MARKOV-NIKOLSKII TYPE INEQUALITIES FOR EXPONENTIAL SUMS ON FINITE INTERVALS

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Abstract. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. The collection of all linear combinations of of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}\,.$$

Motivated by a question of Michel Weber (Strasbourg) we prove the following couple of theorems.

Theorem 1. *Let $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_1 = c_1(p,q,a,b) > 0$ and $c_2 = c_2(p,q,a,b)$ depending only on p, q, a, and b such that*

$$c_1 \left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}} \le \sup_{0 \ne P \in E(\Lambda_n)} \frac{\|P\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \le c_2 \left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}}\,.$$

Theorem 2. *Let $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_1 = c_1(p,q,a,b) > 0$ and $c_2 = c_2(p,q,a,b)$ depending only on p, q, a, and b such that*

$$c_1 \left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right)^{1+\frac{1}{q}-\frac{1}{p}} \le \sup_{0 \ne P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \le c_2 \left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right)^{1+\frac{1}{q}-\frac{1}{p}}\,,$$

where the lower bound holds for all $0 < q \le p \le \infty$, while the upper bound holds when $p \ge 1$ and $0 < q \le p \le \infty$.

1. Introduction and Notation

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. The collection of all linear combinations of of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. For a real-valued function f defined on a set A let

$$\|f\|_{L_\infty A} := \|f\|_A := \sup\{|f(x)| : x \in A\}\,,$$

and let

$$\|f\|_{L_p A} := \left(\int_A |f(x)|^p\,dx\right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. Newman's inequality (see [2] and [6] is an essentially sharp Markov-type inequality for $E(\Lambda_n)$ on $[0,1]$ in the case when each λ_j is non-negative.

Theorem 1.1 (Newman's Inequality)**.** *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. Then*

$$\frac{2}{3}\sum_{j=0}^{n} \lambda_j \le \sup_{0 \ne P \in E(\Lambda_n)} \frac{\|P'\|_{(-\infty,0]}}{\|P\|_{(-\infty,0]}} \le 9 \sum_{j=0}^{n} \lambda_j\,.$$

An L_p, $1 \le p \le \infty$, version of the upper bound in Theorem 1.1 is established in [2], [3], and [4].

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Theorem 1.2. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. Let $1 \leq p \leq \infty$. Then*

$$\|Q'\|_{L_p(-\infty,0]} \leq 9 \left(\sum_{j=0}^{n} \lambda_j \right) \|Q\|_{L_p(-\infty,0]}$$

for every $Q \in E(\Lambda_n)$.

The following $L_p[a,b]$, $1 \leq p \leq \infty$, analog of Theorem 1.2 has been established in [1].

Theorem 1.3. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers, $1 \leq p \leq \infty$, $a, b \in \mathbb{R}$, and $a < b$. There is a constant $c_1 = c_1(a,b)$ depending only on a and b such that*

$$\sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_p[a,b]}} \leq c_1 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right).$$

Theorem 1.3 was proved earlier in [4] under the additional assumptions that $\lambda_j \geq \delta j$ for each j with a constant $\delta > 0$ and with $c_1 = c_1(a,b)$ replaced by $c_1 = c_1(a,b,\delta)$ depending only on a, b, and δ. The novelty of Theorem 1.3 was the fact that

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

is an arbitrary set of real numbers, not even the non-negativity of the exponents λ_j is needed.

In [5] the following Nikolskii-Markov type inequality has been proved for $E(\Lambda_n)$ on $(-\infty, 0]$.

