CHEBYSHEV POLYNOMIALS AND MARKOV–BERNSTEIN TYPE INEQUALITIES FOR RATIONAL SPACES

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Abstract. This paper considers the trigonometric rational system

$$\left\{1, \frac{1 \pm \sin t}{\cos t - a_1}, \frac{1 \pm \sin t}{\cos t - a_2} \dots \right\}$$

on $\mathbb{R}(\text{mod } 2\pi)$ and the algebraic rational system

$$\left\{1, \frac{1}{x - a_1}, \frac{1}{x - a_2}, \dots \right\}$$

on the interval $[-1, 1]$ associated with a sequence of distinct real poles $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$. Chebyshev polynomials for the rational trigonometric system are explicitly found. Chebyshev polynomials of the first and second kinds for the algebraic rational system are also studied, as well as orthogonal polynomials with respect to the weight function $(1 - x^2)^{-1/2}$. Notice that in these situations, the "polynomials" are in fact rational functions. Several explicit expressions for these polynomials are obtained. For the span of these rational systems, an exact Bernstein–Szegő type inequality is proved, whose limiting case gives back the classical Bernstein-Szegő inequality for trigonometric and algebraic polynomials. It gives, for example, the sharp Bernstein–type inequality

$$|p'(x)| \le \frac{1}{\sqrt{1 - x^2}} \sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{|a_k - x|} \max_{y \in [-1,1]} \big|p(y)\big|, \qquad x \in [-1, 1],$$

where p is any real rational function of type (n, n) with poles $a_k \in \mathbb{R} \setminus [-1, 1]$. An asymptotically sharp Markov–type inequality is also established, which is at most a factor of $\frac{n}{n-1}$ away from the best possible result. With proper interpretation of $\sqrt{a_k^2 - 1}$, most of the results are established for $(a_k)_{k=1}^{\infty} \subset \mathbb{C} \setminus [-1, 1]$ in a more general setting.

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§0. Introduction

Let K be either the interval $[-1,1]$ or the unit circle (which is identified as $\mathbb{R} \pmod{2\pi}$ via the mapping $z = e^{it}$). A Chebyshev system $\{u_k\}_{k=0}^N$ on K is a set of $N+1$ continuous functions on K, such that every nontrivial linear combination of them has at most N distinct zeros in K. A Chebyshev polynomial

$$T_N = a_0 u_0 + a_1 u_1 + \cdots + a_N u_N \tag{0.1}$$

for the system is defined by its equioscillation properties (cf. [Ach, Che, DeLo, KaSt, Lor, Riv]). More specifically, when K is the unit circle, N must be even ($N = 2n$, cf. [Lor, p. 26]), T_N has $L^\infty(K)$ norm 1, and it equioscillates N times on K. That is, there are points

$$0 \le x_0 < x_1 < \cdots < x_{N-1} < 2\pi \tag{0.2}$$

so that

$$T_N(x_j) = \pm(-1)^j \max_{x\in K} |T_N(x)| = \pm(-1)^j, \qquad j = 0, 1, 2, \ldots, N-1. \tag{0.3}$$

When K is the interval $[-1,1]$, T_N again has $L^\infty(K)$ norm 1, and it equioscillates $N+1$ times, that is, there are points

$$1 \ge x_0 > x_1 > \cdots > x_N \ge -1 \tag{0.4}$$

so that

$$T_N(x_j) = (-1)^j \max_{x\in K} |T_N(x)| = (-1)^j, \qquad j = 0, 1, 2, \ldots, N. \tag{0.5}$$

In this case the Chebyshev polynomial is also unique and $a_N^{-1} T_N$ can be characterized as the only solution to the extremal problem

$$\min_{\substack{c_j \in \mathbb{R} \\ 0 \le j \le N}} \left\{ \|p\|_{L^\infty[-1,1]} : p = c_0 u_0 + c_1 u_1 + \cdots + c_{N-1} u_{N-1} + c_N u_N,\ c_N = 1 \right\}$$

(cf. [Lor, KaSt]). So the best uniform approximation to u_N by linear combinations of u_0, $u_1, \ldots, u_{N-1}$ is $u_N - a_N^{-1} T_N$.

Typical examples of the Chebyshev systems are the trigonometric system

$$\{1, \cos t, \sin t, \ldots, \cos nt, \sin nt\}, \qquad t \in [0, 2\pi) \tag{0.6}$$

on the unit circle, and the algebraic polynomial system

$$\{1, x, x^2, x^3, \ldots, x^n\}, \qquad x \in [-1,1] \tag{0.7}$$

on the interval $[-1,1]$.

For the trigonometric system (0.6), the Chebyshev polynomials are

$$\cos nt = \left(e^{int}+e^{-int}\right)/2, \qquad \sin nt = \left(e^{int}-e^{-int}\right)/(2i), \tag{0.8}$$

and their linear combinations or shifts

$$V(t)=\cos(nt-\alpha)=\cos\alpha\cos nt+\sin\alpha\sin nt. \tag{0.9}$$

Note that the Chebyshev polynomials for the system (0.6) are not unique. In addition to the equioscillation property, they satisfy various identities

$$(\cos nt)' = -n\sin nt, \quad (\sin nt)' = n\cos nt, \tag{0.10}$$

and

$$\cos^2 nt+\sin^2 nt = 1, \qquad V'(t)^2+n^2V(t)^2 = n^2\max_{\tau\in\mathbb{R}}|V(\tau)|^2, \tag{0.11}$$

where V is a linear combination of $\cos nt$ and $\sin nt$. The Bernstein–Szegő inequality asserts that

$$p'(t)^2+n^2p(t)^2 \le n^2\max_{\tau\in\mathbb{R}}|p(\tau)|^2 \tag{0.12}$$

for all real trigonometric polynomials p of degree at most n, that is, for all p in the real span of (0.6), and the equality holds if and only if p is a linear combination of $\cos nt$ and $\sin nt$.

The Chebyshev polynomial T_n for the system (0.7) on $[-1,1]$ is obtained from a Chebyshev polynomial ($\cos nt$) for the trigonometric system (0.6) by the transformation

$$x=\cos t, \qquad x\in[-1,1],\ t\in[0,\pi], \tag{0.13}$$

and therefore we get

$$T_n(x)=\cos(n\arccos x), \qquad x\in[-1,1]. \tag{0.14}$$

This is the unique Chebyshev polynomial for the algebraic polynomial system (0.7). Indeed, it is easy to verify that T_n equioscillates $n+1$ times on $[-1,1]$, since

$$T_n(x_j)=(-1)^j\max_{-1\le x\le 1}|T_n(x)|=(-1)^j, \qquad x_j=\cos(j\pi/n),\ j=0,1,2,\dots n.$$

The Chebyshev polynomial of the second kind is defined by

$$U_n(x)=\sin((n+1)t)/\sin t, \qquad x=\cos t,\, x\in[-1,1],\ t\in[0,\pi] \tag{0.15}$$

and $(1-x^2)^{1/2}U_n(x)$ satisfies the equioscillation property. The Bernstein–Szegő inequality (0.12) can be converted to the algebraic system (0.7) by the transformation (0.13) and so we have

$$(1-x^2)p'(x)^2 + n^2 p(x)^2 \le n^2 \max_{y\in[-1,1]} |p(y)|^2 \tag{0.16}$$

for all real algebraic polynomials of degree at most n, where the equality holds if and only if p is a constant multiple of T_n in (0.14). This inequality combined with an interpolation formula can be used to obtain the Markov inequality (cf. [Lor, Riv])

$$\max_{x\in[-1,1]} |p'(x)| \le n^2 \max_{x\in[-1,1]} |p(x)| \tag{0.17}$$

for all real algebraic polynomials of degree at most n. Since $T_n'(1) = n^2$, (0.17) is sharp. The Chebyshev polynomials T_n also form an orthogonal system on $[-1,1]$ with respect to the weight function
$(1-x^2)^{-1/2}$. That is,

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = c_n\delta_{m,n}, \qquad n,m = 0,1,2,\ldots, \tag{0.18}$$

where $c_0 = \pi$, and $c_n = \pi/2$ for $n = 1,2,3,\ldots$.

It seems that (0.8) and (0.14) are essentially the only families of Chebyshev polynomials with known explicit expressions. However, explicit formulae for the Chebyshev polynomials for the trigonometric rational system

$$\left\{1, \frac{1\pm\sin t}{\cos t - a_1}, \frac{1\pm\sin t}{\cos t - a_2}, \ldots, \frac{1\pm\sin t}{\cos t - a_n}\right\}, \qquad t\in[0,2\pi) \tag{0.19}$$

and therefore also for the rational system

$$\left\{1, \frac{1}{x-a_1}, \frac{1}{x-a_2}, \frac{1}{x-a_3}, \ldots, \frac{1}{x-a_n}\right\}, \qquad x\in[-1,1] \tag{0.20}$$

with distinct real poles outside $[-1,1]$ are implicitly contained in [Ach, p. 250]. By constructing a finite Blaschke product (which corresponds to e^{int} in (0.8)), we can derive analogue Chebyshev polynomials of the first and second kinds for these systems. We encounter a problem of language that our Chebyshev "polynomials" here are actually rational functions. A pleasant surprise is that almost all properties parallel to (0.8)–(0.18) hold in this case. The classical results are the limiting cases of the results on letting all the poles go to $\pm\infty$.

