

ON A PROBLEM OF BOURGAIN CONCERNING THE L_p NORMS OF EXPONENTIAL SUMS

STEPHEN CHOI AND TAMÁS ERDÉLYI

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ABSTRACT. For $n \geq 1$ let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^n z^{k_j} : 0 \leq k_1 < k_2 < \dots < k_n, k_j \in \mathbb{Z} \right\},$$

that is, \mathcal{A}_n is the collection of all sums of n distinct monomials. These polynomials are also called Newman polynomials. Let

$$M_p(Q) := \left(\int_0^1 |Q(e^{i2\pi t})|^p dt \right)^{1/p}, \quad p > 0.$$

We define

$$S_{n,p} := \sup_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \quad \text{and} \quad S_p := \liminf_{n \rightarrow \infty} S_{n,p} \leq \Sigma_p := \limsup_{n \in \mathbb{N}} S_{n,p}.$$

We show that

$$\Sigma_p \geq \Gamma(1 + p/2)^{1/p}, \quad p \in (0, 2).$$

The special case $p = 1$ recaptures a recent result of Aistleitner [1], the best known lower bound for Σ_1 .

1. INTRODUCTION

Let

$$M_p(Q) := \left(\int_0^1 |Q(e^{i2\pi t})|^p dt \right)^{1/p}, \quad p > 0.$$

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For $n \geq 1$ let

$$\mathcal{A}_n := \left\{ P : P(z) = \sum_{j=1}^n z^{k_j} : 0 \leq k_1 < k_2 < \cdots < k_n, k_j \in \mathbb{Z} \right\},$$

that is, \mathcal{A}_n is the collection of all sums of n distinct monomials. We define

$$S_{n,p} := \sup_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \quad \text{and} \quad S_p := \liminf_{n \rightarrow \infty} S_{n,p} \leq \Sigma_p := \limsup_{n \in \mathbb{N}} S_{n,p}.$$

We also define

$$I_{n,p} := \inf_{Q \in \mathcal{A}_n} \frac{M_p(Q)}{\sqrt{n}} \quad \text{and} \quad I_p := \limsup_{n \rightarrow \infty} I_{n,p} \geq \Omega_p := \liminf_{n \in \mathbb{N}} I_{n,p}.$$

The problem of calculating Σ_1 appears in a paper of Bourgain [5]. Deciding whether $\Sigma_1 < 1$ or $\Sigma_1 = 1$ would be a major step toward confirming or disproving other important conjectures. Karatsuba [7] observed that $\Sigma_1 \geq 1/\sqrt{2} \geq 0.707$. Indeed, taking, for instance,

$$P_n(z) = \sum_{k=0}^{n-1} z^{2^k}, \quad n = 1, 2, \dots,$$

it is easy to see that

$$(1.1) \quad M_4(P_n)^4 = 2n(n-1) + n,$$

and as Hölder's inequality implies

$$n = M_2(P_n)^2 \leq M_1(P_n)^{2/3} M_4(P_n)^{4/3},$$

we conclude

$$(1.2) \quad M_1(P_n) \geq \sqrt{\frac{n^2}{2n-1}} \geq \frac{\sqrt{n}}{\sqrt{2}}.$$

Similarly, if $S_n := \{a_1 < a_2 < \cdots < a_n\}$ is a Sidon set (that is, S_n is a subset of integers such that no integer has two essentially distinct representations as the sum of two elements of S_n), then the polynomials

$$P_n(z) = \sum_{a \in S_n} z^a, \quad n = 1, 2, \dots,$$

satisfy (1.1) and (1.2). In fact, it was observed in [4] that

$$(1.3) \quad \min_{P \in \mathcal{A}_n} M_4(P)^4 = 2n(n-1) + n,$$

and such minimal polynomials in \mathcal{A}_n are precisely constructed by Sidon sets as above.

Improving Karatsuba's result, by using a probabilistic method Aistleitner [1] proved that $\Sigma_1 \geq \sqrt{\pi}/2 \geq 0.886$. We note that P. Borwein and Lockhart [3] investigated the asymptotic behavior of the mean value of normalized L_p norms of Littlewood polynomials for arbitrary $p > 0$. Using the Lindeberg Central Limit Theorem and dominated convergence, they proved that

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_p(f))^p}{n^{p/2}} = \Gamma(1 + p/2)$$

where $\mathcal{L}_n := \left\{ P : P(z) = \sum_{j=0}^n a_j z^j, a_j \in \{-1, +1\} \right\}$. It follows simply from the $p = 1$ case of the the result in [3] quoted above that $\Sigma_1 \geq \sqrt{\pi/8} \geq 0.626$. Moreover, this can be achieved by taking the sum of approximately half of the monomials of $\{x^0, x^1, \dots, x^{2n}\}$ and letting n tend to ∞ .

