QUESTIONS ABOUT POLYNOMIALS WITH $\{0, -1, +1\}$ COEFFICIENTS

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We are interested in problems concerning the location and multiplicity of zeros of polynomials with small integer coefficients. We are also interested in some of the approximation theoretic properties of such polynomials. Let

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 0, 1\} \right\}$$

and let

$$\mathcal{A}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{0, 1\} \right\} \qquad \text{and} \qquad \mathcal{B}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1, 1\} \right\}.$$

Throughout this paper the uniform norm on a set $A \subset \mathbb{R}$ is denoted by $\|.\|_A$.

1. Number of Zeros

The following results are proved in [3].

Theorem 1.1. *Every polynomial p_n of the form*

$$p_n(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_0| = 1\,, \quad |a_j| \leq 1\,, \quad a_j \in \mathbb{C} \tag{1.1}$$

has at most $\left\lfloor \frac{16}{7}\sqrt{n} \right\rfloor + 4$ zeros at 1.

Theorem 1.2. *For every $n \in \mathbb{N}$, there exists a polynomial p_n of the form (1.1) with real coefficients so that p_n has a zero at 1 with multiplicity at least $\lfloor \sqrt{n} \rfloor - 1$.*

A natural question is whether or not it is possible to find a $p \in \mathcal{F}_n$ showing the sharpness of Theorem 1.1. In this direction the following is well known [3,5].

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Theorem 1.3. *There is an absolute constant $c > 0$ such that for every $n \in \mathbb{N}$ there is a $p \in \mathcal{F}_n$ having at least $c\sqrt{n/\log(n+1)}$ zeros at 1.*

Theorems 1.1 and 1.3 show that the right upper bound for the number of zeros a polynomial $p \in \mathcal{F}_n$ can have at 1 is somewhere between $c_1\sqrt{n/\log(n+1)}$ and $c_2\sqrt{n}$ with absolute constants $c_1 > 0$ and $c_2 > 0$.

Problem 1. How many zeros can a polynomial $p \in \mathcal{F}_n$ have at 1? Close the gap between Theorems 1.1 and 1.3. Any improvements would be interesting.

The following pair of theorems is from [3]. Theorem 1.4 improves the old bound $c\sqrt{n \log n}$ given by Schur [7] in 1933, and up to the constant c this is the best possible result.

Theorem 1.4. *There is an absolute constant $c > 0$ such that every polynomial p of the form (1.1) has at most $c\sqrt{n}$ zeros in $[-1, 1]$.*

Theorem 1.5. *There is an absolute constant $c > 0$ such that every polynomial p_n of the form (1.1) has at most c/a zeros at $[-1 + a, 1 - a]$ whenever $a \in (0, 1)$.*

Problem 2. How many *distinct* zeros can a polynomial p_n of the form (1.1) (or $p_n \in \mathcal{F}_n$) have in $[-1, 1]$? In particular, is it possible to give a sequence (p_n) of polynomials of the form (1.1) (or maybe $(p_n) \subset \mathcal{F}_n$) so that p_n has $c\sqrt{n}$ *distinct* zeros in $[-1, 1]$, where $c > 0$ is an absolute constant? If not, what is the sharp analogue of Theorem 1.4 for *distinct* zeros in $[-1, 1]$?

Problem 3. How many *distinct* zeros can a polynomial $p_n \in \mathcal{F}_n$ have in the interval $[-1 + a, 1 - a]$, $a \in (0, 1)$? In particular, is it possible to give a sequence (p_n) of polynomials of the form (1.1) (or maybe $(p_n) \subset \mathcal{F}_n$) so that p_n has c/a *distinct* zeros in $[-1 + a, 1 - a]$, where $c > 0$ is an absolute constant and $a \in [n^{-1/2}, 1)$? If not, what is the sharp analogue of Theorem 2.4 for *distinct real* zeros?

It is easy to prove (see [3]) that a polynomial $p \in \mathcal{A}_n$ can have at most $\log_2 n$ zeros at -1.

Problem 4. Is it true that there is an absolute constant $c > 0$ such that every $p \in \mathcal{A}_n$ with $p(0) = 1$ has at most $c \log n$ *real* zeros? If not, what is the best possible upper bound for the number of *real* zeros of polynomials $p \in \mathcal{A}_n$? What is the best possible upper bound for the number of *distinct real* zeros of polynomials $p \in \mathcal{A}_n$?