In this paper, we give several expressions of the Chebyshev polynomials associated with the rational systems with fixed poles $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}\setminus[-1,1]$ and the orthogonal polynomials with respect to the weight function $(1-x^2)^{-1/2}$. A mixed recursion formula is then obtained for the Chebyshev polynomials. Most of the results in this paper are formulated in a more general setting, allowing arbitrary (possibly repeated) poles in $\mathbb{C}\setminus[-1,1]$.

Highlights of this paper include an exact Bernstein–Szegő type inequality for the rational systems (0.19) and (0.20) which generalizes the classical inequality from the trigonometric polynomials to the rational trigonometric functions (and which contains the classical inequality as a limiting case). An asymptotically sharp Markov–type inequality for the rational system (0.20) is also established, which is at most a factor of $\frac{n}{n-1}$ away from best possible.

Chebyshev polynomials are ubiquitous and have numerous applications, ranging from analysis, statistics, numerical methods, to number theory (cf. [Ach, Che, DeLo, GoVa, IsKe, KaSt, Lor, Riv]), and so their rational analogues should also be of interest.

§1. Chebyshev Polynomials of the First and Second Kinds

We are primarily interested in the linear span of (0.20) and its trigonometric counter part obtained with the substitution $x=\cos t$. Denote by $\mathcal{P}_n$ the set of all real algebraic polynomials of degree at most n, and let $\mathcal{T}_n$ be the set of all real trigonometric polynomials of degree at most n. Let

$$\mathcal{P}_n(a_1,a_2,\dots,a_n)=\left\{\frac{p(x)}{\prod_{k=1}^{n}|x-a_k|}:p\in\mathcal{P}_n\right\} \tag{1.1}$$

and

$$\mathcal{T}_n(a_1,a_2,\dots,a_n)=\left\{\frac{p(t)}{\prod_{k=1}^{n}|\cos t-a_k|}:p\in\mathcal{T}_n\right\}, \tag{1.2}$$

where $\{a_k\}_{k=1}^n\subset\mathbb{C}\backslash[-1,1]$ is a fixed set of poles. (This will be an assumption we put on $\{a_k\}_{k=1}^n$ throughout this paper.) When all poles $\{a_k\}_{k=1}^n$ are distinct and real, (1.1) and (1.2) are simply the real span of the following two systems

$$\left\{1,\frac{1}{x-a_1},\frac{1}{x-a_2},\dots,\frac{1}{x-a_n}\right\}, \tag{1.3}$$

and

$$\left\{1,\frac{1\pm\sin t}{\cos t-a_1},\frac{1\pm\sin t}{\cos t-a_2},\dots,\frac{1\pm\sin t}{\cos t-a_n}\right\}, \tag{1.4}$$

respectively.

We can construct the Chebyshev polynomials of the first and second kinds for the spaces $\mathcal{P}_n(a_1, a_2, \dots, a_n)$ and $\mathcal{T}_n(a_1, a_2, \dots, a_n)$ as follows. Given $\{a_k\}_{k=1}^n \subset \mathbb{C}\backslash[-1, 1]$, we define the numbers $\{c_k\}_{k=1}^n$ by

$$a_k = \frac{1}{2}(c_k + c_k^{-1}), \quad |c_k| < 1, \tag{1.5}$$

that is,

$$c_k = a_k - \sqrt{a_k^2 - 1}, \quad |c_k| < 1. \tag{1.6}$$

Note that $(a_k + \sqrt{a_k^2 - 1})(a_k - \sqrt{a_k^2 - 1}) = 1$. In what follows, $\sqrt{a_k^2 - 1}$ will always be defined by (1.5) or (1.6) (this specifies the choice of root). Let $D = \{z \in \mathbb{C} : |z| < 1\}$,

$$M_n(z) = \left(\prod_{k=1}^{n} (z - c_k)(z - \bar{c}_k) \right)^{1/2}, \tag{1.7}$$

where the square root is defined so that $M_n^*(z) = z^n M_n(z^{-1})$ is analytic in a neighborhood of the closed unit disk $\bar{D}$, and let

$$f_n(z) = \frac{M_n(z)}{z^n M_n(z^{-1})}. \tag{1.8}$$

Note that f_n^2 is actually a finite Blaschke product. The Chebyshev polynomials of the first kind for the systems $\mathcal{P}_n(a_1, a_2, \dots, a_n)$ and $\mathcal{T}_n(a_1, a_2, \dots, a_n)$ are defined by

$$T_n(x) = \frac{1}{2}\left(f_n(z) + f_n(z)^{-1}\right), \quad x = \frac{1}{2}(z + z^{-1}), \ |z| = 1, \tag{1.9}$$

and

$$\widetilde{T}_n(t) = T_n(\cos t), \quad t \in \mathbb{R}, \tag{1.10}$$

respectively. While the Chebyshev polynomials of the second kind for these two systems are defined by

$$U_n(x) = \frac{f_n(z) - f_n(z)^{-1}}{z - z^{-1}}, \quad x = \frac{1}{2}(z + z^{-1}), \ |z| = 1, \tag{1.11}$$

and

$$\widetilde{U}_n(t) = U_n(\cos t) \sin t \tag{1.12}$$

(compare with [Ach, pp. 250–251]). As we will see, these Chebyshev polynomials preserve almost all the elementary properties of the classical trigonometric and algebraic Chebyshev polynomials. This is the content of the next three results.

Theorem 1.1. *Let $\widetilde{T}_n$ and $\widetilde{U}_n$ be defined by (1.10) and (1.12) from $\{a_k\}_{k=1}^n \subset \mathbb{C}\setminus[-1,1]$. Then*

(a) $\widetilde{T}_n \in \mathcal{T}_n(a_1, a_2, \dots, a_n)$ *and* $\widetilde{U}_n \in \mathcal{T}_n(a_1, a_2, \dots, a_n)$.

(b) $\max_{-\pi\le t\le\pi} |\widetilde{T}_n(t)| = 1$, *and* $\max_{-\pi\le t\le\pi} |\widetilde{U}_n(t)| = 1$.

(c) *There are* $0 = t_0 < t_1 < \cdots < t_n = \pi$ *so that*

$$\widetilde{T}_n(t_j) = \widetilde{T}_n(-t_j) = (-1)^j, \quad j = 0, 1, 2, \dots, n.$$

(d) *There are* $0 < s_1 < s_2 < \cdots < s_n < \pi$ *so that*

$$\widetilde{U}_n(s_j) = -\widetilde{U}_n(-s_j) = (-1)^{j-1}, \quad j = 1, 2, \dots, n.$$

(e) $\widetilde{T}_n(t)^2 + \widetilde{U}_n(t)^2 = 1$ *holds for every* $t \in \mathbb{R}$.

Proof. It is easy to see that there are polynomials $p_1 \in \mathcal{P}_n$, $p_2 \in \mathcal{P}_n$ and $p_3 \in \mathcal{P}_{n-1}$ so that

$$\widetilde{T}_n(t) = T_n(\cos t) = \frac{1}{2}\frac{e^{-int}M_n^2(e^{it}) + e^{int}M_n^2(e^{-it})}{M_n(e^{it})M_n(e^{-it})} = \frac{p_1(\cos t)}{\prod_{k=1}^n |\cos t - a_k|} \tag{1.13}$$

and

$$\begin{aligned}\widetilde{U}_n(t) = U_n(\cos t)\sin t &= \frac{e^{-int}M_n^2(e^{it}) - e^{int}M_n^2(e^{-it})}{2iM_n(e^{it})M_n(e^{-it})} \\ &= \frac{p_2(\sin t)}{\prod_{j=1}^k |\cos t - a_j|} = \frac{p_3(\cos t)\sin t}{\prod_{j=1}^k |\cos t - a_j|},\end{aligned} \tag{1.14}$$

thus (a) is proved. Since $|c_k| < 1$ and f_n^2 is a finite Blaschke product (cf. (1.8)), we have

$$|f_n(z)| = 1 \quad \text{whenever } |z| = 1. \tag{1.15}$$

Now (b) follows immediately from (1.8) – (1.12) and (1.15). Note that $\widetilde{T}_n(t)$ is the real part and $\widetilde{U}_n(t)$ is the imaginary part of $f_n(e^{it})$, that is,

$$f_n(e^{it}) = \widetilde{T}_n(t) + i\,\widetilde{U}_n(t), \qquad t \in \mathbb{R}, \tag{1.16}$$

which, together with (1.15) implies (e). To prove (c) and (d), we first note that $\widetilde{T}_n(t) = \pm 1$ if and only if $f_n(e^{it}) = \pm 1$ and $\widetilde{U}_n(t) = \pm 1$ if and only if $f_n(e^{it}) = \pm i$. Since $|c_k| < 1$, for $k = 1, 2, \dots, n$, f_n^2 has exactly $2n$ zeros in the open unit disk D. Since f_n^2 is analytic in a region containing the closed unit disk $\bar{D}$, (c) and (d) follow by the Argument Principle. □

With the transformation $x = \cos t = (z + z^{-1})/2$, and $z = e^{it}$, Lemma 1.1 can be reformulated as follows.