Theorem 1.4. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers and $0 < q \leq p \leq \infty$. Let μ be a non-negative integer. There are constants $c_2 = c_2(p,q,\mu) > 0$ and $c_3 = c_3(p,q,\mu)$ depending only on p, q, and μ such that*

$$c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q} - \frac{1}{p}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P^{(\mu)}\|_{L_p(-\infty,0]}}{\|P\|_{L_q(-\infty,0]}} \leq c_3 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q} - \frac{1}{p}},$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ and $\mu \geq 0$, while the upper bound holds when $\mu = 0$ and $0 < q \leq p \leq \infty$, and when $\mu \geq 1$, $p \geq 1$, and $0 < q \leq p \leq \infty$. Also, there are constants $c_2 = c_2(q,\mu) > 0$ and $c_3 = c_3(q,\mu)$ depending only on q and μ such that

$$c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{|P^{(\mu)}(y)|}{\|P\|_{L_q(-\infty,y]}} \leq c_3 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q}}$$

for all $0 < q \leq \infty$, $\mu \geq 1$, and $y \in \mathbb{R}$.

2. New Results

Motivated by a question of Michel Weber (Strasbourg) we prove the following couple of theorems.

Theorem 2.1. *Suppose $0 < q \leq p \leq \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_4 = c_4(p,q,a,b) > 0$ and $c_5 = c_5(p,q,a,b)$ depending only on p, q, a, and b such that*

$$c_4 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q} - \frac{1}{p}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \leq c_5 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q} - \frac{1}{p}}.$$

Theorem 2.2. *Suppose $0 < q \leq p \leq \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_6 = c_6(p,q,a,b) > 0$ and $c_7 = c_7(p,q,a,b)$ depending only on p, q, a, and b such that*

$$c_6 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1+\frac{1}{q}-\frac{1}{p}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \leq c_7 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1+\frac{1}{q}-\frac{1}{p}},$$

where the lower bound holds for all $0 < q \leq p \leq \infty$, while the upper bound holds when $p \geq 1$ and $0 < q \leq p \leq \infty$.

3. Lemmas

Our first lemma can be proved by a simple compactness argument and may be viewed as a simple exercise.

Lemma 3.1. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$, $a < b$. Let w be a not identically 0 continuous function defined on $[a,b]$. Let $q \in (0, \infty]$. Then there exists a $0 \neq T \in E(\Delta_n)$ such that*

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{0 \neq P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}},$$

and there exists a $0 \neq S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{0 \neq P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Our next lemma is an essential tool in proving our key lemmas, Lemmas 3.3 and 3.4.

Lemma 3.2. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$, $a < b < c$. Let $q \in (0, \infty]$. Let T and S be the same as in Lemma 3.1. Then T has exactly n zeros in $[a,b]$ by counting multiplicities. If $\delta_n \geq 0$, then S also has exactly n zeros in $[a,b]$ by counting multiplicities.*

The heart of the proof of our theorems is the following pair of comparison lemmas. The proof of the next couple of lemmas is based on basic properties of Descartes systems, in particular on Descartes' Rule of Signs, and on a technique used earlier by P.W. Smith and Pinkus. Lorentz ascribes this result to Pinkus, although it was P.W. Smith [7] who published it. I have learned about the method of proofs of these lemmas from Peter Borwein, who also ascribes it to Pinkus. This is the proof we present here. Section 3.2 of [2], for instance, gives an introduction to Descartes systems. Descartes' Rule of Signs is stated and proved on page 102 of [2].

Lemma 3.3. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of real numbers satisfying $\delta_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b, c \in \mathbb{R}$, $a < b \leq c$. Let w be a not identically 0 continuous function defined on $[a,b]$. Let $q \in (0, \infty]$. Then*

$$\sup_{0 \neq P \in E(\Delta_n)} \frac{|(P(c)|}{\|Pw\|_{L_q[a,b]}} \leq \sup_{0 \neq P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Under the additional assumption $\delta_n \geq 0$ we also have

$$\sup_{0 \neq P \in E(\Delta_n)} \frac{|(P'(c)|}{\|Pw\|_{L_q[a,b]}} \leq \sup_{0 \neq P \in E(\Gamma_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Lemma 3.4. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of real numbers satisfying $\delta_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b, c \in \mathbb{R}$, $c \leq a < b$. Let w be a not identically 0 continuous function defined on $[a,b]$. Let $q \in (0, \infty]$. Then*

$$\sup_{0 \neq P \in E(\Delta_n)} \frac{|(P(c)|}{\|Pw\|_{L_q[a,b]}} \geq \sup_{0 \neq P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}.$$

