

# SUMS OF MONOMIALS WITH LARGE MAHLER MEASURE

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*Dedicated to Richard Askey on the occasion of his 80th birthday*

ABSTRACT. For  $n \geq 1$  let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^n z^{k_j} : 0 \leq k_1 < k_2 < \cdots < k_n, k_j \in \mathbb{Z} \right\},$$

that is,  $\mathcal{A}_n$  is the collection of all sums of  $n$  distinct monomials. These polynomials are also called Newman polynomials. If  $\alpha < \beta$  are real numbers then the Mahler measure  $M_0(Q, [\alpha, \beta])$  is defined for bounded measurable functions  $Q(e^{it})$  on  $[\alpha, \beta]$  as

$$M_0(Q, [\alpha, \beta]) := \exp \left( \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \log |Q(e^{it})| dt \right).$$

Let  $I := [\alpha, \beta]$ . In this paper we examine the quantities

$$L_n^0(I) := \sup_{P \in \mathcal{A}_n} \frac{M_0(P, I)}{\sqrt{n}} \quad \text{and} \quad L^0(I) := \liminf_{n \rightarrow \infty} L_n^0(I)$$

with  $0 < |I| := \beta - \alpha \leq 2\pi$ .

## 1. INTRODUCTION

The large sieve of number theory [M-78] asserts that if

$$P(z) = \sum_{k=-n}^n a_k z^k$$

is a trigonometric polynomial of degree at most  $n$ ,

$$0 \leq t_1 < t_2 < \cdots < t_m \leq 2\pi,$$

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and

$$\delta := \min\{t_1 - t_0, t_2 - t_1, \dots, t_m - t_{m-1}\}, \quad t_0 := t_m - 2\pi,$$

then

$$\sum_{j=1}^m |P(e^{it_j})|^2 \leq \left(\frac{n}{2\pi} + \delta^{-1}\right) \int_0^{2\pi} |P(e^{it})|^2 dt.$$

There are numerous extensions of this to  $L_p$  norm (or involving  $\psi(|P(e^{it})|^p)$ , where  $\psi$  is a convex function),  $p > 0$ , and even to subarcs. See [LMN-87] and [GLN-01]. There are versions of this that estimate Riemann sums, for example, with  $t_0 := t_m - 2\pi$ ,

$$\sum_{j=1}^m |P(e^{it_j})|^2 (t_j - t_{j-1}) \leq C \int_0^{2\pi} |P(e^{it})|^2 dt,$$

with a constant  $C$  independent of  $n$ ,  $P$ , and  $\{t_1, t_2, \dots, t_m\}$ . These are often called forward Marcinkiewicz-Zygmund inequalities. Converse Marcinkiewicz-Zygmund inequalities provide estimates for the integrals above in terms of the sums on the left-hand side, see [L-98], [MR-99], [ZZ-95], [KL-04]. A particularly interesting case is that of the  $L_0$  norm. A result in [EL-07] asserts that if  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  are the  $n$ -th roots of unity, and  $P$  is a polynomial of degree at most  $n$ , then

$$(1.1) \quad \prod_{j=1}^n |P(\alpha_j)|^{1/n} \leq 2M_0(P),$$

where

$$M_0(P) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{it})| dt\right)$$

is the Mahler measure of  $P$ . In [EL-07] we were focusing on showing that methods of subharmonic function theory provide a simple and direct way to generalize previous results. We also extended (1.1) to points other than the roots of unity and exponentials of logarithmic potentials of the form

$$P(z) = c \exp\left(\int \log |z - t| d\nu(t)\right),$$

where  $c \geq 0$  and  $\nu$  is a positive Borel measure of compact support with  $\nu(\mathbb{C}) \geq 0$ . Inequalities for exponentials of logarithmic potentials and generalized polynomials were studied by several authors, see [E-91], [E-92], [EN-92], [ELS-94], [EMN-92], [BE-95], and [EL-07], for instance.

