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THE RESOLUTION OF SAFFARI'S PHASE PROBLEM

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Abstract. We prove a conjecture of Saffari on the distribution of the angular speed of ultraflat sequences of unimodular polynomials.

0. Version française abrégée

Soit $\mathcal{K}_n := \left\{p_n : p_n(z) = \sum_{k=0}^n a_k z^k,\ \ a_k \in \mathbb{C}\,,\ \ |a_k| = 1\right\}$.

Définition. *Étant donnée une suite* (ε_{n_k}) *de nombres positifs tendant vers* 0*, on dit qu'une suite* (P_{n_k}) *de polynômes unimodulaires* $P_{n_k} \in \mathcal{K}_{n_k}$ *est* (ε_{n_k})*-ultraplate si*

$$(1-\varepsilon_{n_k})\sqrt{n_k+1} \le |P_{n_k}(z)| \le (1+\varepsilon_{n_k})\sqrt{n_k+1}\,, \qquad z \in \partial D\,,$$

ou encore

$$\max_{z\in\partial D} \big||P_{n_k}(z)| - \sqrt{n_k+1}\big| \le \varepsilon_{n_k}\sqrt{n_k+1}\,.$$

Soit (ε_n) une suite de nombres positifs tendant vers 0. L'hypothèse que la suite (P_n) de polynômes unimodulaires $P_n \in \mathcal{K}_n$ est (ε_n)-ultraplate sera notée par $(P_n) \in \mathrm{UF}((\varepsilon_n))$. Soit $(P_n) \in \mathrm{UF}((\varepsilon_n))$. On écrit

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,.$$

C'est un exercice facile de montrer que α_n peut être choisie différentiable sur $\mathbb{R}$. Avec ces notations, Saffari [Sa] a formulé la conjecture suivante:

Conjecture (Conjecture de distribution uniforme pour la vitesse angulaire).
Supposons $(P_n) \in \mathrm{UF}((\varepsilon_n))$*. Alors, dans l'intervalle* $[0, 2\pi]$*, la distribution de la vitesse angulaire normalisée* $\alpha_n'(t)/n$ *converge vers la distribution uniforme quand* $n \to \infty$*. Plus précisément, on a*

$$\mathrm{meas}\{t \in [0, 2\pi] : 0 \le \alpha_n'(t) \le nx\} = 2\pi x + o_n(x)$$

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pour tout $x \in [0,1]$*, où* $o_n(x)$ *converge vers* 0 *uniformément sur* $[0,1]$*. Il en résulte que* $|P_n'(e^{it})|/n^{3/2}$ *converge aussi vers la distribution uniforme quand* $n \to \infty$*.Plus précisément, on a*

$$\mathrm{meas}\{t \in [0,2\pi] : 0 \leq |P_n'(e^{it})| \leq n^{3/2}x\} = 2\pi x + o_n(x)$$

pour tout $x \in [0,1]$*, où* $o_n(x)$ *converge vers* 0 *uniformément sur* $[0,1]$.

Dans [Er1], nous démontrons cette conjecture. La présente note a pour but d'annoncer ces résultats.

1. Ultraflat Sequences of Unimodular Polynomials: The Theorem of Kahane

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k , \;\; a_k \in \mathbb{C} \, , \;\; |a_k| = 1 \right\} .$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k , \;\; a_k \in \{-1, 1\} \right\} .$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. We can easily see by using Parseval's formula that

$$\min_{z \in \partial D} |P_n(z)| < \sqrt{n+1} < \max_{z \in \partial D} |P_n(z)| \tag{1.1}$$

for all $P_n \in \mathcal{K}_n$.