Under the additional assumption $\gamma_0 \leq 0$ we also have

$$\sup_{0 \neq P \in E(\Delta_n)} \frac{|(Q'(c)|}{\|Qw\|_{L_q[a,b]}} \geq \sup_{0 \neq P \in E(\Gamma_n)} \frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}}\,.$$

Let $\mathcal{P}_n$ denote the collection of all algebraic polynomials of degree at most n with real coefficients. The following sharp Nikolskii-type inequalities for $\mathcal{P}_n$ hold.

Lemma 3.5. *Let $0 < q < p \leq \infty$, $a, b \in \mathbb{R}$, and $a < b$. Suppose μ is a nonnegative integer. There are constants $c_8 = c_8(p,q,\mu) > 0$ and $c_9 = c_9(p,q,\mu)$ such that*

$$c_8 \left(\frac{n^2}{b-a}\right)^{\mu+1/q-1/p} \leq \sup_{0 \neq P \in \mathcal{P}_n} \frac{\|P^{(\mu)}\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \leq c_9 \left(\frac{n^2}{b-a}\right)^{\mu+1/q-1/p}$$

and

$$c_8 \left(\frac{n^2}{b-a}\right)^{\mu+1/q} \leq \sup_{0 \neq P \in \mathcal{P}_n} \frac{|P^{(\mu)}(y)|}{\|P\|_{L_q[a,b]}} \leq c_9 \left(\frac{n^2}{b-a}\right)^{\mu+1/q}$$

for both $y = a$ and $y = b$.

Lemma 3.5 may be viewed well known as well, yet, it is hard to find a direct reference especially to the lower bounds. So in the next section we present the arguments briefly deriving this lemma from explicitly referenced results.

4. Proofs of the Lemmas

Proof of Lemma 3.1. Since Δ_n is fixed, the proof is a standard compactness argument. We omit the details. □

To prove Lemma 3.2 we need the following two facts. (a) Every $f \in E(\Delta_n)$ has at most n real zeros by counting multiplicities. (b) If $t_1 < t_2 < \cdots < t_m$ are real numbers and $k_1, k_2, \ldots, k_m$ are positive integers such that $\sum_{j=1}^m k_j = n$, then there is a $0 \neq f \in E(\Delta_n)$ having a zero at t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$.

Proof of Lemma 3.2. We prove the statement for T first. Suppose to the contrary that

$$t_1 < t_2 < \cdots < t_m$$

are real numbers in $[a,b]$ such that t_j is a zero of T with multiplicity k_j for each $j = 1, 2, \ldots, m$, $k := \sum_{j=1}^m k_j < n$, and T has no other zeros in $[a,b]$ different from $t_1, t_2, \ldots, t_m$. Let $t_{m+1} := c$ and $k_{m+1} := n - k \geq 1$. Choose an $0 \neq R \in E(\Delta_n)$ such that R has a zero at t_j with multiplicity k_j for each $j = 1, 2, \ldots, m+1$, and normalize so that $T(t)$ and $R(t)$ have the same sign at every $t \in [a,b]$. Let $T_\varepsilon := T - \varepsilon R$. Note that T and R are of the form

$$T(t) = \widetilde{T}(t) \prod_{j=1}^m (t - t_j)^{k_j} \qquad \text{and} \qquad R(t) = \widetilde{R}(t) \prod_{j=1}^m (t - t_j)^{k_j}\,,$$

where both $\widetilde{T}$ and $\widetilde{R}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Hence, if $\varepsilon > 0$ is sufficiently small, then $|T_\varepsilon(t)| < |T(t)|$ at every $t \in [a,b] \setminus \{t_1, t_2, \ldots, t_m\}$, so

$$\|T_\varepsilon w\|_{L_q[a,b]} < \|Tw\|_{L_q[a,b]}\,.$$

This, together with $T_\varepsilon(c) = T(c)$, contradicts the maximality of T.