Let  $\alpha < \beta$  be real numbers. The Mahler measure  $M_0(Q, [\alpha, \beta])$  is defined for bounded measurable functions  $Q(e^{it})$  defined on  $[\alpha, \beta]$  as

$$M_0(Q, [\alpha, \beta]) := \exp\left(\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \log |Q(e^{it})| dt\right).$$

It is well known (see [HLP-52], for instance) that

$$M_0(Q, [\alpha, \beta]) = \lim_{p \rightarrow 0^+} M_p(Q, [\alpha, \beta]),$$

where

$$M_p(Q, [\alpha, \beta]) := \left( \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |Q(e^{it})|^p dt \right)^{1/p}, \quad p > 0.$$

It is a simple consequence of the Jensen formula that

$$M_0(Q) := M_0(Q, [0, 2\pi]) = |c| \prod_{k=1}^n \max\{1, |\alpha_k|\}$$

for every polynomial of the form

$$Q(z) = c \prod_{k=1}^n (z - \alpha_k), \quad c, \alpha_k \in \mathbb{C}.$$

Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The class

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the class

$$\mathcal{K}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \quad a_k \in \mathbb{C}, \quad |a_k| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. We also let  $\mathcal{P}_n$  be the set of polynomials of degree  $n$  with complex coefficients. Beller and Newman [BN-73] constructed unimodular polynomials of degree  $n$  whose Mahler measure is at least  $\sqrt{n} - c/\log n$  with an absolute constant  $c > 0$ . For a prime number  $p$ , the  $p$ -th Fekete polynomial is defined as

$$f_p(z) := \sum_{k=1}^{p-1} \left( \frac{k}{p} \right) z^k,$$

where

$$\left( \frac{k}{p} \right) = \begin{cases} 1, & \text{if } x^2 \equiv k \pmod{p} \text{ has a nonzero solution,} \\ 0, & \text{if } p \text{ divides } k, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since  $f_p$  has constant coefficient 0, it is not a Littlewood polynomial, but  $g_p$  defined by  $g_p(z) := f_p(z)/z$  is a Littlewood polynomial, and has the same Mahler measure as  $f_p$ . Fekete polynomials are examined in detail in [B-02]. In [EL-07] we proved the following result.

**Theorem 1.1.** *For every  $\varepsilon > 0$  there is a constant  $c_\varepsilon$  such that*

$$M_0(f_p, [0, 2\pi]) \geq \left(\frac{1}{2} - \varepsilon\right) \sqrt{p}$$

for all primes  $p \geq c_\varepsilon$ .

One of the key lemmas in the proof of the above theorem formulates a remarkable property of the Fekete polynomials. A simple proof is given in [B-02, pp. 37-38], [H-82].

**Lemma 1.2 (Gauss).** *We have*

$$f_p(z_p^j) = \varepsilon_p \left(\frac{j}{p}\right) p^{1/2}, \quad j = 1, 2, \dots, p-1,$$

and  $f_p(1) = 0$ , where

$$z_p := \exp\left(\frac{2\pi i}{p}\right)$$

is the first  $p$ -th root of unity, and

$$\varepsilon_p = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4} \\ i, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

The distribution of the zeros of Littlewood polynomials plays a key role in the study of the Mahler measure of Littlewood polynomials. There are many papers on the distribution of zeros of polynomials with constraints on their coefficients, see [ET-50], [BE-95], [BE-97], [B-97], [B-02], [BEK-99], [E-08], and [P-11], for example. Results of this variety have been exploited in [EL-07] to obtain Theorem 1.1.

From Jensen's inequality,

$$M_0(f_p, [0, 2\pi]) \leq M_2(f_p, [0, 2\pi]) = \sqrt{p-1}.$$

However, as it is observed in [EL-07],  $(1/2 - \varepsilon)$  in Theorem 1.1 cannot be replaced by  $(1 - \varepsilon)$ . Indeed if  $p$  is prime of the form  $p = 4m + 1$ , then the polynomial  $f_p$  is self-reciprocal, that is,  $z^p f_p(1/z) = f_p(z)$ , and hence

$$f_p(e^{2it}) = e^{ipt} \sum_{k=0}^{(p-3)/2} a_k \cos((2k+1)t), \quad a_k \in \{-2, 2\}.$$

A result of Littlewood [L-66] implies that

$$M_0(f_p, [0, 2\pi]) \leq \frac{1}{2\pi} \int_0^{2\pi} |f_p(e^{it})| dt = \frac{1}{2\pi} \int_0^{2\pi} |f_p(e^{2it})| dt \leq (1 - \varepsilon) \sqrt{p-1},$$

for some absolute constant  $\varepsilon > 0$ . A similar argument shows that the same estimate holds when  $p$  is a prime of the form  $p = 4m + 3$ . It is an interesting open question whether or not there is a sequence of Littlewood polynomials  $(f_n)$  such that

$$M_0(f_n, [0, 2\pi]) \geq (1 - \varepsilon) \sqrt{n}$$

for all  $\varepsilon > 0$  and sufficiently large  $n \geq N_\varepsilon$ .

