LECTURE 11: UNDETERMINED COEFFICIENTS (II) 
+ VARIATION OF PARAMETERS

1. Resonance (section 3.5)

Here is an important exception to undetermined coefficients:

**Example 1:**

\[ y'' - 5y' + 6y = e^{2t} \]

**STEP 1: Homogeneous Solution**

\[ r^2 - 5r + 6 = 0 \Rightarrow r = 2 \text{ or } r = 3 \Rightarrow y_0 = Ae^{2t} + Be^{3t} \]

**STEP 2: Particular Solution**

From last time, our guess is \( y_p = Ae^{2t} \)

**BUT** this doesn’t work, since \( Ae^{2t} \) is already part of the homogeneous solution! Plugging it in would just give you 0.

To fix this, guess \( y_p = Ate^{2t} \)

\[
(y_p)' = Ae^{2t} + 2Ate^{2t} \\
(y_p)'' = (Ae^{2t} + 2Ate^{2t})' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} = 4Ae^{2t} + 4Ate^{2t}
\]
\[(y_p)'' - 5 (y_p)' + 6 (y_p) = e^{2t}\]
\[4Ae^{2t} + 4Ate^{2t} - 5 (Ae^{2t} + 2Ate^{2t}) + 6 (Ate^{2t}) = e^{2t}\]
\[4Ae^{2t} + 4Ate^{2t} - 5 Ae^{2t} + 10Ate^{2t} + 6Ate^{2t} = e^{2t}\]
\[-Ae^{2t} = e^{2t}\]
\[A = -1\]

\[y_p = Ate^{2t} = -te^{2t}\]

**STEP 3: General Solution**

\[y = Ae^{2t} + Be^{3t} - te^{2t}\]

**Application:** This “exception” is important in real life and is called resonance. It explains, for example, why bridges collapse. See more at the end of the chapter.

**Example 2:**

Guess the form of the particular solution

(a) \(y'' + 3y' - 4y = e^{2t}\)

**Aux:** \(r^2 + 3r - 4 = 0 \Rightarrow (r - 1)(r + 4) = 0 \Rightarrow r = 1 \text{ or } r = -4\)

\[y_0 = Ae^t + Be^{-4t}\]

\(e^{2t}\) (right-hand-side) corresponds to \(r = 2\), which does not coincide, so
\[ y_p = A e^{2t} \]

(b) \[ y'' + 3y' - 4y = e^t \]

\( e^t \) corresponds to \( r = 1 \), which coincides, so \( y_p = Ate^t \) (resonance)

(c) \[ y'' + 3y' - 4y = t^2 e^{-4t} \]

\( e^{-4t} \) corresponds to \( r = -4 \), which coincides, so

\[ y_p = t(At^2 + Bt + C)e^{-4t} \]

The \( t^2 \) term plays no role here.

(d) \[ y'' + 3y' - 4y = e^{-4t} \cos(t) \]

You have to look at \( e^{-4t} \cos(t) \) as a whole. It corresponds with \( -4 + i \), which does **not** coincide, so

\[ y_p = Ae^{-4t} \cos(t) + Be^{-4t} \sin(t) \]

(e) \[ y'' - 4y' + 13y = e^{2t} \]

\[ r^2 - 4r + 13 = 0 \Rightarrow r = 2 \pm 3i \]

\[ y_0 = Ae^{2t} \cos(3t) + Be^{2t} \sin(3t) \]

\( e^{2t} \) corresponds to \( r = 2 \), which does not coincide, so \( y_p = Ae^{2t} \)