Now we prove the statement for S. Without loss of generality we may assume that $S'(c) > 0$. Suppose to the contrary that $t_1 < t_2 < \cdots < t_m$ are real numbers in $[a,b]$ such that t_j is a zero of

S with multiplicity k_j for each $j = 1, 2, \dots, m$, $k := \sum_{j=1}^m k_j < n$, and S has no other zeros in $[a, b]$ different from $t_1, t_2, \dots, t_m$. Choose a

$$0 \neq Q \in \operatorname{span}\{e^{\delta_{n-k}t}, e^{\delta_{n-k+1}t}, \dots, e^{\delta_n t}\} \subset E(\Delta_n)$$

such that Q has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m$, and normalize so that $S(t)$ and $Q(t)$ have the same sign at every $t \in [a, b]$. Note that S and Q are of the form

$$S(t) = \widetilde{S}(t) \prod_{j=1}^m (t - t_j)^{k_j} \qquad \text{and} \qquad Q(t) = \widetilde{Q}(t) \prod_{j=1}^m (t - t_j)^{k_j}\,,$$

where both $\widetilde{S}$ and $\widetilde{Q}$ are continuous functions on $[a, b]$ having no zeros on $[a, b]$. Let $t_{m+1} := c$ and $k_{m+1} := 1$. Choose an

$$0 \neq R \in \operatorname{span}\{e^{\delta_{n-k-1}t}, e^{\delta_{n-k}t}, \dots, e^{\delta_n t}\} \subset E(\Delta_n)$$

such that R has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m+1$, and normalize so that $S(t)$ and $R(t)$ have the same sign at every $t \in [a, b]$. Note that S and R are of the form

$$S(t) = \widetilde{S}(t) \prod_{j=1}^m (t - t_j)^{k_j} \qquad \text{and} \qquad R(t) = \widetilde{R}(t) \prod_{j=1}^m (t - t_j)^{k_j}\,,$$

where both $\widetilde{S}$ and $\widetilde{R}$ are continuous functions on $[a, b]$ having no zeros on $[a, b]$. Since $\delta_n \geq 0$, it is easy to see that $Q'(c)R'(c) < 0$, so the sign of $Q'(c)$ is different from the sign of $R'(c)$. Let $U := Q$ if $Q'(c) < 0$, and let $U := R$ if $R'(c) < 0$. Let $S_\varepsilon := S - \varepsilon U$. Hence, if $\varepsilon > 0$ is sufficiently small, then $|S_\varepsilon(t)| < |T(t)|$ at every $t \in [a, b] \setminus \{t_1, t_2, \dots, t_m\}$, so

$$\|S_\varepsilon w\|_{L_q[a,b]} < \|Sw\|_{L_q[a,b]}\,.$$

This, together with $S'_\varepsilon(c) > S'(c) > 0$, contradicts the maximality of S. □

Proof of Lemma 3.3. We prove the first inequality first. We may assume that $a < b < c$. The general case when $a < b \leq c$ follows by a standard continuity argument. Let $k \in \{0, 1, \dots, n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n\,, \qquad \gamma_j = \delta_j\,, \quad j \neq k\,, \qquad \text{and} \qquad \delta_k < \gamma_k < \delta_{k+1}$$

(let $\delta_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 3.1 and 3.2, there is a $0 \neq T \in E(\Delta_n)$ such that

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{0 \neq P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}\,,$$

where T has exactly n zeros in $[a, b]$ by counting multiplicities. Denote the distinct zeros of T in $[a, b]$ by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of T with multiplicity k_j for each $j = 1, 2, \dots, m$, and $\sum_{j=1}^m k_j = n$. Then T has no other zeros in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$. Let

$$T(t) =: \sum_{j=0}^n a_j e^{\delta_j t}\,, \qquad a_j \in \mathbb{R}\,.$$

Without loss of generality we may assume that $T(c) > 0$. We have $T(t) > 0$ for every $t > c$, otherwise, in addition to its n zeros in $[a, b]$ (by counting multiplicities), T would have at least one more zero in (c, ∞), which is impossible. Hence

$$a_n := \lim_{t \to \infty} T(t)e^{-\delta_n t} \geq 0\,.$$

Since $E(\Delta_n)$ is the span of a Descartes system on $(-\infty,\infty)$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j}a_j > 0\,, \qquad j = 0,1,\dots,n\,.$$