In [E-11] we proved the following sieve type lower bound for the Mahler measure of polynomials of degree at most  $n$  on subarcs of the unit circle.

**Theorem 1.3.** *Let  $\omega_1 < \omega_2 \leq \omega_1 + 2\pi$ ,*

$$\omega_1 \leq t_0 < t_1 < \cdots < t_m \leq \omega_2,$$

$$t_{-1} := \omega_1 - (t_0 - \omega_1), \quad t_{m+1} := \omega_2 + (\omega_2 - t_m),$$

$$\delta := \max\{t_0 - t_{-1}, t_1 - t_0, \dots, t_{m+1} - t_m\} \leq \frac{1}{2} \sin \frac{\omega_2 - \omega_1}{2}.$$

*There is an absolute constant  $c_1 > 0$  such that*

$$\sum_{j=0}^m \frac{t_{j+1} - t_{j-1}}{2} \log |P(e^{it_j})| \leq \int_{\omega_1}^{\omega_2} \log |P(e^{it})| dt + c_1 E(N, \delta, \omega_1, \omega_2)$$

*for every polynomial  $P$  of the form*

$$P(z) = \sum_{j=0}^N b_j z^j, \quad b_j \in \mathbb{C}, \quad b_0 b_N \neq 0,$$

*where*

$$E(N, \delta, \omega_1, \omega_2) := (\omega_2 - \omega_1)N\delta + N\delta^2 \log(1/\delta) + \sqrt{N \log R} \left( \delta \log(1/\delta) + \frac{\delta^2}{\omega_2 - \omega_1} \right)$$

*and*

$$R := |b_0 b_N|^{-1/2} \max_{t \in \mathbb{R}} |P(e^{it})|.$$

Observe that  $R$  appearing in the above theorem can be easily estimated by

$$R \leq |b_0 b_N|^{-1/2} (|b_0| + |b_1| + \cdots + |b_N|).$$

As a reasonably straightforward consequence of our sieve-type inequality above, the lower bound for the Mahler measure of Fekete polynomials below follows.

**Theorem 1.4.** *There is an absolute constant  $c_2 > 0$  such that*

$$M_0(f_p, [\alpha, \beta]) \geq c_2 \sqrt{p}$$

*for all prime numbers  $p$  and for all  $\alpha, \beta \in \mathbb{R}$  such that*

$$\frac{4\pi}{p} \leq \frac{(\log p)^{3/2}}{p^{1/2}} \leq \beta - \alpha \leq 2\pi.$$

It looks plausible that Theorem 1.4 holds whenever  $4\pi/p \leq \beta - \alpha \leq 2\pi$ , but we do not seem to be able to handle the case  $4\pi/p \leq \beta - \alpha \leq (\log p)^{3/2} p^{-1/2}$  in this paper.

In [E-12] we proved the following.

**Theorem 1.5.** *There are constants  $c_3(q, \varepsilon)$  depending only on  $q > 0$  and  $\varepsilon > 0$  such that*

$$M_0(f_p, [\alpha, \beta]) \leq M_q(f_p, [\alpha, \beta]) \leq c_3(q, \varepsilon)\sqrt{p},$$

whenever  $2p^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$ .