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *Examine that how close a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ can come to satisfying*

$$|P_n(z)| = \sqrt{n+1} \, , \qquad z \in \partial D \, . \tag{1.2}$$

Obviously (1.2) is impossible if $n \geq 1$. So one must look for less than (1.2), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definitions. In the rest of the paper we assume that (n_k) is a strictly increasing sequence of positive integers.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1} \, , \qquad z \in \partial D \, , \tag{1.3}$$

or equivalently

$$\max_{z \in \partial D} \big| |P_n(z)| - \sqrt{n+1} \big| \leq \varepsilon \sqrt{n+1} \, .$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to* 0*, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if*

$$(1.4) \qquad (1-\varepsilon_{n_k})\sqrt{n_k+1} \le |P_{n_k}(z)| \le (1+\varepsilon_{n_k})\sqrt{n_k+1}\,, \qquad z \in \partial D\,,$$

or equivalently

$$\max_{z \in \partial D} \big| |P_{n_k}(z)| - \sqrt{n_k+1} \big| \le \varepsilon_{n_k}\sqrt{n_k+1}\,.$$

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er3]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \ge 1$,

$$(1.5) \qquad \max_{z \in \partial D} |P_n(z)| \ge (1+\varepsilon)\sqrt{n+1}\,,$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$(1.6) \qquad \varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)\,.$$

Thus the Erdős conjecture (1.5) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no sequence of ultraflat unimodular polynomials $P_n \in \mathcal{L}_n$.

An interesting related result to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2].

In [Er1] we study ultraflat sequences (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ in general, not necessarily those produced by Kahane in his paper [Ka]. With trivial modifications our results remain valid even if we study ultraflat sequences (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$. It is left to the reader to formulate these analogue results. In [Er1], among others, we prove a conjecture of Saffari on the distribution of the angular speed of ultraflat sequences of unimodular polynomials.

2. The Phase Problem: the Conjecture of Saffari

Let (ε_n) be a sequence of positive numbers tending to 0. The assumption that the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is (ε_n)-ultraflat will be denoted by $(P_n) \in \mathrm{UF}((\varepsilon_n))$. Let $(P_n) \in \mathrm{UF}((\varepsilon_n))$. We write

$$(2.1) \qquad P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,.$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable on $\mathbb{R}$. This is going to be our understanding throughout the paper. Saffari [Sa] conjectures the following:

Conjecture 2.1 (Uniform Distribution Conjecture for the Angular Speed). *Suppose* $(P_n) \in \mathrm{UF}((\varepsilon_n))$. *Then, with the notation (2.1), in the interval* $[0, 2\pi]$, *the distribution of the normalized angular speed* $\alpha_n'(t)/n$ *converges to the uniform distribution as* $n \to \infty$. *More precisely, we have*

$$\operatorname{meas}\{t \in [0, 2\pi] : 0 \le \alpha_n'(t) \le nx\} = 2\pi x + o_n(x) \tag{2.2}$$

for every $x \in [0, 1]$, *where* $o_n(x)$ *converges to* 0 *uniformly on* $[0, 1]$. *As a consequence,* $|P_n'(e^{it})|/n^{3/2}$ *also converges to the uniform distribution as* $n \to \infty$. *More precisely, we have*

$$\operatorname{meas}\{t \in [0, 2\pi] : 0 \le |P_n'(e^{it})| \le n^{3/2}x\} = 2\pi x + o_n(x) \tag{2.3}$$

for every $x \in [0, 1]$, *where* $o_n(x)$ *converges to* 0 *uniformly on* $[0, 1]$.

In both statements the convergence of $o_n(x)$ *is uniform on* $[0, 1]$ *by Dini's Theorem.*

The basis of this conjecture was that for the special ultraflat sequences of unimodular polynomials produced by Kahane [Ka], (2.2) is indeed true.

In Section 4 of [Er1] we prove this conjecture in general.

In the general case (2.2) can, by integration, be reformulated (equivalently) in terms of the moments of the angular speed $\alpha_n'(t)$. This was observed and recorded by Saffari [Sa]. For completeness we will present the proof of this equivalence in Section 4 of [Er1] and we will settle Conjecture 2.1 by proving the following result.

Theorem 2.2 (Reformulation of the Uniform Distribution Conjecture). *Let* $(P_n) \in \mathrm{UF}((\varepsilon_n))$. *Then, for any* $q > 0$ *we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + o_{n,q} n^q \, . \tag{2.4}$$

with suitable constants $o_{n,q}$ *converging to* 0 *as* $n \to \infty$ *for every fixed* $q > 0$.