Theorem 1.2. *Let T_n and U_n be defined by (1.9) and (1.11) from $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$. Then*

(a) $T_n \in P_n(a_1, a_2, \dots, a_n)$ *and* $U_n \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$.
(b) $\max_{-1 \le x \le 1} |T_n(x)| = \max_{-1 \le x \le 1} |\sqrt{1-x^2}\, U_n(x)| = 1$.
(c) *There are* $1 = x_0 > x_1 > \cdots > x_n = -1$ *so that*

$$T_n(x_j) = (-1)^j, \quad j = 0, 1, 2, \dots, n.$$

(d) *There are* $1 > y_1 > y_2 > \cdots > y_n > -1$ *so that*

$$\sqrt{1-y_j^2}\, U_n(y_j) = (-1)^j, \quad j = 1, 2, \dots, n.$$

(e) $(T_n(x))^2 + (\sqrt{1-x^2}\, U_n(x))^2 = 1, x \in [-1,1]$.

Part (d) of Theorems 1.1 and 1.2 is the equioscillation property of the Chebyshev polynomials, which extends to linear combinations of Chebyshev polynomials. In the polynomial case this is the fact that $\cos\alpha \cos nt + \sin\alpha \sin nt = \cos(nt - \alpha)$ equioscillates $2n$ times on the unit circle $[0, 2\pi]$. Our next theorem characterizes the Chebyshev polynomials of $\mathcal{T}_n(a_1, a_2, \dots, a_n)$ and record a monotonicity property of them.

Theorem 1.3. *Let $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$. Then (i) and (ii) below are equivalent.*

(i) *There is an* $\alpha \in \mathbb{R}$ *so that*

$$V = \cos\alpha\, \widetilde{T}_n + \sin\alpha\, \widetilde{U}_n,$$

where $\widetilde{T}_n$ *and* $\widetilde{U}_n$ *are defined by (1.10) and (1.12).*

(ii) $V \in \mathcal{T}_n(a_1, a_2, \dots, a_n)$ *has supremum norm* 1 *on the unit circle and it equioscillates* $2n$ *times on the unit circle. That is, there are* $0 \le t_0 < t_1 < t_2 \cdots < t_{2n-1} < 2\pi$ *so that*

$$V(t_j) = \pm(-1)^j, \qquad j = 0, 1, 2, \dots, 2n-1.$$

Furthermore, if V is of the form in (i) (or characterized by (ii)), then

(iii) $V' = \cos\alpha\, \widetilde{T}_n' + \sin\alpha\, \widetilde{U}_n'$ *is strictly positive or strictly negative between two consecutive points of equioscillation, that is, between* t_{j-1} *and* t_j*, for* $j = 1, 2, \dots, 2n-1$*, and between* t_{2n-1} *and* $2\pi + t_0$.

Proof. (i) $\implies$ (ii). By Theorem 1.1 (e) and Cauchy's inequality, we have

$$|\cos\alpha\, \widetilde{T}_n + \sin\alpha\, \widetilde{U}_n|^2 \le (\cos^2\alpha + \sin^2\alpha)(\widetilde{T}_n^2 + \widetilde{U}_n^2) = 1 \tag{1.17}$$

on the real line. From Theorem 1.1 (c) and (d), we have that $\widetilde{T}_n/\widetilde{U}_n$ oscillates between $+\infty$ and $-\infty$ exactly $2n$ times on the unit circle, and hence it takes the value $\cot\alpha$ exactly $2n$ times. At each such point, (1.17) becomes equality, namely, $\cos\alpha\,\widetilde{T}_n + \sin\alpha\,\widetilde{U}_n = \pm 1$, and the signs change for every two consecutive such points.

(ii) $\implies$ (i). Let V be as specified in part (ii) of the theorem. Let t_* be a point where V achieves its maximum on $\mathbb{R}$, so $V(t_*) = 1$. We want to show that V is equal to $p = \widetilde{T}_n(t_*)\widetilde{T}_n + \widetilde{U}_n(t_*)\widetilde{U}_n$. In fact, $V(t_*) = p(t_*) = 1$ and $V'(t_*) = p'(t_*) = 0$, that means that $V - p$ has a double zero at t_*. There are at least $2n-1$ more zeros (we count every zero without sign change twice) of $V - p$, with one between each pair of consecutive points of equioscillation of p if the first zero of p to the right of t_* is greater than the first zero of V to the right of t_*. (If the first zero of V to the right of t_* is greater than the first zero of p to the right of t_*, then there will be one zero of $p - V$ between each pair of consecutive points of equioscillation of V.) This implies that $V - p$ has at least $2n+1$ zeros (counting multiplicities), proving that $V - p \equiv 0$.

(iii) Let $V \in \mathcal{T}_n(a_1, a_2, \ldots, a_n)$ be so that $\|V\|_{L^\infty(\mathbb{R})} = 1$ and V equioscillates $2n$ times between ± 1. If there is a $t_* \in [0, 2\pi)$ so that $|V(t_*)| < 1$ and $V'(t_*) = 0$. Then there is a trigonometric polynomial q of degree n, so that

$$V(t) - V(t_*) = \frac{q(t)}{\prod_{k=1}^{n} |\cos t - a_k|}.$$

Since q has the same sign as V at those points of equioscillation, there are at least $2n$ distinct zeros of q in $[0, 2\pi)$. One of these zeros is t_*, where $q'(t_*) = 0$ since $q(t_*) = 0$ and $V'(t_*) = 0$. Hence, by counting multiplicities, q has at least $2n+1$ zeros in $[0, 2\pi)$, so $q \equiv 0$, and this is a contradiction. Therefore $V'(t) \neq 0$ if $|V(t)| < 1$, which means that V is strictly monotone between two consecutive points of equioscillation. $\square$

§2. Derivatives of the Chebyshev Polynomials

In this section we calculate the derivative of the Chebyshev polynomials of the first and second kinds. We also study the identities they satisfy. The similarity to the identities satisfied by $\cos nt$ and $\sin nt$ is striking. These identities will help us to examine the size of $\widetilde{T}_n'$ and $\widetilde{U}_n'$ on $\mathbb{R}$ and the magnitude of T_n' and U_n' on $[-1, 1]$. The results of this section will then be applied in Section 3, where we prove the Bernstein–Szegő type inequalities and the Markov–type inequalities.

As in (1.5) or (1.6), $\{c_k\}_{k=1}^n$ is defined from $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$ by

$$c_k = a_k - \sqrt{a_k^2 - 1}, \quad |c_k| < 1. \tag{2.1}$$

We introduce the functions

$$B_n(x) = \sum_{k=1}^{n} \Re \frac{\sqrt{a_k^2 - 1}}{a_k - x} \qquad \text{and} \qquad \widetilde{B}_n(t) = B_n(\cos t) = \sum_{k=1}^{n} \Re \frac{\sqrt{a_k^2 - 1}}{a_k - \cos t}, \tag{2.2}$$

where the choice of $\sqrt{a_k^2 - 1}$ is determined by the restriction $|c_k| < 1$ in (2.1). Because of their role in the Bernstein–type inequalities, we call B_n and $\widetilde{B}_n$ the Bernstein factors. Note that

$$B_n(x) = \sum_{k=1}^{n} \Re \frac{\sqrt{a_k^2 - 1}}{a_k - x} = \sum_{k=1}^{n} \Re \frac{c_k^{-1} - c_k}{(c_k^{-1} + c_k)/2 - x} \geq \sum_{k=1}^{n} \frac{(1 - |c_k|^2)(1 - |c_k|)^2}{|1 + c_k^2 - 2c_k x|^2} > 0$$

for every $x \in [-1, 1]$.