Same for \[ y'' - 4y' + 13y = \cos(3t) \]
(f) $y'' - 4y' + 13y = e^{2t} \sin(3t)$

e$^{2t} \sin(3t)$ corresponds to $r = 2 \pm 3i$, which coincides, so

$$y_p = Ate^{2t} \cos(3t) + Bte^{2t} \sin(3t)$$

(g) $y'' - 4y' + 4y = e^{2t}$

**Aux:** $r^2 - 4r + 4 = 0 \Rightarrow r = 2$ (double root)

$$y = Ae^{2t} + Bte^{2t}$$

e$^{2t}$ corresponds to the double root $r = 2$, so $y_p = At^2e^{2t}$

(h) $y'' - 2y' = t^2$

**Aux:** $r^2 - 2r = 0 \Rightarrow r = 0$ or $r = 2$

Careful, $t^2 = t^2e^{0t}$ corresponds to $r = 0$ which coincides, so

$$y_p = t \left( At^2 + Bt + C \right)$$

**Sums of Terms:** Finally, for sums of terms, like $y'' + 4y = 7e^{3t} + t$, better to guess the particular solutions separately, like $y_p = Ae^{3t}$ and $y_p = At + B$ and then add them up. See end of previous lecture notes.

2. **Interlude: Cramer’s Rule**

**Video:** Cramer’s Rule
Here is a really cool and direct way of solving systems of equations

**Example 3:**

\[
\begin{align*}
\begin{cases}
    x + 2y &= 1 \\
    3x + 4y &= 0
\end{cases}
\end{align*}
\]

**STEP 1:** Write in Matrix form

\[
\begin{bmatrix}
    1 & 2 \\
    3 & 4
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
= 
\begin{bmatrix}
    1 \\
    0
\end{bmatrix}
\]

**STEP 2:** Cramer’s Rule: Gives you \(x\) and \(y\) directly

\[
x = \frac{(1)(4) - (2)(0)}{(1)(4) - (2)(3)} = \frac{4}{4 - 6} = \frac{4}{-2} = -2
\]

The numerator is just the original coefficients, but you replace the first column by the right hand side \(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\).

For \(y\), same thing except you replace the *second* column by the right hand side.

\[
y = \frac{(1)(0) - (1)(3)}{(1)(4) - (2)(3)} = \frac{-3}{-2} = \frac{3}{2}
\]

So \(x = -2\) and \(y = \frac{3}{2}\).
Example 4:

\[
\begin{bmatrix}
-5 & 3 \\
3 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
9 \\
-5
\end{bmatrix}
\]

\[x = \frac{9}{-5} - \frac{3}{-1} = \frac{(9)(-1) - 3(-5)}{(-5)(-1) - (3)(3)} = \frac{6}{-4} = -\frac{3}{2}\]

\[y = \frac{-5}{3} - \frac{9}{-5} = \frac{(-5)(-5) - (9)(3)}{-4} = \frac{25 - 27}{-4} = \frac{-2}{-4} = \frac{1}{2}\]

\[x = -\frac{3}{2}, y = \frac{1}{2}\]

3. Variation of Parameters (section 3.6)

Video: [Variation of Parameters](#)

Welcome to the second method for solving inhomogeneous equations. It’s harder to use but it requires no guessing whatsoever. And we’ll see good friend of ours again 😊

Example 5:

Solve using variation of parameters

\[y'' - 5y' + 6y = e^t\]
**STEP 1:** Put into standard form ✓

**STEP 2:** Homogeneous

\[ y_0 = Ae^{2t} + Be^{3t} \]

**STEP 3:** Particular

**Main Idea:** Guess

\[ y_p = u(t)e^{2t} + v(t)e^{3t} \]

With \( u(t) \) and \( v(t) \) two unknown functions

If you plug this into the ODE and do one little simplification, eventually you get the system (see appendix for a derivation)

\[
\begin{align*}
    e^{2t}u'(t) + e^{3t}v'(t) &= 0 \\
    (2e^{2t})u'(t) + (3e^{3t})v'(t) &= e^t
\end{align*}
\]

Which you can rewrite as:

**Var of Par equations: (memorize)**

\[
\begin{bmatrix}
    e^{2t} & e^{3t} \\
    2e^{2t} & 3e^{3t}
\end{bmatrix}
\begin{bmatrix}
    u'(t) \\
    v'(t)
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    e^t
\end{bmatrix}
\]

