EXTENSIONS OF THE BLOCH – PÓLYA THEOREM ON THE NUMBER OF REAL ZEROS OF POLYNOMIALS

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Abstract. We prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that for every

$$\{a_0, a_1, \dots, a_n\} \subset [1, M]\,, \qquad 1 \le M \le \exp(c_1 n^{1/4})\,,$$

there are

$$b_0, b_1, \dots, b_n \in \{-1, 0, 1\}$$

such that

$$P(z) = \sum_{j=0}^{n} b_j a_j z^j$$

has at least $c_2 n^{1/4}$ distinct sign changes in $(0,1)$. This improves and extends earlier results of Bloch and Pólya.

1. Introduction

Let $\mathcal{F}_n$ denote the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$. Let $\mathcal{L}_n$ denote the set of polynomials of degree n with coefficients from $\{-1, 1\}$. In [6] the authors write

"The study of the location of zeros of these classes of polynomials begins with Bloch and Pólya [2]. They prove that the average number of real zeros of a polynomial from $\mathcal{F}_n$ is at most $c\sqrt{n}$. They also prove that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn \log\log n}{\log n}$$

real zeros. This quite weak result appears to be the first on this subject. Schur [13] and by different methods Szegő [15] and Erdős and Turán [8] improve this to $c\sqrt{n \log n}$ (see also [4]). (Their results are more general, but in this specialization not sharp.)

Our Theorem 4.1 gives the right upper bound of $c\sqrt{n}$ for the number of real zeros of polynomials from a much larger class, namely for all polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}\,.$$

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Schur [13] claims that Schmidt gives a version of part of this theorem. However, it does not appear in the reference he gives, namely [12], and we have not been able to trace it to any other source. Also, our method is able to give $c\sqrt{n}$ as an upper bound for the number of zeros of a polynomial $p \in \mathcal{P}_n^c$ with $|a_0| = 1, |a_j| \le 1$, inside any polygon with vertices in the unit circle (of course, c depends on the polygon). This may be discussed in a later publication.

Bloch and Pólya [2] also prove that there are polynomials $p \in \mathcal{F}_n$ with

$$\frac{cn^{1/4}}{\sqrt{\log n}} \tag{1.1}$$

distinct real zeros of odd multiplicity. (Schur [13] claims they do it for polynomials with coefficients only from $\{-1, 1\}$, but this appears to be incorrect.)

In a seminal paper Littlewood and Offord [11] prove that the number of real roots of a $p \in \mathcal{L}_n$, on average, lies between

$$\frac{c_1 \log n}{\log\log\log n} \qquad \text{and} \qquad c_2 \log^2 n$$

and it is proved by Boyd [7] that every $p \in \mathcal{L}_n$ has at most $c \log^2 n / \log\log n$ zeros at 1 (in the sense of multiplicity).

Kac [10] shows that the expected number of real roots of a polynomial of degree n with random uniformly distributed coefficients is asymptotically $(2/\pi)\log n$. He writes "I have also stated that the same conclusion holds if the coefficients assume only the values 1 and -1 with equal probabilities. Upon closer examination it turns out that the proof I had in mind is inapplicable.... This situation tends to emphasize the particular interest of the discrete case, which surprisingly enough turns out to be the most difficult." In a recent related paper Solomyak [14] studies the random series $\sum \pm\lambda^n$."

In fact, the paper [5] containing the "polygon result" mentioned in the above quote appeared sooner than [6]. The book [4] contains only a few related weeker results. Our Theorem 2.1 in [6] sharpens and generalizes results of Amoroso [1], Bombieri and Vaaler [3], and Hua [9] who give versions of that result for polynomials with integer coefficients.

In this paper we improve the lower bound (1.1) in the result of Bloch and Pólya to $cn^{1/4}$. Moreover we allow a much more general coefficient constraint in our main result. Our approach is quite different from that of Bloch and Pólya.

2. New Result

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that for every*

$$\{a_0, a_1, \dots, a_n\} \subset [1, M], \qquad 1 \le M \le \exp(c_1 n^{1/4}),$$

there are

$$b_0, b_1, \dots, b_n \in \{-1, 0, 1\}$$

such that

$$P(z) = \sum_{j=0}^{n} b_j a_j z^j$$

has at least $c_2 n^{1/4}$ distinct sign changes in $(0, 1)$.