The following theorem generalizes the trigonometric identities $(\cos nt)' = -n \sin nt$, $(\sin nt)' = n \cos nt$, and $[(\cos nt)']^2 + [(\sin nt)']^2 = n^2$, which are limiting cases (note that if $n \in \mathbb{N}$ and $t \in \mathbb{R}$ are fixed, then $\lim \widetilde{B}_n(t) = n$ as all $a_k \to \pm\infty$).

Theorem 2.1. *Let $\widetilde{T}_n$ and $\widetilde{U}_n$ be determined from $\{a_k\}_{k=1}^n$ by (1.10) and (1.12). Then*

$$\widetilde{T}_n'(t) = -\widetilde{B}_n(t)\widetilde{U}_n(t), \quad \widetilde{U}_n'(t) = \widetilde{B}_n(t)\widetilde{T}_n(t), \qquad t \in \mathbb{R} \tag{2.3}$$

and

$$\widetilde{T}_n'(t)^2 + \widetilde{U}_n'(t)^2 = \widetilde{B}_n(t)^2, \qquad t \in \mathbb{R}, \tag{2.4}$$

where the Bernstein factor $\widetilde{B}_n$ is defined by (2.2).

Proof. If we differentiate the Chebyshev polynomials of the first kind (cf. (1.7)–(1.10)), we get

$$\begin{aligned} \widetilde{T}_n'(t) &= \frac{1}{2}\left(f_n'(e^{it}) - \frac{f_n'(e^{it})}{f_n^2(e^{it})} \right) ie^{it} \\ &= -\frac{e^{it} f_n'(e^{it})}{f_n(e^{it})} \frac{f_n(e^{it}) - f_n^{-1}(e^{it})}{2i} = -\widetilde{B}_n(t)\widetilde{U}_n(t), \end{aligned}$$

since

$$\begin{aligned} \frac{e^{it} f_n'(e^{it})}{f_n(e^{it})} &= e^{it} \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{e^{it} - c_k} + \frac{1}{e^{it} - \bar{c}_k} - \frac{1}{e^{it} - c_k^{-1}} - \frac{1}{e^{it} - \bar{c}_k^{-1}} \right) \\ &= \frac{1}{2} \sum_{k=1}^{n} \frac{e^{it}(c_k - c_k^{-1})}{(e^{it} - c_k)(e^{it} - c_k^{-1})} + \frac{1}{2} \sum_{k=1}^{n} \frac{e^{it}(\bar{c}_k - \bar{c}_k^{-1})}{(e^{it} - \bar{c}_k)(e^{it} - \bar{c}_k^{-1})} \\ &= \frac{1}{2} \sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{a_k - \cos t} + \frac{1}{2} \sum_{k=1}^{n} \frac{\sqrt{\bar{a}_k^2 - 1}}{\bar{a}_k - \cos t} = \widetilde{B}_n(t) \end{aligned}$$

(cf. the definition in (2.2)). Note that in the last step, we have used the relation $c_k - c_k^{-1} = 2\sqrt{a_k^2-1}$ and $c_k + c_k^{-1} = 2a_k$. This proves the first part of Theorem 2.1. Similarly, for the derivative of the Chebyshev polynomials of the second kind, we have

$$\widetilde{U}_n'(t) = \frac{1}{2i}\left(f_n'(e^{it}) + \frac{f_n'(t)}{f_n^2(e^{it})}\right) ie^{it} = \frac{e^{it}f_n'(e^{it})}{f_n(e^{it})}\widetilde{T}_n(t) = \widetilde{B}_n(t)\widetilde{T}_n(t),$$

and (2.4) follows from (2.3) and the identity $\widetilde{T}_n^2 + \widetilde{U}_n^2 = 1$ (cf. Theorem 1.1(e)). □

The identities (2.3) and (2.4) can be coupled to get two other identities

$$(\widetilde{T}_n')^2 + \widetilde{B}_n^2\widetilde{T}_n^2 = \widetilde{B}_n^2 \quad \text{and} \quad (\widetilde{U}_n')^2 + \widetilde{B}_n^2\widetilde{U}_n^2 = \widetilde{B}_n^2. \tag{2.5}$$

In fact, a similar formula holds for linear combinations of $\widetilde{T}_n$ and $\widetilde{U}_n$, which will be used in the proof of the Bernstein–Szegő type inequality of Theorem 3.1.

Theorem 2.2. *If $V = \cos\alpha\,\widetilde{T}_n + \sin\alpha\,\widetilde{U}_n$ with some $\alpha \in \mathbb{R}$, then*

$$(V')^2 + \widetilde{B}_n^2 V^2 = \widetilde{B}_n^2, \tag{2.6}$$

holds on the real line, where $\alpha \in \mathbb{R}$ and the Bernstein factor $\widetilde{B}_n$ is defined by (2.2).

Proof. Since on the real line we have

$$\begin{aligned}(V')^2 + \widetilde{B}_n^2V^2 &= (\cos\alpha\,\widetilde{T}_n' + \sin\alpha\,\widetilde{U}_n)^2 + \widetilde{B}_n^2(\cos\alpha\,\widetilde{T}_n + \sin\alpha\,\widetilde{U}_n)^2\\ &= \cos^2\alpha\left((\widetilde{T}_n')^2 + \widetilde{B}_n^2\widetilde{T}_n^2\right) + \sin^2\alpha\left((\widetilde{U}_n')^2 + \widetilde{B}_n^2\widetilde{U}_n^2\right)\\ &\qquad + 2\cos\alpha\sin\alpha(\widetilde{T}_n'\widetilde{U}_n' + \widetilde{B}_n^2\widetilde{T}_n\widetilde{U}_n),\end{aligned}$$

the identities (2.3) and (2.5) yield (2.6). □

We now calculate $T_n'(1)$. This will be used in the proof of the Markov–type inequality of Theorem 3.5.

Theorem 2.3. *Let T_n be defined by (1.9). Then*

$$T_n'(1) = \left(\sum_{k=1}^n \Re\frac{1+c_k}{1-c_k}\right)^2 \quad \text{and} \quad T_n'(-1) = (-1)^n\left(\sum_{k=1}^n \Re\frac{1-c_k}{1+c_k}\right)^2,$$

where the numbers c_k, $k = 1, 2, \ldots, n$, are defined by (2.1).

Proof. We prove only the first equality, the proof of the second one is similar. Since $T_n(\cos t) = \widetilde{T}_n(t)$ for every t in $\mathbb{R}$ (cf. (1.10)), by taking the derivative with respect to t, we have $-T_n'(\cos t)\sin t = \widetilde{T}_n'(t) = -\widetilde{B}_n(t)\widetilde{U}_n(t)$ (cf. (2.3)). Hence

$$T_n'(1) = \lim_{t\to 0}\widetilde{B}_n(t)\frac{\widetilde{U}_n(t)}{t}\,\frac{t}{\sin t} = \widetilde{B}_n(0)\widetilde{U}_n'(0),$$

where $\widetilde{U}_n(0)=0$ (cf. Theorem 1.1 (d)) is used. Note also that $\widetilde{U}_n'=\widetilde{B}_n\widetilde{T}_n$ (cf. (2.3)) and $\widetilde{T}_n(0)=1$, so we have

$$T_n'(1)=\widetilde{B}_n^2(0)=\left(\sum_{k=1}^n \Re\frac{\sqrt{a_k^2-1}}{a_k-1}\right)^2=\left(\sum_{k=1}^n \Re\frac{1+c_k}{1-c_k}\right)^2,$$

where we have used the relations $2\sqrt{a_k^2-1}=c_k^{-1}-c_k$ and $2a_k=c_k^{-1}+c_k$ (cf. (2.1) or (1.5)–(1.6)). The derivative $T_n'(-1)$ can be calculated in exactly the same way. □

§3. Bernstein and Markov Type Inequalities

Bernstein and Markov type inequalities play a central role in approximation theory, and have been much studied (cf. [Ach, BoEr, Che, DeLo, DuSc, Lor, PePo, Riv]). In this section, we first prove a sharp Bernstein–Szegő type inequality with the best possible constant for the spaces $\mathcal{T}_n(a_1,a_2,\dots,a_n)$ defined by (1.2), and $\mathcal{P}_n(a_1,a_2,\dots,a_n)$ defined by (1.1). In the case when all the poles are distinct reals outside $[-1,1]$, (1.1) becomes

$$\mathcal{P}_n(a_1,a_2,\dots,a_n)=\operatorname{span}\left\{1,\frac{1}{x-a_1},\dots,\frac{1}{x-a_n}\right\}. \tag{3.1}$$

The limiting case of the Bernstein–Szegő type inequality (letting the poles approach $\pm\infty$) is the classical Bernstein–Szegő inequality. We also establish an asymptotically sharp Markov–type inequality for the same space. (It is at most a factor $\frac{n}{n-1}$ away from the best possible constant.)