In this note we show that

$$\Sigma_p \geq S_p \geq \Gamma(1 + p/2)^{1/p}, \quad p \in (0, 2),$$

and

$$\Omega_p \leq I_p \leq \Gamma(1 + p/2)^{1/p}, \quad p \in (2, \infty).$$

The special case $p = 1$ recaptures a recent result of Aistleitner [1], the best known lower bound for Σ_1 . Observe that Parseval's formula gives $\Omega_2 = \Sigma_2 = 1$.

2. NEW RESULTS

Theorem 2.1. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j \left(1 + \frac{c_j}{j^{1/2}} \right), \quad j = 1, 2, \dots,$$

where $\lim_{j \rightarrow \infty} c_j = \infty$. Let

$$P_n(z) = \sum_{j=1}^n z^{k_j}, \quad n = 1, 2, \dots$$

We have

$$\lim_{n \rightarrow \infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}$$

for every $p \in (0, 2)$.

Theorem 2.2. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > qk_j, \quad j = 1, 2, \dots,$$

where $q > 1$. Let

$$P_n(z) = \sum_{j=1}^n z^{k_j}, \quad n = 1, 2, \dots$$

We have

$$\lim_{n \rightarrow \infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}$$

for every $p \in [1, \infty)$.

Corollary 2.3. *We have $\Sigma_p \geq S_p \geq \Gamma(1 + p/2)^{1/p}$ for all $p \in (0, 2)$.*

The special case $p = 1$ of Corollary 2.3 recaptures a recent result of Aistleitner, and it is the best known lower bound in the problem of Bourgain mentioned in the introduction.

Corollary 2.4. *We have $\Sigma_1 \geq S_1 \geq \sqrt{\pi}/2$.*

Corollary 2.5. *We have $\Omega_p \leq I_p \leq \Gamma(1 + p/2)^{1/p}$ for all $p \in (2, \infty)$.*

We remark here that the same results also hold for the polynomials with coefficients $\sum_{j=1}^n a_j z^{k_j}$ if a general form of Salem-Zygmund theorem is used (e.g see (2) in [6]).

Our final result shows that the upper bound $\Gamma(1 + p/2)^{1/p}$ in Corollary 2.5 is optimal at least for even integers.

Corollary 2.6. *For any even integer $p = 2m \geq 2$, we have*

$$\lim_{n \rightarrow \infty} \min_{P \in \mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}.$$

Observe that a standard way to prove a Nikolskii-type inequality for trigonometric polynomials [2, p. 394] applies to the classes \mathcal{A}_n . Indeed,

$$\begin{aligned} M_p(P) &= \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^p dt \right)^{1/p} \\ &\leq \left(\left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^2 dt \right) \left(\max_{t \in [0, 2\pi]} |P(e^{it})| \right)^{p-2} \right)^{1/p} \\ &= (nn^{p-2})^{1/p} = n^{1-1/p}, \end{aligned}$$

for every $P \in \mathcal{A}_n$ and $p \geq 2$, and the Dirichlet kernel $D_n(z) := 1 + z + \dots + z^n$ shows the sharpness of this upper bound up to a multiplicative factor constant $c > 0$. So if we study the original Bourgain problem in the case of $p > 2$, we should normalize by dividing by $n^{1-1/p}$ rather than $n^{1/2}$.

3. PROOFS

Let $m(A)$ denote the Lebesgue measure of measurable sets $A \subset [0, 1]$. To prove Theorem 2.1 we need the complex-valued analogue of the following result of Erdős [6] (note that there is a typo in (4) in [6], the term $N^{1/2}$ should be $(N/2)^{1/2}$).

Theorem 3.1. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j \left(1 + \frac{c_j}{j^{1/2}}\right), \quad j = 1, 2, \dots,$$

where $\lim_{j \rightarrow \infty} c_j = \infty$. Let

$$Q_n(t) = \sum_{j=1}^n \cos(2\pi k_j(t - \theta_j)). \quad \theta_j \in \mathbb{R}.$$

Then

$$\lim_{n \rightarrow \infty} m(\{t \in [0, 1] : Q_n(t) < x(n/2)^{1/2}\}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$

for every $x \in \mathbb{R}$.