Notice the left hand side is precisely the Wronskian (matrix) ⊗

**Mnemonic:**

\[
\begin{bmatrix}
    \text{Wronskian Matrix} \\
    \text{Inhomogeneous}
\end{bmatrix}
\begin{bmatrix}
    u' \\
    v'
\end{bmatrix}
= \begin{bmatrix}
    0
\end{bmatrix}
\]
STEP 4: Solve using Cramer’s rule

\[
\begin{bmatrix}
  e^{2t} & e^{3t} \\
  2e^{2t} & 3e^{3t}
\end{bmatrix}
\begin{bmatrix}
  u'(t) \\
  v'(t)
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  e^t
\end{bmatrix}
\]

\[
u'(t) = \frac{\begin{vmatrix}
  0 & e^{3t} \\
  e^{2t} & 3e^{3t}
\end{vmatrix}}{\begin{vmatrix}
  e^{2t} & e^{3t} \\
  2e^{2t} & 3e^{3t}
\end{vmatrix}} = \frac{0 - e^{3t}e^t}{(3e^{3t})e^{2t} - e^{3t}(2e^{2t})} = \frac{-e^{4t}}{e^{5t}} = -e^{-t}
\]

\[
v'(t) = \frac{\begin{vmatrix}
  e^{2t} & 0 \\
  2e^{2t} & e^{3t}
\end{vmatrix}}{\begin{vmatrix}
  e^{2t} & e^{3t} \\
  2e^{2t} & 3e^{3t}
\end{vmatrix}} = \frac{e^{2t}e^t - 0}{e^{5t}} = \frac{e^{3t}}{e^{5t}} = e^{-2t}
\]

STEP 5: Integrate

\[
u'(t) = -e^{-t} \Rightarrow u(t) = \int -e^{-t} dt = e^{-t}
\]

\[
v'(t) = e^{-2t} \Rightarrow v(t) = \int e^{-2t} = -\frac{1}{2} e^{-2t}
\]

\[
y_p(t) = u(t)e^{2t} + v(t)e^{3t} = e^{-t}e^{2t} - \frac{1}{2} e^{-2t}e^{3t} = e^t - \frac{1}{2} e^t = \frac{1}{2} e^t
\]

STEP 6: Answer

\[
y(t) = Ae^{2t} + Be^{3t} + \frac{1}{2} e^t
\]

Remark: Even though undetermined coefficients here is easier, this method works every time and doesn’t require any guessing.
Example 6:

\[ y'' + y = \tan(t) \]

**STEP 1:** Standard form ✓

**STEP 2:** Homogeneous Solution

\[ r^2 + 1 = 0 \Rightarrow r = \pm i \]

\[ y_0 = A\cos(t) + B\sin(t) \]

**STEP 3:** Variation of Parameters

\[ y_p = u(t)\cos(t) + v(t)\sin(t) \]

\[
\begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix}
\begin{bmatrix}
u'(t) \\
v'(t)
\end{bmatrix} =
\begin{bmatrix}
0 \\
\tan(t)
\end{bmatrix}
\]

**STEP 4:** Cramer’s Rule

\[
u'(t) = \frac{\begin{vmatrix}
0 & \sin(t) \\
\tan(t) & \cos(t)
\end{vmatrix}}{\begin{vmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{vmatrix}} = \frac{0 - \sin(t)\tan(t)}{\cos^2(t) + \sin^2(t)} = -\sin(t)\tan(t)\]

\[
v'(t) = \frac{\begin{vmatrix}
\cos(t) & 0 \\
-\sin(t) & \tan(t)
\end{vmatrix}}{1} = \tan(t)\cos(t) = \frac{\sin(t)}{\cos(t)}\cos(t) = \sin(t)\]