Theorem 3.1. *Let $\{a_k\}_{k=1}^n\subset\mathbb{C}\setminus[-1,1]$ and let the Bernstein factor $\widetilde{B}_n$ be defined by (2.2). Then*

$$p'(t)^2+\widetilde{B}_n(t)^2\,p(t)^2\le\widetilde{B}_n(t)^2\max_{\tau\in\mathbb{R}}|p(\tau)|^2,\qquad t\in\mathbb{R}. \tag{3.2}$$

for every p in $\mathcal{T}_n(a_1,a_2,\dots,a_n)$, and equality holds in (3.2) if and only if t is a maximum point of $|p|$ (i.e. $p(t)=\pm\|p\|_{L^\infty(\mathbb{R})}$), or p is a linear combination of $\widetilde{T}_n$ and $\widetilde{U}_n$.

If we drop the second term in the left–hand side of (3.2), we have the Bernstein–type inequality.

Corollary 3.2. *Let $\{a_k\}_{k=1}^n$ be as in Theorem 3.1. Then*

$$|p'(t)|\le\widetilde{B}_n(t)\max_{\tau\in\mathbb{R}}|p(\tau)|,\qquad t\in\mathbb{R} \tag{3.3}$$

for every $p\in\mathcal{T}_n(a_1,a_2,\dots,a_n)$, where the Bernstein factor is defined by (2.2). Equality holds in (3.3) if and only if p is a linear combination of $\widetilde{T}_n$ and $\widetilde{U}_n$ and $p(t)=0$.

Proof of Theorem 3.1. Let $p \in \mathcal{T}_n(a_1, a_2, \ldots, a_n)$ be arbitrary with infinite norm not larger than 1. That is, $0 < ||p||_{L^\infty(\mathbb{R})} < 1$. It is sufficient to show that

$$p'(t)^2 + \widetilde{B}_n(t)^2 p(t)^2 \leq \widetilde{B}_n(t)^2 \tag{3.4}$$

for every fixed $t \in \mathbb{R}$. Then a scaling and limiting process imply that (3.2) holds for p with arbitrary norm. First we claim that for every fixed $t \in \mathbb{R}$ there is an $\alpha \in \mathbb{R}$, so that

$$V = \cos\alpha\, \widetilde{T}_n + \sin\alpha\, \widetilde{U}_n \tag{3.5}$$

has the same value as p at the point t, and their derivative signs at t also match, that is,

$$V(t) = p(t) \quad \text{and} \quad p'(t)V'(t) \geq 0. \tag{3.6}$$

Indeed, since t is fixed, we may view V as a function of α. Let

$$\phi(\alpha) = \cos\alpha\, \widetilde{T}_n(t) + \sin\alpha\, \widetilde{U}_n(t).$$

Then $\phi(\alpha) = \cos(\alpha - \theta)$, where θ is determined by $\cos\theta = \widetilde{T}_n(t)$ and $\sin\theta = \widetilde{U}_n(t)$ (recall that $|\widetilde{T}_n|^2 + |\widetilde{U}_n|^2 \equiv 1$ on $\mathbb{R}$ by Theorem 1.1 (e)). Since $|p(t)| < 1$, $\phi(\alpha)$ takes the value of $p(t)$ twice on every translation of the interval $[0, 2\pi)$. Hence there are α_1 and α_2 in $\mathbb{R}$ so that $\phi(\alpha_1) = \phi(\alpha_2) = p(t)$, and $(\alpha_1 - \theta) + (\alpha_2 - \theta) = 2\pi$. We thus get two linear combinations

$$V_j(\cdot) = \cos\alpha_j \widetilde{T}_n(\cdot) + \sin\alpha_j \widetilde{U}_n(\cdot), \qquad j = 1, 2,$$

such that $V_j(t) = p(t)$, $j = 1$, 2. To see that one of V_1 or V_2 is a suitable choice to satisfy (3.5) and (3.6), it is sufficient to show that $V_1'(t)V_2'(t) < 0$. This can be verified quite easily. Now $V_j'(t) = \cos\alpha_j \widetilde{T}_n'(t) + \sin\alpha_j \widetilde{U}_n'(t)$ and by (2.3) and the choice of θ, we get

$$V_j'(t) = \widetilde{B}_n(t)[-\cos\alpha_j \widetilde{U}_n(t) + \sin\alpha_j \widetilde{T}_n(t)] = \widetilde{B}_n(t)\sin(\alpha_j - \theta).$$

Consequently, $V_1'(t)V_2'(t) = \widetilde{B}_n(t)^2 \sin(\alpha_1 - \theta)\sin(\alpha_2 - \theta)$. Hence $V_1'(t)V_2'(t) = -\widetilde{B}_n(t)^2 \sin^2(\alpha_1 - \theta) = -\widetilde{B}_n(t)^2(1 - p(t)^2) < 0$ since $(\alpha_1 - \theta) + (\alpha_2 - \theta) = 2\pi$ and $|p(t)| < 1$. Therefore there is a real α, so that (3.5) and (3.6) hold. ¿From now on let V be a function of the form (3.5) satisfying (3.6) ($t \in \mathbb{R}$ is fixed). We now prove that

$$|p'(t)| \leq |V'(t)|. \tag{3.7}$$

If the above does not hold, then by Theorem 1.3 (iii) we have, without loss of generality, that $p'(t) > V'(t) > 0$, hence there is a $\delta > 0$ such that $p - V > 0$ on $(t, t + \delta)$ and $p - V < 0$ on $(t - \delta, t)$ since $p(t) - V(t) = 0$. Let t_j and t_{j+1} be the two consecutive equioscillation points of V so that $t_j < t < t_{j+1}$ (cf. Theorem 1.3(iii)). Then $V(t_j) = -1$ and $V(t_{j+1}) = 1$, and so $p - V > 0$ at t_j and $p - V < 0$ at t_{j+1}. Thus, there are 3 zeros of $p - V$ in (t_j, t_{j+1}). It is easy to see that there are $2n-1$ zeros of $p - V$ outside (t_j, t_{j+1}) in a period of length 2π, since $p - V$ has the same sign as V when $V = \pm 1$. This gives rise to $3 + (2n-1) = 2n+2$ zeros of $p - V$ in a period of length 2π, which is a contradiction, since every non-zero element in $\mathcal{T}_n(a_1, a_2, \ldots, a_n)$ has at most $2n$ zeros in an interval of length 2π. This finishes the proof of (3.7).

¿From (3.6), (3.7) and Theorem 2.2, we have

$$p'(t)^2 + \widetilde{B}_n(t)^2 p(t)^2 \le V'(t)^2 + \widetilde{B}_n(t)^2 V(t)^2 = \widetilde{B}_n(t)^2.$$

Thus (3.4) is proved. As pointed out earlier, this finishes the proof of (3.2).

¿From Theorem 2.3 we know that (3.2) holds with equality sign when p is a linear combination of $\widetilde{T}_n$ and $\widetilde{U}_n$. To prove the converse, let $\|p\|_{L^\infty(\mathbb{R})} = 1$, and assume that there is a $t \in \mathbb{R}$, such that $|p(t)| < 1$. By the above argument, there is an $\alpha \in R$, so that p and $V = \cos\alpha\, \widetilde{T}_n + \sin\alpha\, \widetilde{U}_n$ have the same value at t, and $p'V'$ is positive at t. Since both p and V satisfy (3.2) with equality, and $|p(t)| = |V(t)| < 1$, we have $|p'(t)| = |V'(t)| > 0$. Therefore we may assume that $p'(t) = V'(t) (> 0)$ Consequently, $p - V$ has a zero at t with multiplicity at least 2. Since V equioscillates $2n$ times on $K = \mathbb{R} (\text{mod } 2\pi)$ with $L^\infty(\mathbb{R})$ norm 1, and $\|p\|_{L^\infty(\mathbb{R})} = 1$, it is easy to see that $p - V$ has at least $2n-1$ zeros (by counting multiplicities) in $(\mathbb{R} \setminus \{t\})$ (mod 2π). Hence $p - V$ has at least $2n+1$ zeros (by counting multiplicities) on $[0, 2\pi)$, which yields $p - V \equiv 0$. □

Using the fact that $p \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$ implies $p(\cos(\cdot)) \in \mathcal{T}_n(a_1, a_2, \ldots, a_n)$, from Corollary 3.2 we immediately obtain

Corollary 3.3. *Let $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$. Then for every $x \in [-1, 1]$,*

$$(1 - x^2) p'(x)^2 + B_n(x)^2 p(x)^2 \le B_n(x)^2 \max_{y \in [-1,1]} |p(y)|^2, \qquad x \in [-1, 1] \tag{3.8}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, where B_n is defined by (2.2). The above holds with equality if and only if $p(x) = \pm \|p\|_{L^\infty[-1,1]}$ or p is a constant multiple of T_n.