Choose $R \in E(\Gamma_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j e^{\gamma_j t}\,, \qquad b_j \in \mathbb{R}\,,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1,2,\dots,m$, and normalize so that $R(c) = T(c)(>0)$ (this $R \in E(\Gamma_n)$ is uniquely determined). Similarly to $a_n \geq 0$ we have $b_n \geq 0$. Since $E(\Gamma_n)$ is the span of a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs yields,

$$(-1)^{n-j}b_j > 0\,, \qquad j = 0,1,\dots,n\,.$$

We have

$$(T-R)(t) = a_k e^{\delta_k t} - b_k e^{\gamma_k t} + \sum_{\substack{j=0\\ j\neq k}}^{n} (a_j - b_j)e^{\delta_j t}\,.$$

Since $T-R$ has altogether at least $n+1$ zeros at $t_1,t_2,\dots,t_m$, and c (by counting multiplicities), it does not have any zero on $\mathbb{R}$ different from $t_1,t_2,\dots,t_m$, and c. Since

$$(e^{\delta_0 t}, e^{\delta_1 t}, \dots, e^{\delta_k t}, e^{\gamma_k t}, e^{\delta_{k+1} t}, \dots, e^{\delta_n t})$$

is a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs implies that the sequence

$$(a_0 - b_0,\;\; a_1 - b_1,\;\; \dots,\;\; a_{k-1} - b_{k-1},\;\; a_k,\;\; -b_k,\;\; a_{k+1} - b_{k+1},\;\; \dots,\;\; a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k}a_k > 0$, this implies that $a_n - b_n < 0$ if $k < n$, and $-b_n < 0$ if $k = n$, so

$$(T-R)(t) < 0\,, \qquad t > c\,.$$

Since each of T, R, and $T-R$ has a zero at t_j with multiplicity k_j for each $j = 1,2,\dots,m$; $\sum_{j=1}^m k_j = n$, and $T-R$ has a sign change (a zero with multiplicity 1) at c, we can deduce that each of T, R, and $T-R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0,1,\dots,m$ with $t_0 := -\infty$ and $t_{m+1} := c$. Hence $|R(t)| \leq |T(t)|$ holds for all $t \in [a,b] \subset [a,c]$ with strict inequality at every t different from $t_1,t_2,\dots,t_m$. Combining this with $R(c) = T(c)$, we obtain

$$\frac{|R(c)|}{\|Rw\|_{L_q[a,b]}} \geq \frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{0\neq P\in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}\,.$$

Since $R \in E(\Gamma_n)$, the first conclusion of the lemma follows from this.

Now we start the proof of the second inequality of the lemma. Although it is quite similar to that of the first inequality, we present the details. We may assume that $a < b < c$ and $\delta_n > 0$. The general case when $a < b \leq c$ and $\delta_n \geq 0$ follows by a standard continuity argument. Let $k \in \{0,1,\dots,n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n\,, \qquad \gamma_j = \delta_j\,, \quad j \neq k\,, \qquad \text{and} \qquad \delta_k < \gamma_k < \delta_{k+1}$$

(let $\delta_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 3.1 and 3.2, there is an $S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{0\neq P\in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}\,,$$

where S has exactly n zeros in $[a,b]$ by counting multiplicities. Denote the distinct zeros of S in $[a,b]$ by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of S with multiplicity k_j for each $j = 1,2,\dots,m$, and $\sum_{j=1}^m k_j = n$. Then S has no other zeros in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$. Let

$$S(t) =: \sum_{j=0}^{n} a_j e^{\delta_j t}\,, \qquad a_j \in \mathbb{R}\,.$$

Without loss of generality we may assume that $S(c) > 0$. Since $\delta_n > 0$, we have $\lim_{t\to\infty} S(t) = \infty$, otherwise, in addition to its n zeros in (a,b), S would have at least one more zero in (c,∞), which is impossible.