Odlyzko asked the next question after observing computationally that no $p \in \mathcal{A}_n$ with $n \leq 25$ had a repeat root of modulus greater than one.

Problem 5. Prove or disprove that a polynomial $p \in \mathcal{A}_n$ has all its repeated zeros at 0 or on the unit circle.

One can show, not completely trivially, that there are polynomials $p \in \mathcal{F}_n$ with repeated zeros in $(0, 1)$ up to mutiplicity 4.

Problem 6. Can the multiplicity of a zero of a $p \in \cup_{n=1}^{\infty} \mathcal{F}_n$ in $\{z \in \mathbb{C} : 0 < |z| < 1\}$ be arbitrarily large?

A negative answer to the above question would resolve an old conjecture of Lehmer concerning Mahler's measure. (See [1].)

Boyd [6] shows that there is an absolute constant c such that every $p \in \mathcal{B}_n$ can have at most $c \log^2 n / \log\log n$ zeros at 1. Is is easy to give polynomials $p \in \mathcal{B}_n$ with $c \log n$ zeros at 1.

Problem 7. Prove or disprove that there is an absolute constant c such that every polynomial $p \in \mathcal{B}_n$ can have at most $c \log n$ zeros at 1.

Problems 8 and 9 are about the spacing of the zeros of polynomials from $\mathcal{F}_n$.

Problem 8. As a function of n, give a lower bound for the minimal distance between two consecutive *distinct real* zeros of a polynomial from $\mathcal{F}_n$.

Problem 9. As a function of n, give a lower bound for the minimal distance between two *distinct* complex zeros of a polynomial from $\mathcal{F}_n$.

The next question is a version of an old and hard unsolved problem known as the Tarry-Escott Problem.

Problem 10. Let $N \in \mathbb{N}$ be fixed. Let $a(N)$ be the smallest value of k for which there is a polynomial $p \in \cup_{n=1}^{\infty} \mathcal{F}_n$ with exactly k nonzero terms in it and with a zero at 1 with multiplicity at least N. Prove or disprove that $a(N) = 2N$.

To prove that $a(N) \geq 2N$ is simple. The fact that $a(N) \leq 2N$ is known for $N = 1, 2, \ldots, 10$, but the problem is open for every $N \geq 11$. The best known upper bound for $a(N)$ in general seems to be $a(N) \leq cN^2 \log N$ with an absolute constant $c > 0$. See [4]. Even improving this (like dropping the factor $\log N$) would be a significant achievement.

2. The Chebyshev Problem on $[0,1]$

The following two theorems are proved in [3].

Theorem 2.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right).$$

Theorem 2.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1 \log^2(n+1)\right) \leq \inf_{0 \neq p \in \mathcal{A}_n} \|p(-x)\|_{[0,1]} \leq \exp\left(-c_2 \log^2(n+1)\right).$$

In the light of the above two theorems, it is natural to ask the following questions.

Problem 11. Does

$$\lim_{n\to\infty} \frac{\log\left(\inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]}\right)}{\sqrt{n}}$$

exist? If it does, what is it?

Problem 12. Does

$$\lim_{n\to\infty} \frac{\log\left(\inf_{0\neq p\in\mathcal{A}_n} \|p(-x)\|_{[0,1]}\right)}{\log^2(n+1)}$$

exist? If it does, what is it?

3. Markov- and Bernstein-Type Inequalities for $\mathcal{F}_n$

In [4] the following results are obtained.

Theorem 3.1 (Markov-Type Inequality for $\mathcal{F}_n$). *There is an absolute constant $c > 0$ such that*

$$\|p'\|_{[0,1]} \le cn\log(n+1)\,\|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$.

Theorem 3.2 (Bernstein-Type Inequality) for $\mathcal{F}_n$. *There is an absolute constant $c > 0$ such that*

$$|p'(y)| \le \frac{c}{(1-y)^2}\,\|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$ and $y \in [0,1)$.

Problem 13. Is it possible to improve Theorem 3.1? It is tempting to think that the factor $\log(n+1)$ can be dropped.

Problem 14. Is it possible to improve Theorem 3.2? It is tempting to think that the the factor $(1-y)^{-2}$ can be replaced by $(1-y)^{-1}$.

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