For  $n \geq 1$  let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^n z^{k_j} : 0 \leq k_1 < k_2 < \cdots < k_n, k_j \in \mathbb{Z} \right\},$$

that is,  $\mathcal{A}_n$  is the collection of all sums of  $n$  distinct monomials. Let  $I := [\alpha, \beta]$ . We define

$$L_n^1(I) := \sup_{P \in \mathcal{A}_n} \frac{M_1(P, [\alpha, \beta])}{\sqrt{n}} \quad \text{and} \quad L^1(I) := \liminf_{n \rightarrow \infty} L_n^1(I) \leq \Sigma(I) := \sup_{n \in \mathbb{N}} L_n^1(I).$$

In the case of  $I = [0, 2\pi]$  the problem of calculating  $\Sigma(I)$  appears in a paper of Bourgain [B-93]. Deciding whether  $\Sigma([0, 2\pi]) < 1$  or  $\Sigma([0, 2\pi]) = 1$  would be a major step toward confirming or disproving other important conjectures. Karatsuba [K-98] observed that  $\Sigma([0, 2\pi]) \geq 1/\sqrt{2} \geq 0.707$ . Indeed, taking, for instance,

$$P_n(z) = \sum_{k=0}^{n-1} z^{2^k}, \quad n = 1, 2, \dots,$$

it is easy to see that

$$(1.2) \quad M_4(P_n, [0, 2\pi])^4 = 2n(n-1) + n,$$

and as

$$n = M_2(P_n, [0, 2\pi])^2 \leq M_1(P_n, [0, 2\pi])^{2/3} M_4(P_n, [0, 2\pi])^{4/3},$$

the inequality

$$(1.3) \quad M_1(P_n, [0, 2\pi]) \geq \sqrt{\frac{n^2}{2n-1}} \geq \frac{\sqrt{n}}{\sqrt{2}}$$

follows. Similarly, if  $S_n := \{a_1 < a_2 < \cdots < a_n\}$  is a Sidon set (that is,  $S_n$  is a subset of integers such that no integer has two essentially distinct representations as the sum of two elements of  $S_n$ ), then the polynomials

$$P_n(z) = \sum_{a \in S_n} z^a, \quad n = 1, 2, \dots,$$

satisfy (1.2) and (1.3).

Improving Karatsuba's result, by using a probabilistic method Aistleitner [A-13] proved that  $\Sigma([0, 2\pi]) \geq \sqrt{\pi}/2 \geq 0.886$ . We note that P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of normalized  $L_p$  norms of Littlewood polynomials for arbitrary  $p > 0$ . Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_p(f, [0, 2\pi]))^p}{n^{p/2}} = \Gamma\left(1 + \frac{p}{2}\right).$$

An analogue of this result does not seem to be known for  $p = 0$  (the Mahler measure). However, it follows simply from the  $p = 1$  case of the result in [BL-01] quoted above that  $\Sigma([0, 2\pi]) \geq \sqrt{\pi}/8 \geq 0.626$ . Moreover, this can be achieved by taking the sum of approximately half of the monomials of  $\{x^0, x^1, \dots, x^{2n}\}$  and letting  $n$  tend to  $\infty$ .

Let  $Q_n(z) := \sum_{k=1}^n a_{k,n} z^{2^k}$ . In [CE-13] we showed that

$$\lim_{n \rightarrow \infty} \frac{M_p(Q_n)}{\sqrt{n}} = \Gamma\left(1 + \frac{p}{2}\right)^{1/p},$$

and we have recaptured Aistleitner's result  $\Sigma([0, 2\pi]) \geq \Gamma(3/2) = \sqrt{\pi}/2 \geq 0.886$ , as the special case  $p = 1$ .

Littlewood asked how small the ratio

$$\frac{M_4(Q, [0, 2\pi])}{M_2(Q, [0, 2\pi])} = \frac{M_4(Q, [0, 2\pi])}{\sqrt{n+1}}$$

can be for polynomials  $Q \in \mathcal{L}_n$  as the degree tends to infinity. As it is remarked in [JKS-13], since 1988, the least known asymptotic value of this ratio has been  $\sqrt[4]{7/6}$  which was conjectured to be minimum. In [JKS-13] this conjecture was disproved by showing that there is a sequence  $(Q_{n_k})$  of Littlewood polynomials  $Q_{n_k} \in \mathcal{L}_{n_k}$ , derived from the Fekete polynomials, for which

$$\lim_{k \rightarrow \infty} \frac{M_4(Q_{n_k}, [0, 2\pi])}{\sqrt{n_k+1}} < \sqrt[4]{22/19}.$$

Hence, it follows by a simple application of Hölder's inequality that

$$\liminf_{k \rightarrow \infty} \frac{M_1(Q_{n_k}, [0, 2\pi])}{\sqrt{n_k+1}} \geq \sqrt{19/22} > 0.929.$$

In this paper we examine the size of

$$L_n^0(I) := \sup_{P \in \mathcal{A}_n} \frac{M_0(P, [\alpha, \beta])}{\sqrt{n}} \quad \text{and} \quad L^0(I) := \liminf_{n \rightarrow \infty} L_n^0([\alpha, \beta])$$

with  $0 < |I| := \beta - \alpha \leq 2\pi$ .