An immediate consequence of (2.4) is the remarkable fact that for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein factors

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt}$$

of the elements of ultraflat sequences (P_n) of unimodular polynomials are essentially independent of the polynomials. More precisely (2.4) implies the following result.

Theorem 2.3 (The Bernstein Factors). *Let* q *be an arbitrary positive real number. Let* $(P_n) \in \mathrm{UF}((\varepsilon_n))$. *We have*

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt} = \frac{n^{q+1}}{q+1} + o_{n,q} n^{q+1} \, ,$$

and as a limit case,

$$\frac{\max_{0\leq t\leq 2\pi}|P_n'(e^{it})|}{\max_{0\leq t\leq 2\pi}|P_n(e^{it})|}=n+o_n n\,.$$

with suitable constants $o_{n,q}$ and o_n converging to 0 as $n\to\infty$ for every fixed q.

In Section 3 [Er1] we prove the following result which turns out to be stronger than Theorem 2.2.

Theorem 2.4 (Negligibility Theorem for Higher Derivatives). *Let $(P_n)\in$ UF$((\varepsilon_n))$. For every integer $r\geq 2$, we have*

$$\max_{0\leq t\leq 2\pi}|\alpha_n^{(r)}(t)|=o_{n,r}n^r$$

with suitable constants $o_{n,r}$ converging to 0 for every fixed $r=2,3,\dots$. As a consequence,

$$\max_{z\in\partial D}|P_n^{(r)}(z)|=(1+o_{n,r}^*)n^{r+1/2}$$

and

$$\min_{z\in\partial D}|P_n^{(r)}(z)|=o_{n,r}^{**}n^{r+1/2}$$

with suitable constants $o_{n,r}^*$ and $o_{n,r}^{**}$ converging to 0 for every fixed $r=1,2,\dots$.

We show in Section 4 of [Er1] how Theorem 2.1 follows from Theorem 2.4.

3. Structural Properties

As it was mentioned earlier, the existence of ultraflat sequences of unimodular polynomials was questionable for a long time; its only known proof due to Kahane [Ka] is built on ideas coming from probability theory and even experts find it a very difficult reading; and although a constructive proof would be pretty much desirable, until today nobody has succeeded in finding an algorithm establishing ultraflat sequences of unimodular polynomials. Yet, the class of ultraflat sequences of unimodular polynomials seems to be quite large, and it shows a beautiful structure. One of the properties is encapsulated by the Uniform Distribution Theorem of the angular speed of ultraflat sequences of unimodular polynomials. This proves a basic conjecture of Saffari. The structural properties of ultraflat sequences of unimodular polynomials may lead closer to settle other conjectures such as the existence (or non-existence) of ultraflat sequences of polynomials with only ± 1 coefficients. Also, we may search for ultraflat sequences of unimodular polynomials with some other additional properties. For example, if Q_n is a polynomial of degree n of the form

$$Q_n(z)=\sum_{k=0}^n a_k z^k\,,\qquad a_k\in\mathbb{C}\,,$$

then its conjugate polynomial is defined by

$$Q_n^*(z):=z^n\overline{Q}_n(1/z):=\sum_{k=0}^n\overline{a}_{n-k}z^k\,.$$

In a forthcoming paper [Er2] it is shown that the following is true.

Theorem 3.1. *If (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$, then*

$$\int_{\partial D} (|P_n'(z)| - |P_n^{*\prime}(z)|)^2 \, |dz| = 2\pi \left(\frac{1}{3} + \gamma_n \right) n^3 \,,$$

where (γ_n) is a sequence of real numbers converging to 0.

Hence there are no ultraflat sequences of "conjugate reciprocal" unimodular polynomials (with the property $P_n = P_n^*$ for each $n = 1, 2, \dots$). Moreover, Theorem 3.1 measures how far an ultraflat sequence of unimodular polynomials is from being "conjugate reciprocal".

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