Because of the extremal property of S, we have $S'(c) \neq 0$. We show that $S'(c) > 0$. To see this observe that Rolle's Theorem implies that $S' \in E(\Delta_n)$ has at least $n-1$ zeros in $[t_1, t_m]$. If $S'(c) < 0$, then $S(t_m) = 0$ and $\lim_{t\to\infty} S(t) = \infty$ imply that S' has at least 2 more zeros in (t_m,∞) (by counting multiplicities). Thus $S'(c) < 0$ would imply that S' has at least $n+1$ zeros in $[a,\infty)$, which is impossible. Hence $S'(c) > 0$, indeed. Also $a_n := \lim_{t\to\infty} S(t)e^{-\delta_n t} \geq 0$. Since $E(\Delta_n)$ is the span of a Descartes system on $(-\infty.\infty)$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j} a_j > 0\,, \qquad j = 0,1,\dots,n\,.$$

Choose $R \in E(\Gamma_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j e^{\gamma_j t}\,, \qquad b_j \in \mathbb{R}\,,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1,2,\dots,m$, and normalize so that $R(c) = S(c)(>0)$ (this $R \in E(\Gamma_n)$ is uniquely determined). Similarly to $a_n \geq 0$ we have $b_n \geq 0$. Since $E(\Gamma_n)$ is the span of a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs implies that

$$(-1)^{n-j} b_j > 0\,, \qquad j = 0,1,\dots,n\,.$$

We have

$$(S-R)(t) = a_k e^{\delta_k t} - b_k e^{\gamma_k t} + \sum_{\substack{j=0\\ j\neq k}}^{n} (a_j - b_j) e^{\delta_j t}\,.$$

Since $S-R$ has altogether at least $n+1$ zeros at $t_1, t_2, \dots, t_m$, and c (by counting multiplicities), it does not have any zero on $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$, and c. Since

$$(e^{\delta_0 t}, e^{\delta_1 t}, \dots, e^{\delta_k t}, e^{\gamma_k t}, e^{\delta_{k+1} t}, \dots, e^{\delta_n t})$$

is a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs implies that the sequence

$$(a_0 - b_0,\ a_1 - b_1,\ \dots,\ a_{k-1} - b_{k-1},\ a_k,\ -b_k,\ a_{k+1} - b_{k+1},\ \dots,\ a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k} a_k > 0$, this implies that $a_n - b_n < 0$ if $k < n$, and $-b_n < 0$ if $k = n$, so

$$(S-R)(t) < 0\,, \qquad t > c\,.$$

Since each of S, R, and $S-R$ has a zero at t_j with multiplicity k_j for each $j = 1,2,\dots,m$; $\sum_{j=1}^m k_j = n$, and $S-R$ has a sign change (a zero with multiplicity 1) at c, we can deduce that each of S, R, and $S-R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0,1,\dots,m$ with $t_0 := -\infty$ and $t_{m+1} := c$. Hence $|R(t)| \leq |S(t)|$ holds for all $t \in [a,b] \subset [a,c]$ with strict inequality at every t different from $t_1, t_2, \dots, t_m$. Combining this with $0 < S'(c) < R'(c)$ (recall that $R(c) = S(c) > 0$), we obtain

$$\frac{|R'(c)|}{\|Rw\|_{L_q[a,b]}} \geq \frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{0\neq P\in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}\,.$$

Since $R \in E(\Gamma_n)$, the second conclusion of the lemma follows from this. $\square$

Proof of Lemma 3.4. The lemma follows from Lemma 3.3 by the substitution $u = -t$. $\square$