**STEP 5:** Integrate
\[ u'(t) = -\sin(t) \tan(t) \Rightarrow u(t) = \int -\sin(t) \tan(t) dt \quad \text{Sorry} \]

\[
\int -\sin(t) \tan(t) = \int -\sin(t) \left( \frac{\sin(t)}{\cos(t)} \right) dt \\
= \int -\frac{\sin^2(t)}{\cos(t)} dt \\
= \int -\left( \frac{1 - \cos^2(t)}{\cos(t)} \right) dt \\
= \int -\frac{1}{\cos(t)} + \cos(t) dt \\
= \int -\sec(t) + \cos(t) dt \\
\]
\[ u(t) = -\ln |\sec(t) + \tan(t)| + \sin(t) \]

\[ v'(t) = \sin(t) \Rightarrow v(t) = \int \sin(t) = -\cos(t) \]

**STEP 6:**

\[ y_p = u(t) \cos(t) + v(t) \sin(t) \]
\[ = (-\ln |\sec(t) + \tan(t)| + \sin(t)) \cos(t) + (-\cos(t) \sin(t)) \]
\[ = -\ln |\sec(t) + \tan(t)| \cos(t) + \frac{\sin(t) \cos(t) - \cos(t) \sin(t)}{\cos(t)} \]
\[ = -\ln |\sec(t) + \tan(t)| \cos(t) \]

\[ y = A \cos(t) + B \sin(t) - \ln |\sec(t) + \tan(t)| \cos(t) \]

Good luck guessing that with undetermined coefficients!
4. Appendix: Why Variation of Parameters Works

In this optional appendix, I explain how to derive the var of par equations:

**Example 7:** \[y'' + y = \tan(t)\]

**STEP 1:** The homogeneous solution is \(y_0 = A\cos(t) + B\sin(t)\) and the idea is to guess
\[ y_p = u(t) \cos(t) + v(t) \sin(t) \]

**STEP 2:** First Equation

\[
(y_p)' = u'(t) \cos(t) + u(t) (-\sin(t)) + v'(t) \sin(t) + v(t) (\cos(t))
\]
\[= \cos(t)u'(t) + \sin(t)v'(t) - u(t) \sin(t) + v(t) \cos(t) \]

Since we’re just looking for one particular solution, it’s ok to simplify our work by assuming that the term in blue is 0. We don’t have to do that, but it will make our life easier.

This gives us our first equation:

\[ \cos(t)u'(t) + \sin(t)v'(t) = 0 \]

And \((y_p)’\) simplifies to

\[ (y_p)' = -u(t) \sin(t) + v(t) \cos(t) \]

**STEP 3:** Second Equation

\[
(y_p)'' = (-u(t) \sin(t) + v(t) \cos(t))'
\]
\[= -u'(t) \sin(t) - u(t) \cos(t) + v'(t) \cos(t) - v(t) \sin(t) \]

\[
(y_p)'' + y_p
\]
\[= -u'(t) \sin(t) - u(t) \cos(t) + v'(t) \cos(t) - v(t) \sin(t) + (u(t) \cos(t) + v(t) \sin(t)) \]
\[= -u'(t) \sin(t) + \cos(t)v'(t) + u(t) (-\cos(t) + \cos(t)) + v(t) (-\sin(t) + \sin(t)) \]
\[= -u'(t) \sin(t) + \cos(t)v'(t) \]

Although not apparent at first sight, the cancellation in the last step actually follows because \cos and \sin are homogeneous solutions
STEP 4:

Therefore \((y_p)'' + (y_p) = \tan(t)\) just becomes

\[
u'(t)(-\sin(t)) + v'(t)(\cos(t)) = \tan(t)
\]

Which is our second equation!

STEP 5: So to summarize, our equations are

\[
\begin{cases}
\cos(t)u'(t) + \sin(t)v'(t) = 0 \\
(-\sin(t))u'(t) + (\cos(t))v'(t) = \tan(t)
\end{cases}
\]

Which becomes

\[
\begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix}
\begin{bmatrix}
u'(t) \\
v'(t)
\end{bmatrix}
= \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}
\]