## 2 NEW RESULTS

**Theorem 2.1.** *There are polynomials  $S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  with  $N = 2n + o(n)$  such that*

$$M_0(S_n, [0, 2\pi]) \geq \left( \frac{1}{2\sqrt{2}} + o(1) \right) \sqrt{n}, \quad n \in \mathbb{N}.$$

*That is,*

$$L^0([0, 2\pi]) \geq \frac{1}{2\sqrt{2}}.$$

**Theorem 2.2.** *There are polynomials  $S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  with  $N = 2n + o(n)$ , an absolute constant  $c_4 > 0$ , and a constant  $c_5(\varepsilon) > 0$  depending only on  $\varepsilon > 0$  such that*

$$M_0(S_n, [\alpha, \beta]) \geq c_4 \sqrt{n}$$

*for all  $n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$  such that*

$$\frac{4\pi}{n} \leq \frac{(\log n)^{3/2}}{n^{1/2}} \leq \beta - \alpha \leq 2\pi,$$

*while*

$$M_1(S_n, [\alpha, \beta]) \leq c_5(\varepsilon) \sqrt{n}$$

*for all  $n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$  such that  $(n/2)^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$ .*

**Theorem 2.3.** *There are polynomials  $P_n \in \mathcal{L}_n$  such that*

$$M_0(P_n, [0, 2\pi]) \geq \left( \frac{1}{2} + o(1) \right) \sqrt{n}, \quad n \in \mathbb{N}.$$

**Theorem 2.4.** *There are polynomials  $P_n \in \mathcal{L}_n$ , and absolute constant  $c_4 > 0$ , and a constant  $c_5(\varepsilon) > 0$  depending only on  $\varepsilon > 0$  such that*

$$M_0(P_n, [\alpha, \beta]) \geq c_4 \sqrt{n}$$

*for all  $n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$  such that*

$$\frac{4\pi}{n} \leq \frac{(\log n)^{3/2}}{n^{1/2}} \leq \beta - \alpha \leq 2\pi,$$

*while*

$$M_1(P_n, [\alpha, \beta]) \leq c_5(\varepsilon) \sqrt{n}$$

*for all  $n \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{R}$  such that  $(n/2)^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$ .*



### 3. LEMMAS

Theorem 1.3 will be used as a key lemma in the proof of Theorem 2.2. Theorem 1.5 will play a key role in the proof of Theorem 2.2. In addition to these we need the following lemmas. We present the short proof of each lemma right after the lemma. The new results stated in Section 2 will be proved in Section 4.

**Lemma 3.1.** *We have*

$$\left( \prod_{k=0}^{p-1} |Q(z_p^k)| \right)^{1/p} \leq 2^{N/p} M_0(Q)$$

for all polynomials  $Q$  of degree  $N$  with complex coefficients.

*Proof.* Let

$$Q(z) = c \prod_{k=1}^N (z - \alpha_k), \quad c, \alpha_k \in \mathbb{C}.$$

Note that

$$|\alpha_k^p - 1|^{1/p} \leq (2|\alpha_k|^p)^{1/p} = 2^{1/p} |\alpha_k|, \quad |\alpha_k| \geq 1,$$

while

$$|\alpha_k^p - 1|^{1/p} \leq 2^{1/p}, \quad |\alpha_k| < 1.$$

Multiplying these inequalities for  $k = 1, 2, \dots, N$ , we obtain

$$\left( \prod_{k=0}^{p-1} |Q(z_p^k)| \right)^{1/p} = c \left( \prod_{k=1}^N |\alpha_k^p - 1| \right)^{1/p} \leq 2^{N/p} c \prod_{k=1}^N \max\{|\alpha_k|, 1\} \leq 2^{N/p} M_0(Q).$$

□

For  $r \in \{1, 3\}$ , let  $\pi(x, 4, r)$  denote the number of prime numbers of the form  $p = 4k + r$  in  $[1, x]$ . A well known result (e.g. See [A-76]) states that

$$\lim_{x \rightarrow \infty} \frac{2\pi(x, 4, 1)}{x/\log x} = \lim_{x \rightarrow \infty} \frac{2\pi(x, 4, 3)}{x/\log x} = 1.$$

An immediate consequence of this is the following.