Again, if we drop the second term in the left–hand side of the above, we have another form of a Bernstein–type inequality.

Corollary 3.4. *Let $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1, 1]$. Then,*

$$|p'(x)| \le \frac{1}{\sqrt{1 - x^2}} \sum_{k=1}^n \Re \frac{\sqrt{a_k^2 - 1}}{a_k - x} \max_{y \in [-1,1]} |p(y)|, \qquad x \in (-1, 1) \tag{3.9}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, where $\sqrt{a_k^2 - 1}$ is determined by (2.1). Equality holds in (3.9) if and only if p is a constant multiple of T_n and $p(x) = 0$.

Remark. An immediate consequence of (3.9) is that if $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$ and

$$\sum_{k=1}^{\infty} \frac{\sqrt{a_k^2 - 1}}{a_k - x} < \infty \quad \text{for some } x \in (-1, 1), \text{ i.e.} \quad \sum_{k=1}^{\infty} \sqrt{1 - |a_k|^{-2}} < \infty,$$

then the real span of

$$\left\{1, \frac{1}{x - a_1}, \frac{1}{x - a_2}, \frac{1}{x - a_3}, \ldots\right\}$$

is not dense in $C[-1, 1]$ (cf. [Ach, p. 250]).

The Bernstein–type inequality (3.9) does not give good estimates of the derivatives when x is close to ± 1. The following Markov–type inequality remedies this, at least when the poles are real.

Theorem 3.5. *Let $\{a_k\}_{k=1}^{n} \subset \mathbb{R} \setminus [-1, 1]$. Then*

$$\max_{-1 \le x \le 1} |p'(x)| \le \frac{n}{n-1} \left(\sum_{k=1}^{n} \frac{1 + |c_k|}{1 - |c_k|} \right)^2 \max_{-1 \le x \le 1} |p(x)|$$

holds for every $p \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$, where the numbers $\{c_k\}_{k=1}^{n}$ are defined from $\{a_k\}_{k=1}^{n}$ by (2.1).

The following lemma will be used in the proof of Theorem 3.5.

Lemma 3.6. *Let $\{a_k\}_{k=1}^{n} \subset \mathbb{C} \setminus [-1, 1]$, let*

$$a_k(y) = \begin{cases} \frac{2a_k}{1+y} + \frac{1-y}{1+y} & \text{if } 0 \le y \le 1 \\ \frac{2a_k}{1-y} + \frac{1+y}{1-y} & \text{if } -1 \le y \le 0, \ k = 1, 2, \ldots, n, \end{cases} \tag{3.10}$$

and let $c_k(y)$, $k = 1, 2, \ldots, n$, be defined by

$$a_k(y) = \frac{1}{2}\left(c_k(y) + c_k(y)^{-1}\right), \quad |c_k(y)| < 1. \tag{3.11}$$

Then

$$|p'(y)| \le \frac{2}{1 + |y|} \left(\sum_{k=1}^{n} \frac{1 + c_k(y)}{1 - c_k(y)} \right)^2 \max_{-1 \le x \le 1} |p(x)|$$

for every $p \in \mathcal{P}_n(a_1, a_2, \ldots, a_n)$.

Proof. It can be shown by a simple variational method (cf. [KaSt]) that

$$\sup_p \frac{|p'(1)|}{\max_{-1 \le x \le 1} |p(x)|} = |T_n'(1)| \tag{3.12}$$

and

$$\sup_p \frac{|p'(-1)|}{\max_{-1\le x\le 1}|p(x)|} = |T_n'(-1)|, \tag{3.13}$$

where the supremums in (3.12) and (3.13) are taken for all $p \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and T_n is the Chebyshev polynomial defined by (1.9). Now the lemma follows from Theorem 2.3 by a linear transformation (we shift from $[-1,1]$ to $[-1,y]$ if $0 \le y \le 1$ or to $[y,1]$ if $-1 \le y \le 0$). $\square$

Applying the Bernstein–type inequality (3.9) at 0, we get

$$|p'(0)| \le \sum_{k=1}^{n} \frac{\sqrt{a_k^2-1}}{a_k} \max_{-1\le x\le 1}|p(x)| \tag{3.14}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, where the values $\sqrt{a_k^2-1}$, $k = 1, 2, \dots, n$, are defined by (2.1). Note that if $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ is an arbitrary set of real poles, then (3.14) yields

$$|p'(0)| \le n \max_{-1\le x\le 1}|p(x)| \tag{3.15}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, and from this, by a linear transformation, we obtain

Corollary 3.7. *Let $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$ be an arbitrary set of poles. Then*

$$|p'(y)| \le \frac{n}{1-|y|} \max_{-1\le x\le 1}|p(x)| \tag{3.16}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and $y \in (-1,1)$.

Proof. This follows from (3.15) by a linear transformation (we shift from $[-1,1]$ to $[2y-1,1]$ if $0 \le y < 1$, or to $[-1,2y+1]$ if $-1 < y \le 0$). $\square$

Now we prove Theorem 3.5.

Proof of Theorem 3.5. Since $\{a_k\}_{k=1}^n \subset \mathbb{R}\backslash[-1,1]$, it follows from (3.10), (3.11), and (1.1) that

$$|a_k(y)| > |a_k| \quad \text{and} \quad |c_k(y)| < |c_k| < 1, \quad k = 1, 2, \dots, n \tag{3.17}$$

hold for every $y \in [-1,1]$. Therefore Lemma 3.6 yields

$$\begin{aligned} |p'(y)| &\le \frac{2}{1+|y|}\left(\sum_{k=1}^{n}\frac{1+|c_k(y)|}{1-|c_k(y)|}\right)^2 \max_{-1\le x\le 1}|p(x)| \\ &\le \frac{n}{n-1}\left(\sum_{k=1}^{n}\frac{1+|c_k|}{1-|c_k|}\right)^2 \max_{-1\le x\le 1}|p(x)| \end{aligned} \tag{3.18}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$ and for every y with $1 - 2n^{-1} \le |y| \le 1$. If $|y| < 1 - 2n^{-1}$, then Corollary 3.7 gives

$$\begin{aligned} |p'(y)| &\le \frac{n}{1-|y|} \max_{-1\le x\le 1} |p(x)| \qquad (3.19) \\ &\le n^2 \max_{-1\le x\le 1} |p(x)| \le \left(\sum_{k=1}^{n} \frac{1+|c_k|}{1-|c_k|} \right)^2 \max_{-1\le x\le 1} |p(x)| \end{aligned}$$

for every $p \in \mathcal{P}_n(a_1, a_2, \dots, a_n)$, which, together with (3.18), yields the theorem. □

§4. Chebyshev and Orthogonal Polynomials

In this section, we study some additional properties of the Chebyshev polynomials with respect to the rational system (0.20) with distinct real poles outside $[-1,1]$ and their orthogonalizations with respect to the measure $(1-x^2)^{-1/2}$ on $[-1,1]$. We start with an explicit partial fraction formula for the Chebyshev polynomials, then we record a contour integral form of the Chebyshev polynomials, from which a mixed recursion formula follows. The rest of the section will be devoted to orthogonality. Many aspects of orthogonal rationals and their applications can be found in the literature, for examples, in [Ach, BGHN, Djrb, VaVa, Wal]. The novelty of our approach is that we derive the orthogonal polynomials from the Chebyshev "polynomials" (cf. §1).

If $(a_k)_{k=1}^{\infty}$ is a sequence of real numbers outside $[-1,1]$, then the related $(c_k)_{k=1}^{\infty} \subset (-1,1)$ is defined by

$$a_k = \frac{1}{2}(c_k + c_k^{-1}), \quad c_k = a_k - \sqrt{a_k^2 - 1}, \quad c_k \in (-1,1), \qquad (4.1)$$

where the choice $\sqrt{a_k^2-1}$ is determined by $c_k \in (-1,1)$, and the associated Chebyshev polynomials of the first and second kinds are defined by (cf. (1.9) and (1.11))

$$T_n(x) = \frac{1}{2}\left(\frac{M_n(z)}{z^n M_n(z^{-1})} + \frac{z^n M_n(z^{-1})}{M_n(z)} \right) \qquad (4.2)$$

and

$$U_n(x) = \frac{1}{z - z^{-1}}\left(\frac{M_n(z)}{z^n M_n(z^{-1})} + \frac{z^n M_n(z^{-1})}{M_n(z)} \right), \qquad (4.3)$$

respectively, where $M_n(z) = \prod_{k=1}^{n}(z - c_k)$ and $x = (z + z^{-1})/2$. First we can easily get the partial fraction forms of the Chebyshev polynomials.