3. Lemmas

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Denote by $\mathcal{S}_M$ the collection of all analytic functions f on the open unit disk D that satisfy

$$|f(z)| \leq \frac{M}{1-|z|}\,, \qquad z \in D\,.$$

Let $\|f\|_A := \sup_{x \in A} |f(x)|$. To prove Theorem 2.1 our first lemma is the following.

Lemma 3.1. *There is an absolute constants $c_3 > 0$ such that*

$$\|f\|_{[\alpha,\beta]} \geq \exp\left(\frac{-c_3(1+\log M)}{\beta-\alpha}\right)$$

for every $f \in \mathcal{S}_M$ and $0 < \alpha < \beta \leq 1$ with $|f(0)| \geq 1$ and for every $M \geq 1$.

This follows from the lemma below by a linear scaling:

Lemma 3.2. *There are absolute constants $c_4 > 0$ and $c_5 > 0$ such that*

$$|f(0)|^{c_5/a} \leq \exp\left(\frac{c_4(1+\log M)}{a}\right) \|f\|_{[1-a,1]}$$

for every $f \in \mathcal{S}_M$ and $a \in (0,1]$.

To prove Lemma 3.2 we need some corollaries of the following well known result.

Hadamard Three Circles Theorem. *Let $0 < r_1 < r_2$. Suppose f is regular in*

$$\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}\,.$$

For $r \in [r_1, r_2]$, let

$$M(r) := \max_{|z|=r} |f(z)|\,.$$

Then

$$M(r)^{\log(r_2/r_1)} \leq M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}\,.$$

Corollary 3.3. *Let $a \in (0,1]$. Suppose f is regular inside and on the ellipse E_a with foci at $1-a$ and $1-a+\frac{1}{4}a$ and with major axis*

$$\left[1-a-\frac{9a}{64}\,, 1-a+\frac{25a}{64}\right].$$

Let $\widetilde{E}_a$ be the ellipse with foci at $1-a$ and $1-a+\frac{1}{4}a$ and with major axis

$$\left[1-a-\frac{a}{32}\,, 1-a+\frac{9a}{32}\right].$$

Then

$$\max_{z \in \widetilde{E}_a} |f(z)| \leq \left(\max_{z \in \left[1-a, 1-a+\frac{1}{4}a\right]} |f(z)|\right)^{1/2} \left(\max_{z \in E_a} |f(z)|\right)^{1/2}.$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution

$$w = \frac{a}{8}\left(\frac{z+z^{-1}}{2}\right) + \left(1-a+\frac{a}{8}\right).$$

The Hadamard Three Circles Theorem is applied with $r_1 := 1, r := 2$, and $r_2 := 4$. □

Corollary 3.4. *For every $f \in \mathcal{S}_M$ and $a \in (0,1]$ we have*

$$\max_{z \in \widetilde{E}_a} |f(z)| \le \left(\frac{64M}{39a}\right)^{1/2} \left(\max_{z \in [1-a,1]} |f(z)|\right)^{1/2}.$$

Proof of Lemma 3.2. Let $f \in \mathcal{S}_M$ and $h(z) = \frac{1}{2}(1-a)(z+z^2)$. Observe that $h(0) = 0$, and there are absolute constants $c_6 > 0$ and $c_7 > 0$ such that

$$|h(e^{it})| \le 1 - c_6 t^2\,, \qquad -\pi \le t \le \pi\,,$$

and for $t \in [-c_7 a, c_7 a]$, $h(e^{it})$ lies inside the ellipse $\widetilde{E}_a$. Now let $m := \lfloor \pi/(c_7 a) \rfloor + 1$. Let $\xi := \exp(2\pi i/(2m))$ be the first $2m$-th root of unity, and let

$$g(z) = \prod_{j=0}^{2m-1} f(h(\xi^j z))\,.$$

Using the Maximum Principle and the properties of h, we obtain

$$\begin{aligned} |f(0)|^{2m} = |g(0)| \le \max_{|z|=1} |g(z)| &\le \left(\max_{z \in \widetilde{E}_a} |f(z)|\right)^2 \prod_{k=1}^{m-1} \left(\frac{M}{c_6(\pi k/m)^2}\right)^2 \\ &= \left(\max_{z \in \widetilde{E}_a} |f(z)|\right)^2 M^{2m-2} \exp(c_8(m-1)) \left(\frac{m^{m-1}}{(m-1)!}\right)^4 \\ &< \left(\max_{z \in \widetilde{E}_a} |f(z)|\right)^2 (Me)^{c_9(m-1)} \end{aligned}$$

with absolute constants c_8 and c_9, and the result follows from Corollary 3.4. □