**Lemma 3.2.** *For every  $n \geq 5$  there is a prime of the form  $p = 4k + 3$  and there are integers  $q > 0$  and  $r \in \{0, 1\}$  such that*

$$n = \mu + 1 + 2q + r, \quad \mu = \frac{p-1}{2}, \quad q = o(n).$$

*Proof.* We first choose  $r$  so that  $n - r$  is even. Then we choose a prime  $p = 4k + 3$  so close to  $2(n - r) - 1$  that

$$0 < q := \frac{2(n - r) - (p + 1)}{4} = o(n).$$

This can be done because of the asymptotic estimation of  $\pi(x, 4, 3)$ .  $\square$

Associated with a prime number  $p$  let  $0 < q \leq \mu$  be an integer. Let

$$0 < a_1 < a_2 < \cdots < a_{\mu-1} < a_\mu \leq p - 1$$

denote the quadratic residues and

$$0 < b_1 < b_2 < \cdots < b_{\mu-1} < b_\mu \leq p - 1$$

denote the quadratic non-residues modulo  $p$ . Let

$$G_q(z) := \sum_{j=1}^q (z^{a_j} + z^{-a_j}) .$$

Note that as a simple consequence of the Pólya-Vinogradov inequality (see Ch. 23, pp. 135–137 in [D-80]), we have  $a_q = 2q + O(\sqrt{p} \log p)$ . For an integer  $k \in \mathbb{N}$  we also define

$$D_k(z) := \sum_{j=0}^{k-1} z^j .$$

Using a possible decomposition given by Lemma 3.2, for any integer  $n \geq 5$  we define  $S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  with  $N := p + a_q + r = 2n + o(n)$  by

$$(3.1) \quad S_n(z) := \frac{1}{2} (f_p(z) + D_p(z) + 1) + z^p G_q(z) + r z^{p+a_q+1}$$

Note that the selection of the numbers  $a_j \in \mathbb{N}$  ensures that the terms in  $\frac{1}{2} (f_p(z) + D_p(z) + 1)$  and  $z^p G_q(z)$  do not overlap. So  $S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  with  $N = 2n + o(n)$ , indeed.

Combining (3.1), Lemma 3.2 we can easily deduce the following.

**Lemma 3.3.** *Let  $S_n \in \mathcal{A}_n$  be defined by (3.1). We have*

$$|S_n(z_p^j)| \geq \frac{\sqrt{p}}{2} - 1, \quad j = 1, 2, \dots, p-1,$$

where  $z_p := \exp(2\pi i/p)$  is the first  $p$ -th root of unity. We also have  $S_n(1) = n$ .

*Proof.* When  $j = 0$  we have  $S_n(z_p^0) = \frac{p+1}{2} + 2q + r = n$ , as stated in the lemma. Now let  $j \in \{1, 2, \dots, p-1\}$ . Lemma 1.2 implies that  $|\operatorname{Im}(f_p(z_p^j))| = \sqrt{p}$ . Also  $D_p(z_p^j) = 0$ ,  $1 + z_p^{jp} G_q(z_p^j) \in \mathbb{R}$ , and  $\operatorname{Im}(r z_p^{j(p+a_q+1)}) \leq 1$ . Hence

$$|S_n(z_p^j)| \geq |\operatorname{Im}(S_n(z_p^j))| \geq \frac{\sqrt{p}}{2} - 1, \quad j = 1, 2, \dots, p-1, \quad k \geq \sqrt{p}. \quad \square$$

**Lemma 3.4** (see [IP-84]). *There is always a prime number in the interval  $[n - n^{23/42}, n]$  for all sufficiently large  $n \in \mathbb{N}$ .*

Lemma 3.4 has the following straightforward consequence.