Following the proof of Theorem 1 in Erdős's paper [6], we can calculate the moments of $|P_n(e^{i2\pi t})|^2$ on $[0, 1]$ in the same way, and the limit distribution function

$$F(x) := \lim_{n \rightarrow \infty} m(\{t \in [0, 1] : |P_n(e^{i2\pi t})|^2 < xn\})$$

can be identified as $F(x) = 1 - e^{-x}$ on $[0, \infty)$. Hence the following complex-valued version of Erdős's result can be obtained. While Erdős could have easily claimed it in [6], our Theorem 3.2 below seems to be a new result.

Theorem 3.2. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > k_j \left(1 + \frac{c_j}{j^{1/2}}\right), \quad j = 1, 2, \dots,$$

where $\lim_{j \rightarrow \infty} c_j = \infty$. Let

$$P_n(z) = \sum_{j=1}^n z^{k_j}, \quad n = 1, 2, \dots$$

Then

$$\lim_{n \rightarrow \infty} m(\{t \in [0, 1] : |P_n(e^{i2\pi t})|^2 < xn\}) = 1 - e^{-x}$$

for every $x \in [0, \infty)$.

We also need the following result from [8, p. 215].

Theorem 3.3. *Let (k_j) be a strictly increasing sequence of nonnegative integers satisfying*

$$k_{j+1} > qk_j, \quad j = 1, 2, \dots,$$

where $q > 1$. Let

$$Q_n(t) = \sum_{j=1}^n \cos(2\pi k_j(t - \theta_j)).$$

Then for every $r > 0$ there are constants (depending only on r and q) $A_{r,q} > 0$ and $B_{r,q} > 0$ such that

$$A_{r,q}\sqrt{n} \leq M_r(Q_n) \leq B_{r,q}\sqrt{n}$$

for every $n \in \mathbb{N}$ and $r > 0$.

Proof of Theorem 2.1. Let

$$Z_n(t) := \frac{1}{\sqrt{n}} P_n(e^{i2\pi t}), \quad n = 1, 2, \dots$$

Observe that the functions $|Z_n|^p, n = 1, 2, \dots$, are uniformly integrable on $[0, 1]$. To see this let $a \geq 1$,

$$E := E_{n,p} := \{t \in [0, 1] : |Z_n(t)|^p \geq a\}$$

and

$$F := F_n := \{t \in [0, 1] : |Z_n(t)|^2 \geq a\}.$$

Using $p \in (0, 2)$, we have $E \subset F$. This, together with

$$\int_0^1 |Z_n(t)|^2 dt = 1$$

gives $m(E) \leq m(F) \leq a^{-1}$ for every $a > 1, p \in (0, 2)$, and $n \in \mathbb{N}$. Using Hölder's inequality we obtain that

$$\int_E |Z_n(t)|^p dt \leq \left(\int_E |Z_n(t)|^2 dt \right)^{p/2} (m(E))^{(2-p)/2} \leq a^{(p-2)/2}$$

for every $a > 1, p \in (0, 2)$, and $n \in \mathbb{N}$, which shows that the functions $|Z_n|^p, n = 1, 2, \dots$, are uniformly integrable on $[0, 1]$, indeed.

By Theorem 3.2 we have

$$U_n(x) := m(\{t \in [0, 1] : |Z_n(t)|^2 < x\})$$

converges to $F(x) := 1 - e^{-x}$ pointwise on $[0, \infty)$ as $n \rightarrow \infty$. Combining this with the uniform integrability of $|Z_n|^p, n = 1, 2, \dots$, on $[0, 1]$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 |Z_n(t)|^p dt &= \int_0^\infty x^{p/2} dF(x) = \int_0^\infty x^{p/2} F'(x) dx = \int_0^\infty x^{p/2} e^{-x} dx \\ &= \Gamma(1 + p/2). \quad \square \end{aligned}$$

Proof of the Theorem 2.2. Let, as before,

$$Z_n(t) := \frac{1}{\sqrt{n}} P_n(e^{i2\pi t}), \quad n = 1, 2, \dots$$

Introducing

$$X_n(t) := \frac{1}{\sqrt{n}} \operatorname{Re}(P_n(e^{i2\pi t})) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \cos(2\pi k_j t),$$

and

$$Y_n(t) := \frac{1}{\sqrt{n}} \operatorname{Im}(P_n(e^{i2\pi t})) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \sin(2\pi k_j t).$$

we have $Z_n(t) := X_n(t) + iY_n(t)$. Observe that the functions $|Z_n|^p, n = 1, 2, \dots$, are uniformly integrable on $[0, 1]$. To see this let $a > 0$ and

$$E := E_{n,p} := \{t \in [0, 1] : |Z_n(t)|^{2p} \geq a\}.$$

By Theorem 3.3 (recall that $p \geq 1$) we have

$$\begin{aligned} m(E)a &\leq \int_0^1 |Z_n(t)|^{2p} dt = \int_0^1 (|X_n(t)|^2 + |Y_n(t)|^2)^p dt \\ &\leq 2^{p-1} \int_0^1 (|X_n(t)|^{2p} + |Y_n(t)|^{2p}) dt \leq 2^p B_{2p,q}^{2p}. \end{aligned}$$

