu(y),$$

and G is the Gaussian.