**Lemma 3.5.** *For every  $n \geq 5$  there are primes  $p$  and  $q$  and  $r \in \mathbb{N}$  such that*

$$n = p + q + r \quad q = O(n^{2/3}), \quad \text{and} \quad r = O(n^{4/9}).$$

*Proof.* We first choose a prime  $p$  such that  $1 \ll n - p \ll n^{23/42}$ . Then we choose another prime  $q$  such that  $0 \leq n - p - q \leq (n - p)^{23/42} \ll n^{(23/42)^2} \ll n^{4/9}$ . Now  $q \leq n - p \ll n^{23/42} \ll n^{2/3}$  and  $r := n - p - q \ll n^{4/9}$ .  $\square$

Using a possible decomposition given by Lemma 3.5, for any integer  $n \geq 5$  we define

$$(3.2) \quad P_n := 1 + f_p(z) + z^p f_q(z) + z^{p+q} D_r \in \mathcal{L}_n$$

The following property of the Fekete polynomials  $f_p$  is due to Montgomery [M-80].

**Lemma 3.6.** *There are absolute constants  $c_1 > 0$  and  $c_2 > 0$  such that*

$$c_1 \sqrt{p} \log \log p \leq \max_{t \in \mathbb{R}} |f_p(e^{it})| \leq c_2 \sqrt{p} \log p.$$

Combining (3.2), Lemma 3.5, and the upper bound of Lemma 3.6 we can easily deduce the following.

**Lemma 3.7.** *Let  $P_n \in \mathcal{L}_n$  be defined by (3.2). We have*

$$|P_n(z_p^j)| = (1 + o(1)) \sqrt{p}, \quad j = 1, 2, \dots, p-1,$$

and  $P_n(1) = 1 + r \geq 1$ , where  $z_p := \exp(2\pi i/p)$  is the first  $p$ -th root of unity.

#### 4. PROOF OF THE THEOREMS

*Proof of Theorem 2.1.* The theorem follows from Lemmas 3.1 and 3.3 in a straightforward fashion. Let  $S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  be defined by (3.1). Let

$$0 \leq t_0 < t_1 < t_2 < \dots < t_{p-1} \leq 2\pi$$

be chosen so that  $z_p^k = e^{it_k}$ ,  $k = 0, 1, \dots, p-1$ . Using Lemma 3.3 and then applying Lemmas 3.1 with  $Q := S_n \in \mathcal{A}_n \cap \mathcal{P}_N$ , and recalling that  $N = p + o(p)$ , we obtain

$$((1 + o(1)) \sqrt{n/2})^{(p-1)/p} \leq \left( \prod_{k=0}^{p-1} |S_n(e^{it_k})| \right)^{1/p} \leq 2^{N/p} M_0(S_n),$$

and hence

$$M_0(S_n) \geq \frac{1 + o(1)}{2\sqrt{2}} \sqrt{n},$$

and the theorem follows.  $\square$

*Proof of Theorem 2.2.* Let  $S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  be defined by (3.1). Note that

$$(M_0(f, [\alpha, \beta]))^{\beta-\alpha} = (M_0(f, [\alpha, \gamma]))^{\gamma-\alpha} (M_0(f, [\gamma, \beta]))^{\beta-\gamma},$$

for all  $\alpha < \gamma < \beta \leq \alpha + 2\pi$  and for all continuous functions  $f$  on  $[\alpha, \beta]$ . Hence, to prove the lower bound of the theorem, without loss of generality we may assume that  $\beta - \alpha \leq \pi$ . The lower bound of the theorem now follows from Theorem 2.1 and Lemmas 3.3 and 3.5 in a straightforward fashion. Let

$$(4.1) \quad \omega_1 := \alpha \leq t_0 < t_1 < t_2 < \cdots < t_m \leq \beta =: \omega_2$$

be chosen so that  $e^{it_j}$ ,  $j = 0, 1, \dots, m$ , ( $m \leq p - 1$ ), are exactly the primitive  $p$ -th roots of unity lying on the arc connecting  $e^{i\alpha}$  and  $e^{i\beta}$  on the unit circle counterclockwise. The assumption on  $n$  together with  $p = 2n + o(n)$  guarantees that the value of  $\delta$  defined in Theorem 1.3 is at most  $2\pi/p$ . Observe also that  $R \leq n$ . Applying Theorem 1.3 with  $P := S_n \in \mathcal{A}_n \cap \mathcal{P}_N$  and (4.1), we obtain

