REVIEW OF THE BOOK "THE CAUCHY-SCHWARZ MASTER CLASS" BY J. MICHAEL STEELE

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According to the book's subtitle, the author offers "An Introduction to the Art of Mathematical Inequalities". The book is written in the style of Pólya's classic "How to solve it". Much attention is paid to presenting possibly the shortest and simplest proofs of various inequalities. In addition, the reader can also learn why certain seemingly natural approaches fail. The book could be a pleasant reading for everyone with a solid real analysis background at undergraduate level, even before reading Pólya-Szegő. In fact, even researchers working on topics close to those in this book can find much to add to their repertoire. The author offers standard discussions as well as less standard ones of well-known classical inequalities such as the Cauchy-Schwarz a.k.a. Bunyakovskii (Cauchy was first, and stated it for sums; Bunyakovskii was next, and stated it for integrals; Schwarz was third, and stated it for arbitrary inner products), Minkowski, Hölder, Rogers, Chebyshev's Order, AM-GM, Power Mean, Jensen, Carleman, Abel, Newton-Maclaurin, van der Corput, Hardy, Hilbert, Muirhead, Young inequalities and some of their applications. However, the book is special for some other reasons. The book deals with many choice problems the solutions to which require more than a routine application of the theoretical background of the given section. An example for this is a discussion of the solution to an American Mathematical Monthly problem proposed by M. Mazur, where it is demonstrated quite transparently that an effective use of Jensen's inequality calls for one to find a function that is convex on the positive real axis and that is never larger than a given function f, see Fig. 6.3. I particularly like Pólya's proof of the celebrated Carleman inequality. Section 10 was also a highlight for me. Proving that π in Hilbert's Inequality is the best possible, while 4 is the best constant in its max version is exceptionally educational. I have learned much from Section 12 about symmetric sums. The proof of the Newton-Maclaurin inequalities presented here is really beautiful. Section 13 is about majorization and Schur convexity, which are viewed by the author as two of the most productive concepts in the theory of inequalities. Indeed, they unify the understanding of many familiar bounds. Since they are not as well known as they should be, they can become one's secret weapon. A nice treatment of Birkhoff's Theorem on doubly stochastic matrices shows connections to areas such as probability and operator theory, while the proof of the "marriage lemma" given by Halmos and Vaughan in 1950 shows relations to discrete mathematics. This "marriage

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lemma" is a cornerstone of the large and active field of matching theory which is beautifully surveyed by Lovász and Plummer (1986).

The historical component of the book is also remarkable. The aim of this paragraph is to illustrate where the presentation of a topic in this book starts and ends. The problem whether it is always possible to write a real polynomial of d variables that is nonnegative on its domain as a sum of squares of some other polynomials of d variables, turns out to be wonderfully rich. When $d = 1$ the answer is yes, and this can be shown rather simply, but it was Hilbert who first proved that this is not always possible when $d \geq 2$, as it was conjectured earlier by Minkowski. Hilbert's proof was long, subtle, and indirect. The first explicit example of a nonnegative polynomial that cannot be written as the sum of the squares of real polynomials was given in 1967, almost eighty years after Hilbert proved the existence of such polynomials. The explicit example this book presents was discovered by T.S. Motzkin. The 17th problem of Hilbert's great list is a direct descendant of Minkowski's conjecture. In this problem Hilbert asked whether every nonnegative real polynomial in d variables has a representation as a sum of squares of ratios of polynomials. This modification of Minkowski's problem makes a difference, and Hilbert's question was answered affirmatively in 1927 by Emil Artin. Artin's solution to Hilbert's 17th problem is now widely considered to be one of the crown jewels of modern algebra, but it is clearly beyond the scope of this book.

Each section contains several interesting and challenging exercises. Even when some of these do not prove to be some of the most important problems of the day, the reader has a chance to learn their short and elegant solutions in the closing section. In this respect the author must have followed Pólya-Szegő. Participants in the 2002 Canadian Mathematics Olympiad may find not only a familiar problem, Exercise 2.4, but an arsenal to prove inequalities which would be far too difficult even in the International Mathematics Olympiad, and not only because of the time constraint. A large mathematics department with a functional graduate program could easily consider to offer a master course based on this book.

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