Proof of Lemma 3.5. The upper bound follows as a combination of two results from [2]: Theorem A.4.14 on page 402 and Theorem A.4.4 on page 395. For the lower bound we refer to the lower bound in Theorem 1.4. To get the lower bound of the lemma, we can use the lower bound of Theorem 1.4 with $\delta_j := j$, $j = 0, 1, \dots, n$, and then we use a substitution $x = e^{-t}$. $\square$

Proof of the Theorems

Proof of Theorem 2.1. Since the right Markov-type inequality is available for $E(\Lambda_n)$, the proof of the upper bound of the theorem is pretty simple. For the sake of brevity let

$$M := c_1(a,b)\left(n^2 + \sum_{j=0}^{n} |\lambda_j|\right),$$

where the constant $c_1(a,b)$ is the same as in Theorem 1.3. Let $P \in E(\Lambda_n)$. Choose a point $t_0 \in [a,b]$ such that $|P(t_0)| = \max_{t\in[a,b]} |P(t)|$. Combining the Mean Value Theorem and the Markov-type inequality of Theorem 1.3 (we need only the case $p = \infty$), we obtain that

$$|P(u)| \geq \frac{1}{2} \max_{t\in[a,b]} |P(t)|\,, \qquad u \in I := [a,b] \cap [t_0 - (2M)^{-1}, t_0 + (2M)^{-1}]\,.$$

Hence

$$\|P\|^q_{L_q[a,b]} = \int_a^b |P(t)|^q\,dt \geq (2M)^{-1}\left(\frac{1}{2}\max_{t\in[a,b]}|P(t)|\right)^q,$$

that is,

$$\max_{t\in[a,b]} |P(t)| \leq 2(2M)^{1/q} \|P\|_{L_q[a,b]}\,.$$

Therefore

$$\begin{aligned}\|P\|^p_{L_p[a,b]} &= \int_a^b |P(t)|^{p-q}|P(t)|^q\,dt \leq \left(\max_{t\in[a,b]}|P(t)|\right)^{p-q}\int_a^b |P(t)|^q\,dt \leq \\ &\leq 2^{p-q}(2M)^{(p-q)/q}\|P\|^{p-q}_{L_q[a,b]}\|P\|^q_{L_q[a,b]} \leq 2^{p-q}(2M)^{(p-q)/q}\|P\|^p_{L_q[a,b]}\,,\end{aligned}$$

that is,

$$\|P\|_{L_p[a,b]} \leq 2\cdot(2M)^{1/q-1/p}\|P\|_{L_q[a,b]}\,,$$

which finishes the proof of the upper bound of the theorem.

Now we turn to the proof of the lower bound. In the light of the upper bound of the theorem it is sufficient to prove the lower bound of it only in the case when $p = \infty$. Assume that

$$\lambda_0 < \lambda_1 < \cdots < \lambda_m < 0 \leq \lambda_{m+1} < \lambda_{m+2} < \cdots < \lambda_n\,.$$

We distinguish four cases.

Case 1: $\sum_{j=m+1}^n |\lambda_j| \geq \frac{1}{2}\sum_{j=0}^n |\lambda_j| \geq n^2$. In this case the lower bound of Theorem 1.4 with $\mu = 0$ gives the lower bound of the theorem.

Case 2: $\sum_{j=0}^m |\lambda_j| \geq \frac{1}{2}\sum_{j=0}^n |\lambda_j| \geq n^2$. In this case the lower bound of Theorem 1.4 with $\mu = 0$ gives the lower bound of the theorem after the substitution $u = -t$.

Case 3: $\frac{1}{2}\sum_{j=0}^n |\lambda_j| \leq n^2$ and $m \leq n/2$. We apply Lemma 3.3 with $n - m - 1$ in place of n, with

$$\Gamma_{n-m-1} = \{\gamma_0 < \gamma_1 < \cdots < \gamma_{n-m+1}\} := \{\lambda_{m+1} < \lambda_{m+2} < \cdots < \lambda_n\}\,,$$

$$\Delta_{n-m-1} := \{\delta_0 < \delta_1 < \cdots < \delta_{n-m-1}\}\,, \qquad \delta_j := j\varepsilon\,, \quad j = 0, 1, \dots, n-m-1\,,$$

(if $\varepsilon > 0$ is sufficiently small, then the assumptions are satisfied), and $c := b$. By letting $\varepsilon > 0$ tend to 0, the lower bound of the theorem follows from Lemma 3.5 with $\mu = 0$.