Proposition 4.1. *Let $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$ be a sequence of distinct numbers such that its nonreal elements are paired by complex conjugation, and T_n and U_n be the Chebyshev polynomials of first and second kinds defined by (1.9) and (1.11), respectively. Then*

$$T_n(x) = A_{0,n} + \frac{A_{1,n}}{x-a_1} + \cdots + \frac{A_{n,n}}{x-a_n} \tag{4.4}$$

and

$$U_n(x) = \frac{B_{1,n}}{x-a_1} + \cdots + \frac{B_{n,n}}{x-a_n}, \tag{4.5}$$

where

$$A_{0,n} = \frac{(-1)^n}{2}(c_1^{-1}\dots c_n^{-1} + c_1\dots c_n), \tag{4.6}$$

$$A_{k,n} = \left(\frac{c_k^{-1}-c_k}{2}\right)^2 \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1-c_kc_j}{c_k-c_j}, \quad k=1,2,\dots,n, \tag{4.7}$$

and

$$B_{k,n} = \frac{c_k^{-1}-c_k}{2} \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1-c_kc_j}{c_k-c_j}, \quad k=1,2,\dots,n. \tag{4.8}$$

Proof. It follows from Theorem 1.1 (a) and Theorem 1.2 (a) that T_n and U_n can be written as the partial fraction form of (4.4) and (4.5). Now it is quite easy to calculate the coefficients $A_{k,n}$ and $B_{k,n}$. For example,

$$\begin{aligned} A_{0,n} &= \lim_{x\to\infty} T_n(x) = \lim_{z\to 0} \frac{1}{2}\left(\frac{M_n(z)}{z^nM_n(z^{-1})} + \frac{z^nM_n(z^{-1})}{M_n(z)}\right) \\ &= \frac{(-1)^n}{2}(c_1^{-1}\dots c_n^{-1} + c_1\dots c_n), \end{aligned}$$

and for $k=1,2,\dots,n$,

$$\begin{aligned} A_{k,n} &= \lim_{x\to a_k} (x-a_k)T_n(x) \\ &= \lim_{z\to c_k} \frac{1}{4}(z-c_k)(1-c_k^{-1}z^{-1})\left(\frac{M_n(z)}{z^nM_n(z^{-1})} + \frac{z^nM_n(z^{-1})}{M_n(z)}\right) \\ &= \left(\frac{c_k^{-1}-c_k}{2}\right)^2 \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1-c_kc_j}{c_k-c_j}, \quad k=1,2,\dots,n. \end{aligned}$$

The coefficients $B_{k,n}$ can be calculated in the same fashion. □

We now give a contour integral expression for T_n, which can be used to derive a mixed recursion formula.

Lemma 4.2. *Let* $\{a_k\}_{k=1}^n \subset \mathbb{C} \setminus [-1,1]$. *Let* T_n *be defined by (1.9). Then we have*

$$T_n(x) = \frac{1}{2\pi i} \int_\gamma \left(\prod_{j=1}^{n} \frac{(t-c_j)(t-\bar{c}_j)}{(1-c_j t)(1-\bar{c}_j t)} \right)^{1/2} \frac{t-x}{t^2-2tx+1} dt, \qquad x \in [-1,1],$$

where γ *is a circle centered at the origin, with radius* $1 < r < \min\{|c_j^{-1}| : 1 \le j \le n\}$, *and the square root is chosen to be an analytic function of* t *inside gamma.*

Proof. Recalling that with the transformation $x = (z+z^{-1})/2$, we have

$$\begin{aligned} T_n(x) &= \frac{1}{2}\left(\frac{M_n(z)}{z^n M_n(z^{-1})} + \frac{z^n M_n(z^{-1})}{M_n(z)} \right) \\ &= \frac{1}{2\pi i} \int_\gamma \frac{1}{2} \frac{M_n(z)}{z^n M_n(z^{-1})} \left(\frac{1}{t-z} + \frac{1}{t-z^{-1}} \right) dt \\ &= \frac{1}{2\pi i} \int_\gamma \frac{M_n(z)}{z^n M_n(z^{-1})} \frac{t-x}{t^2-2tx+1} dt, \end{aligned}$$

where γ is a circle slightly larger than the unit circle as in the statement of this lemma. □

It is now quite simple to obtain a mixed recursion formula for the Chebyshev polynomials associated with rational systems. To do this, we need some notation. Let S_n denote the Chebyshev polynomials with respect to the rational system

$$\left\{ 1, \frac{1}{x-a_1}, \dots, \frac{1}{x-a_{n-2}}, \frac{1}{x-a_n} \right\}, \tag{4.9}$$

missing the function $\frac{1}{x-a_{n-1}}$, so by Lemma 4.2

$$S_n(x) = \frac{1}{2\pi i} \int_\gamma \prod_{\substack{j=1 \\ j \ne n-1}}^{n} \frac{t-c_j}{1-c_j t} \frac{t-x}{t^2-2tx+1} dt. \tag{4.10}$$

We remark that if n is fixed, then in order to define T_n and S_n correctly, one needs only to assume that a_{n-1} is real, and that the nonreal poles in $\{a_1, \dots, a_{n-2}, a_n\}$ are paired by complex conjugation. However, in order to define T_n and S_n correctly for all $n = 1, 2, \dots$, the assumption that $(a_k)_{k=1}^\infty \subset \mathbb{R} \setminus [-1,1]$ is needed. This remark is valid for most results in this section. Sometimes this assumption is adopted for the purpose of simplicity. We have

Lemma 4.3. *Let $(T_k)_{k=0}^{\infty}$ and $(S_n)_{n=1}^{\infty}$ be defined from $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1,1]$ by (4.2) and (4.10). Then,*

$$T_n = T_{n-2} + \frac{c_n c_{n-1} - 1}{c_n - c_{n-1}}(T_{n-1} - S_n), \qquad n = 2, 3, \ldots, \tag{4.11}$$

where $(c_k)_{k=1}^{\infty}$ is defined from $\{a_k\}_{k=1}^{\infty}$ by (4.1).

Proof. By the contour integral formulae in Lemma 4.2 and (4.10),

$$\begin{aligned} & T_{n-2}(x) + \frac{c_n c_{n-1} - 1}{c_n - c_{n-1}}(T_{n-1} - S_n) \\ =& \frac{1}{2\pi i}\int_\gamma \prod_{j=1}^{n-2} \frac{t - c_j}{1 - tc_j}\left[1 + \frac{c_n c_{n-1} - 1}{c_n - c_{n-1}}\left(\frac{t - c_{n-1}}{1 - tc_{n-1}} - \frac{t - c_n}{1 - tc_n}\right)\right]\frac{t - x}{t^2 - 2tx + 1}dt \\ =& \frac{1}{2\pi i}\int_\gamma \prod_{j=1}^{n-2} \frac{t - c_j}{1 - tc_j}\left[\frac{(t - c_{n-1})(t - c_n)}{(1 - tc_{n-1})(1 - tc_n)}\right]\frac{t - x}{t^2 - 2tx + 1}dt, \end{aligned}$$

which, again by the contour integral expression in Lemma 4.2, is $T_n(x)$. □

The ordinary Chebyshev polynomials $\cos(n \arccos x)$, $n = 0, 1, \ldots$, are orthogonal with respect to the weight function $(1 - x^2)^{-1/2}$ on $[-1, 1]$. For Chebyshev polynomials T_n, $n = 0, 1, \ldots$, defined by (4.2), they are not orthogonal. However they are almost orthogonal in the sense of the following two theorems, and they can be modified to orthogonalize the rational systems

$$\left\{1, \frac{1}{x - a_1}, \frac{1}{x - a_2}, \ldots\right\} \quad \text{and} \quad \left\{\frac{1}{x - a_1}, \frac{1}{x - a_2}, \ldots\right\},$$

respectively, where $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$ is a sequence of distinct numbers.

Lemma 4.4. *Let $(T_k)_{k=1}^{\infty}$ be defined from $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1,1]$ by (4.2). Then*

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2}(-1)^{n+m}(1 + c_1^2 \ldots c_m^2)c_{m+1} \ldots c_n, \quad 0 \le m \le n,$$

where $\{c_k\}_{k=1}^{\infty} \subset (-1, 1)$ is related to $(a_k)_{k=1}^{\infty}$ by (4.1), and the empty product is understood to be 1 for $m = 0$ or n.