$$\sum_{j=0}^m \frac{t_{j+1} - t_{j-1}}{2} \log |S_n(e^{it_j})| \leq \int_{\alpha}^{\beta} \log |S_n(e^{it})| dt + c_1 E(N, 4\pi/p, \alpha, \beta),$$

where the assumption

$$\frac{(\log n)^{3/2}}{n^{1/2}} \leq \beta - \alpha \leq 2\pi$$

together with  $p = 2n + o(n)$ ,  $N = 2n + o(n)$ , and  $R \leq n$  implies that

$$\sqrt{N \log n} \left( \frac{\log p}{p} + \frac{1}{p^2(\beta - \alpha)} \right) \leq c(\beta - \alpha)$$

with an absolute constant  $c > 0$  and hence,

$$\begin{aligned} E(N, 4\pi/p, \alpha, \beta) &\leq c_6 \left( \frac{(\beta - \alpha)N}{p} + \frac{N \log p}{p^2} + \sqrt{N \log n} \left( \frac{\log p}{p} + \frac{1}{p^2(\beta - \alpha)} \right) \right) \\ &\leq c_7(\beta - \alpha) \end{aligned}$$

with absolute constants  $c_6 > 0$  and  $c_7 > 0$ . Therefore

$$M_0(S_n, [\alpha, \beta]) \geq \exp(-c_1 c_7) (1 + o(1)) \sqrt{n/2},$$

and the lower bound of the theorem follows.

Now we prove the upper bound of the theorem. Using the well known estimate

$$M_1(D_\mu, [\alpha, \beta]) \leq \frac{1}{\beta - \alpha} M_1(D_\mu, [0, 2\pi]) \leq \frac{c_8 \log \mu}{\beta - \alpha}$$

with an absolute constant  $c_8 > 0$  and applying Theorem 1.5, we obtain that there is a constant  $c_9(\varepsilon) > 0$  depending only on  $\varepsilon > 0$  such that

$$\begin{aligned}
M_1(S_n, [\alpha, \beta]) &= M_1\left(\frac{1}{2}(f_p + D_p) + \frac{1}{2}z^p(f_q + D_q) + z^{p+q}D_r : [\alpha, \beta]\right) \\
&\leq \frac{1}{2}(M_1(f_p, [\alpha, \beta]) + M_1(f_q, [\alpha, \beta])) \\
&\quad + \frac{1}{2}M_1(D_p, [\alpha, \beta]) + \frac{1}{2}M_1(D_q, [\alpha, \beta]) + M_1(D_r, [\alpha, \beta]) \\
&\leq c_3(\varepsilon)\sqrt{p} + c_3(\varepsilon)\sqrt{q} + \frac{3c_8 \log(2n)}{\beta - \alpha} \\
&\leq c_9(\varepsilon)\sqrt{n}
\end{aligned}$$

whenever  $2(2n + o(n))^{-1/2+\varepsilon} \leq \beta - \alpha \leq 2\pi$ .  $\square$

*Proof of Theorem 2.3.* The theorem follows from Lemmas 3.5 and 3.7 in a straightforward fashion. Let  $P_n \in \mathcal{L}_n$  be defined by (3.2). Using Lemma 3.7 and then applying Lemma 3.1 with  $Q := P_n \in \mathcal{L}_n$ , and recalling that  $n = p + o(p)$ , we obtain

$$\begin{aligned}
((1 + o(1))\sqrt{n})^{(p-1)/p} &\leq \left(\prod_{k=0}^{p-1} |P_n(z_p^k)|\right)^{1/p} \leq 2^{n/p} M_0(P_n), \\
M_0(P_n) &\geq \left(\frac{1}{2} + o(1)\right) \sqrt{n},
\end{aligned}$$

and the theorem follows.  $\square$

*Proof of Theorem 2.4.* The proof is quite similar to that of Theorem 2.4 by replacing Lemma 3.3 with Lemma 3.7. We leave the straightforward modifications to the reader.  $\square$

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