4. Proof of Theorem 2.1

Proof of Theorem 2.1. Let $L \le \frac{1}{2} n^{1/2}$ and

$$\mathcal{M}(P) := (P(1-n^{-1/2}),\, P(1-2n^{-1/2}),\, \ldots,\, P(1-Ln^{-1/2})) \in [-M\sqrt{n}, M\sqrt{n}]^L\,.$$

We consider the polynomials

$$P(z) = \sum_{j=0}^{n-1} b_j a_j z^j\,, \qquad b_j \in \{0,1\}\,.$$

There are 2^n such polynomials. Let $K \in \mathbb{N}$. Using the box principle we can easily deduce that $(2K)^L < 2^n$ implies that there are two different

$$P_1(z) = \sum_{j=0}^{n-1} b_j a_j z^j\,, \qquad b_j \in \{0,1\}\,,$$

and

$$P_2(z) = \sum_{j=0}^{n-1} \widetilde{b}_j a_j z^j\,, \qquad \widetilde{b}_j \in \{0,1\}\,,$$

such that

$$|P_1(1-jn^{-1/2}) - P_2(1-jn^{-1/2})| \le \frac{M\sqrt{n}}{K}\,, \qquad j = 1,2,\dots,L\,.$$

Let

$$P_1(z) - P_2(z) = \sum_{j=m}^{n-1} \beta_j a_j z^j\,, \qquad \beta_j \in \{-1,0,1\}, \quad b_m \neq 0\,.$$

Let $0 \neq Q(z) := z^{-m}(P_1(z) - P_2(z))$. Then Q is of the form

$$Q(z) := \sum_{j=0}^{n-1} \gamma_j a_j z^j\,, \qquad \gamma_j \in \{-1,0,1\}\,, \quad \gamma_0 \in \{-1,1\}\,,$$

and, since $1 - x \ge e^{-2x}$ for all $x \in [0,1/2]$, we have

$$|Q(1-jn^{-1/2})| \le \exp(2Ln^{1/2})\frac{M\sqrt{n}}{K}\,, \qquad j = 1,2,\dots,L\,. \tag{4.1}$$

Also, by Lemma 3.1, there are

$$\xi_j \in I_j := [1-jn^{-1/2}, 1-(j-1)n^{-1/2}]\,, \qquad j = 1,2,\dots,L\,,$$

such that

$$|Q(\xi_j)| \ge \exp\left(-c_3(1+\log M)\sqrt{n}\right)\,, \qquad j = 1,2,\dots,L\,. \tag{4.2}$$

Now let $L := \lfloor (1/16)n^{1/4} \rfloor$ and $2K = \exp(n^{3/4})$. Then $(2K)^L < 2^n$ holds. Also, if $\log M = O(n^{1/4})$, then (4.1) implies

$$|Q(1-jn^{-1/2})| \le \exp(-(3/4)n^{3/4})\,, \qquad j = 1,2,\dots,L\,, \tag{4.3}$$

for all sufficiently large n. Now observe that $1 \le M \le \exp((64c_3)^{-1}n^{1/4})$ yields that

$$\begin{aligned} |a_n x^n| \ge& |x|^n \ge \exp(-2(1-x)) \ge \exp(-2Ln^{1/2}) \\ \ge& \exp(-(1/8)n^{3/4})\,, \qquad x \in [1-Ln^{-1/2}, 1-(L/2)n^{-1/2}]\,, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} |a_n x^n| \le& M\exp(-(L/2)n^{1/2}) \\ \le& \exp(-(1/33)n^{3/4})\,, \qquad x \in [1-Ln^{-1/2}, 1-(L/2)n^{-1/2}]\,, \end{aligned} \tag{4.5}$$

for all sufficiently large n. Observe also that with $\log M \le (64c_3)^{-1}n^{1/4}$ (4.2) implies

$$|Q(\xi_j)| > \exp(-(1/63)n^{3/4})\,, \qquad j = 1,2,\dots,L\,, \tag{4.6}$$

for all sufficiently large n. Now we study the polynomials

$$S_1(z) := Q(z) - a_n z^n \qquad \text{and} \qquad S_2(z) := Q(z) + a_n z^n\,.$$

These are of the requested special form. It follows from (4.3) – (4.6) that either S_1 or S_2 has a sign change in at least half of the intervals $I_j\,, j = L, L-1, \dots \lfloor L/2 \rfloor + 2\,,$ for all sufficiently large n, and the theorem is proved. $\square$

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