Proof. Fix $0 \le m \le n$. By (4.2) and using the transformation $x = (z + z^{-1})/2$, we have

$$\begin{aligned} & \int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1 - x^2}} \\ =& \frac{1}{4}\int_{\gamma^+}\left[\frac{M_n(z)}{z^n M_n(z^{-1})} + \frac{z^n M_n(z^{-1})}{M_n(z)}\right]\left[\frac{M_m(z)}{z^m M_m(z^{-1})} + \frac{z^m M_m(z^{-1})}{M_m(z)}\right]\frac{dz}{iz}, \end{aligned}$$

where γ^+ is the upper half unit circle. On expanding the product in the integrand, keeping two terms over the upper half circle, and converting the other two terms to the lower half circle γ^-, we get

$$\begin{aligned}&\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}\\ =&\frac{\pi}{2}\frac{1}{2\pi i}\int_\gamma \left[\frac{M_n(z)}{z^nM_n(z^{-1})}\frac{M_m(z)}{z^mM_m(z^{-1})}+\frac{M_n(z)}{z^nM_n(z^{-1})}\frac{z^mM_m(z^{-1})}{M_m(z)}\right]\frac{dz}{z}\\ =&\frac{\pi}{2}[\quad]\Big|_{z=0}=\frac{\pi}{2}\left[(-1)^{n+m}c_1\dots c_nc_1\dots c_m+(-1)^{n-m}c_{m+1}\dots c_n\right],\end{aligned}$$

which is the same as stated in Lemma 4.4. □

Some partial orthogonality still holds for T_n (cf. [Ach, p. 250]). First we calculate the following integral.

Lemma 4.5. *Let $(T_k)_{k=0}^\infty$ be defined from $(a_k)_{k=1}^\infty \subset \mathbb{R}\backslash[-1,1]$ by (4.2) and $a \in \mathbb{R}\backslash[-1,1]$. Then*

$$\int_{-1}^{1} T_n(x)\frac{1}{x-a}\frac{dx}{\sqrt{1-x^2}} = \frac{2\pi}{c-c^{-1}}\prod_{j=1}^{n}\frac{c-c_j}{1-cc_j}, \tag{4.12}$$

where $c \in (-1,1)$ is defined by $a = (c+c^{-1})/2$.

Proof. Let γ^+ be the upper half unit circle, and apply the transformation $x = (z+z^{-1})/2$, we get

$$\begin{aligned}\int_{-1}^{1} T_n(x)\frac{1}{x-a}\frac{dx}{\sqrt{1-x^2}} &= \frac{1}{2}\int_{\gamma^+}\left[\frac{M_n(z)}{z^nM_n(z^{-1})}+\frac{z^nM_n(z^{-1})}{M_n(z)}\right]\frac{2}{c+c^{-1}-z-z^{-1}}\frac{dz}{iz}\\ &=\frac{1}{2i}\int_\gamma \frac{M_n(z)}{z^nM_n(z^{-1})}\frac{2dz}{(c-z)(c^{-1}-z)}.\end{aligned}$$

Hence

$$\int_{-1}^{1} T_n(x)\frac{1}{x-a}\frac{dx}{\sqrt{1-x^2}} = \frac{\pi M_n(z)}{z^nM_n(z^{-1})}\frac{2}{c^{-1}-z}\Big|_{z=c} = \frac{2\pi}{c^{-1}-c}\prod_{j=1}^{n}\frac{c-c_j}{1-cc_j}. \quad \square$$

Corollary 4.6. *Let $(T_n)_{n=0}^\infty$ be defined from $(a_k)_{k=1}^\infty$ by (4.2). Then*

$$\int_{-1}^{1} T_n(x)\frac{dx}{\sqrt{1-x^2}} = (-1)^n\pi c_1\dots c_n \tag{4.13}$$

and

$$\int_{-1}^{1} T_n(x)\frac{1}{x-a_k}\frac{dx}{\sqrt{1-x^2}} = 0, \quad k=1,2,\dots,n, \tag{4.14}$$

where $(c_k)_{k=1}^{\infty}$ is related to $(a_k)_{k=1}^{\infty}$ by (4.1).

Proof. The proof of the second part is a direct application of (4.12). To prove (4.13), we can either repeat the proof of Lemma 4.5, or we simply divide both sides of (4.12) by a and let $a \to \infty$, and notice that $a = (c + c^{-1})/2$ implies $c^{-1}/a \to 2$. $\square$

Given a sequence $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$, we define

$$R_0 = 1, \quad R_n = T_n + c_n T_{n-1} \quad n \geq 1 \tag{4.15}$$

and

$$R_0^* = \frac{1}{\sqrt{\pi}}, \quad R_n^* = \sqrt{\frac{2}{\pi(1 - c_n^2)}} \, (T_n + c_n T_{n-1}) \tag{4.16}$$

(cf. (4.2) and (4.3)). The following theorem indicates that these simple linear combinations of T_n and T_{n-1}, $n = 1, 2, \dots$, give the orthogonalization of the rational system

$$\left\{ 1, \frac{1}{x - a_1}, \frac{1}{x - a_2} \dots \right\}, \tag{4.17}$$

where $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1, 1]$ is a sequence of distinct numbers.

Theorem 4.7. *Let $(R_n^*)_{n=0}^{\infty}$ be defined by (4.15) and (4.16). Then*

$$\int_{-1}^{1} R_n^*(x) R_m^*(x) \frac{dx}{\sqrt{1 - x^2}} = \delta_{m,n} \tag{4.18}$$

holds for $n, m = 0, 1, 2, \dots$.

Proof. Let $m \leq n$. By Corollary 4.6,

$$\int_{-1}^{1} R_n(x) \frac{1}{x - a_k} \frac{dx}{\sqrt{1 - x^2}} = 0$$

holds for $k = 0, 1, \dots, n - 1$. Also by Corollary 4.6,

$$\begin{aligned} \int_{-1}^{1} R_n(x) \frac{dx}{\sqrt{1 - x^2}} &= \int_{-1}^{1} (T_n(x) + c_n T_{n-1}(x)) \frac{dx}{\sqrt{1 - x^2}} \\ &= (-1)^n c_1 \dots c_n + c_n (-1)^{n-1} c_1 \dots c_{n-1} = 0. \end{aligned}$$

This implies that

$$\int_{-1}^{1} R_n(x) R_m(x) \frac{dx}{\sqrt{1 - x^2}} = 0, \quad m < n.$$

When $m = n$, we have

$$\int_{-1}^{1} R_n(x)^2 \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} R_n(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} \left(T_n(x) + c_n T_{n-1}(x)\right) T_n(x) \frac{dx}{\sqrt{1-x^2}},$$

which, by Lemma 4.4, is

$$\frac{\pi}{2}(1 + c_1^2 \dots c_n^2) - \frac{\pi}{2} c_n (1 + c_1^2 \dots c_{n-1}^2) c_n = \frac{\pi}{2}(1 - c_n^2).$$

Therefore, $R_n^* = \sqrt{2(1-c_n^2)/\pi} R_n$ is the n–th orthonormal polynomial. □

It is also easy to orthogonalize the system

$$\left\{ \frac{1}{x-a_1}, \frac{1}{x-a_2}, \frac{1}{x-a_3}, \dots \right\} \tag{4.19}$$

with respect to the weight function $1/\sqrt{1-x^2}$ on $[-1,1]$ (where compared with (4.17), the constant function 1 is removed). In fact we only need to take the linear combination of T_n and T_{n-1} so that the partial fraction form (cf. (4.4) and (4.5)) does not have the constant term.

Corollary 4.8. *Let $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1,1]$, and define $(c_k)_{k=1}^{\infty} \subset (-1,1)$ by (4.1). If $(T_n)_{n=1}^{\infty}$ is defined by (4.2), and $(r_n)_{n=1}^{\infty}$ is defined by*

$$r_n = c_n (1 + c_1^2 \dots c_{n-1}^2) T_n + (1 + c_1^2 \dots c_n^2) T_{n-1}, \tag{4.20}$$

then r_n is an element in the real span of the system (4.19) and

$$\int_{-1}^{1} r_n(x) r_m(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(1-c_n^2)(1 + c_1^2 \dots c_{n-1}^2)(1 + c_1^2 \dots c_n^2)\, \delta_{n,m}$$

holds for $n, m = 0, 1, 2, \dots$.

The proof of the above is very similar to that of Theorem 4.7, and we can safely omit it. ¿From the definition of R_n and r_n, and Proposition 4.1, we can get their explicit partial fraction forms.

Finally, by applying [PiZi, Theorem 1.1], and noticing that (4.9) and (4.20) are both Chebyshev systems (cf. [SaSt]), we have

Corollary 4.9. *Assume $(a_k)_{k=1}^{\infty} \subset \mathbb{R} \setminus [-1,1]$. Let $(T_n)_{n=1}^{\infty}$, $(R_n)_{n=1}^{\infty}$, and $(r_n)_{n=1}^{\infty}$ be defined by (4.2),(4.15), and (4.19). Then for every $n = 1, 2, 3, \dots$, T_n and R_n have exactly n zeros in $[-1,1]$, r_n has exactly $n-1$ zeros in $[-1,1]$, and their zeros strictly interlace the zeros of T_{n-1}, R_{n-1}, and r_{n-1}, respectively.*

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