Case 4: $\frac{1}{2}\sum_{j=0}^{n}|\lambda_j| \leq n^2$ and $m \geq n/2$. We apply Lemma 3.4 with m in place of n, with

$$\Delta_m = \{\delta_0 < \delta_1 < \cdots < \delta_m\} := \{\lambda_0 < \lambda_1 < \cdots < \lambda_m\}\,,$$

$$\Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,, \qquad \gamma_j := j\varepsilon\,, \quad j = 0, 1, \ldots, m\,,$$

(if $\varepsilon > 0$ is sufficiently small, then the assumptions are satisfied), and $c := a$. By letting $\varepsilon > 0$ tend to 0, the lower bound of the theorem follows from Lemma 3.5 with $\mu = 0$. □

Proof of Theorem 2.2. The upper bound of the theorem can be obtained by combining Theorem 1.3 and the upper bound of Theorem 2.1.

Now we turn to the proof of the lower bound. In the light of the upper bound of Theorem 2.1 it is sufficient to prove the lower bound of the theorem only in the case when $p = \infty$. Assume that

$$\lambda_0 < \lambda_1 < \cdots < \lambda_m < 0 \leq \lambda_{m+1} < \lambda_{m+2} < \cdots < \lambda_n\,.$$

We distinguish four cases.

Case 1: $\sum_{j=m+1}^{n}|\lambda_j| \geq \frac{1}{2}\sum_{j=0}^{n}|\lambda_j| \geq n^2$. In this case the lower bound of Theorem 1.4 with $\mu = 1$ gives the lower bound of the theorem.

Case 2: $\sum_{j=0}^{m}|\lambda_j| \geq \frac{1}{2}\sum_{j=0}^{n}|\lambda_j| \geq n^2$. In this case the lower bound of Theorem 1.4 with $\mu = 1$ gives the lower bound of the theorem after the substitution $u = -t$.

Case 3: $\frac{1}{2}\sum_{j=0}^{n}|\lambda_j| \leq n^2$ and $m \leq n/2$. We apply Lemma 3.3 with $n-m-1$ in place of n, with

$$\Gamma_{n-m-1} = \{\gamma_0 < \gamma_1 < \cdots < \gamma_{n-m+1}\} := \{\lambda_{m+1} < \lambda_{m+2} < \cdots < \lambda_n\}\,,$$

$$\Delta_{n-m-1} := \{\delta_0 < \delta_1 < \cdots < \delta_{n-m-1}\}\,, \qquad \delta_j := j\varepsilon\,, \quad j = 0, 1, \ldots, n-m-1\,,$$

(if $\varepsilon > 0$ is sufficiently small, then the assumptions are satisfied), and $c := b$. By letting ε tend to 0, the lower bound of the theorem follows from Lemma 3.5 with $\mu = 1$.

Case 4: $\frac{1}{2}\sum_{j=0}^{n}|\lambda_j| \leq n^2$ and $m \geq n/2$. We apply Lemma 3.4 with m in place of n, with

$$\Delta_m = \{\delta_0 < \delta_1 < \cdots < \delta_m\} := \{\lambda_0 < \lambda_1 < \cdots < \lambda_m\}\,,$$

$$\Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,, \qquad \gamma_j := j\varepsilon\,, \quad j = 0, 1, \ldots, m\,,$$

(if $\varepsilon > 0$ is sufficiently small, then the assumptions are satisfied), and $c := a$. By letting ε tend to 0, the lower bound of the theorem follows from Lemma 3.5 with $\mu = 1$. □

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