Hence $m(E) \leq 2^p B_{2p,q}^{2p} a^{-1}$ for every $a > 0, p \geq 1$, and $n \in \mathbb{N}$. Combining this with the Cauchy-Schwarz inequality and Theorem 3.3, we obtain

$$\begin{aligned} \int_E |Z_n(t)|^p dt &\leq \left(\int_E |Z_n(t)|^{2p} dt \right)^{1/2} (m(E))^{1/2} \leq B_{2p,q}^p (2^{p/2} B_{2p,q}^p a^{-1/2}) \\ &= 2^{p/2} B_{2p,q}^{2p} a^{-1/2} \end{aligned}$$

for every $a > 0, p \geq 1$, and $n \in \mathbb{N}$, which shows that the functions $|Z_n|^p, n = 1, 2, \dots$, are uniformly integrable on $[0, 1]$, indeed.

By Theorem 3.2 we have

$$U_n(x) := m(\{t \in [0, 1] : |Z_n(t)|^2 < x\})$$

converges to $F(x) := 1 - e^{-x}$ pointwise on $[0, \infty)$ as $n \rightarrow \infty$. Combining this with the uniform integrability of $|Z_n|^p, n = 1, 2, \dots$, on $[0, 1]$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 |Z_n(t)|^p dt &= \int_0^\infty x^{p/2} dF(x) = \int_0^\infty x^{p/2} F'(x) dx = \int_0^\infty x^{p/2} e^{-x} dx \\ &= \Gamma(1 + p/2). \quad \square \end{aligned}$$

Proof of Corollary 2.6. Let $P \in \mathcal{A}_n$ be of the form $P(z) = \sum_{j=1}^n z^{k_j}$ with some integers $0 \leq k_1 < k_2 < \dots < k_n$. We have

$$(P(z))^m = \left(\sum_{j=1}^n z^{k_j} \right)^m = \sum_{k=0}^{\infty} \left(\sum_{\substack{1 \leq j_1, j_2, \dots, j_m \leq n, \\ k_{j_1} + k_{j_2} + \dots + k_{j_m} = k}} 1 \right) z^k.$$

Hence

$$M_p(P)^p = M_2(P^m)^2 = \sum_{k=0}^{\infty} \left(\sum_{\substack{1 \leq j_1, j_2, \dots, j_m \leq n, \\ k_{j_1} + k_{j_2} + \dots + k_{j_m} = k}} 1 \right)^2 \geq \sum_{k=0}^{\infty} \left(\sum_{\substack{1 \leq j_1, j_2, \dots, j_m \leq n, j_\ell \neq j_i, \\ k_{j_1} + k_{j_2} + \dots + k_{j_m} = k}} 1 \right)^2.$$

Now, as the number of permutations of distinct values $k_{j_1}, k_{j_2}, \dots, k_{j_m}$ is $m!$, it follows that

$$M_p(P)^p \geq (m!)^2 \binom{n}{m} = m!n(n-1)\dots(n-m+1).$$

Hence, we have

$$\min_{P \in \mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} \geq \Gamma(1 + p/2)^{1/p} \left(1 \cdot \left(1 - \frac{1}{n} \right) \dots \left(1 - \frac{m-1}{n} \right) \right)^{1/p}.$$

Therefore

$$(3.1) \quad \liminf_{n \rightarrow \infty} \min_{P \in \mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} \geq \Gamma(1 + p/2)^{1/p}.$$

By Theorem 2.2, there are polynomials $P_n \in \mathcal{A}_n$ such that

$$\lim_{n \rightarrow \infty} \frac{M_p(P_n)}{\sqrt{n}} = \Gamma(1 + p/2)^{1/p}.$$

Hence

$$(3.2) \quad \limsup_{n \rightarrow \infty} \min_{P \in \mathcal{A}_n} \frac{M_p(P)}{\sqrt{n}} \leq \Gamma(1 + p/2)^{1/p}.$$

The corollary now follows from (3.1) and (3.2). \square

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DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, B.C., CANADA V5A 1S6
(S. CHOI)

E-mail address: `schoia@sfu.ca`

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843 (T.
ERDÉLYI)

E-mail address: `terdelyi@math.tamu.edu`