THE FULL MÜNTZ THEOREM IN $C[0,1]$ AND $L_1[0,1]$

PETER BORWEIN AND TAMÁS ERDÉLYI

ABSTRACT. The main result of this paper is the establishment of the "full Müntz Theorem" in $C[0,1]$. This characterizes the sequences $\{\lambda_i\}_{i=1}^{\infty}$ of distinct, positive real numbers for which

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$. The novelty of this result is the treatment of the most difficult case when $\inf_i \lambda_i = 0$ while $\sup_i \lambda_i = \infty$. The paper settles the L_∞ and L_1 cases of the following.

Conjecture (Full Müntz Theorem in $L_p[0,1]$). *Let $p \in [1,\infty]$. Suppose $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_p[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \infty.$$

1. INTRODUCTION

Müntz's beautiful, classical theorem characterizes sequences $\{\lambda_i\}_{i=1}^{\infty}$ with $\lambda_0 = 0$ and $\inf_i \lambda_i > 0$ for which the Müntz space $\operatorname{span}\{1, x^{\lambda_0}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$. Here, and in what follows, $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C[A]$ is the space of all real-valued continuous functions on $A \subset [0,\infty)$ equipped with the uniform norm. Throughout we assume that the exponents, λ_i, are real. Müntz's Theorem [12, 18, 25, 28] states the following.

Müntz's Theorem. *Let $\Lambda := \{\lambda_i\}_{i=1}^{\infty}$ be a sequence with $\inf_i \lambda_i > 0$. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_1}, \dots\}$$

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is dense in $C[0,1]$ *if and only if*

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_i}=\infty.$$

The original Müntz Theorem proved by Müntz [18] in 1914, by Szász [25] in 1916, and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. Later works, see, for example, [22] and [19], include the above result, as well as a treatment of the case when $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct, positive real numbers tending to 0. The novelty in this paper is the treatment of the case when $\inf_i \lambda_i = 0$, while $\sup_i \lambda_i = \infty$. This "full Müntz Theorem in $C[0,1]$" is formulated by Theorem 2.1. For the sake of completeness, we present a proof in all of the cases.

When $\lambda_i \geq 1$ for each $i = 1, 2, \dots$, the above theorem follows by a simple trick from the $L_2[0,1]$ version of Müntz's Theorem. This "full Müntz Theorem in $L_2[0,1]$ formulated by Theorem 2.2 was obtained by Szász. Most of the standard proofs available in the literature deal only with special cases (like assuming monotonicity of the sequence), despite the fact that the proof of the full version is not harder. We present its proof in Section 4 both because it is short and for reasons of completeness.

Theorem 2.3 establishes the full $L_1[0,1]$ version of Müntz's Theorem. Based on the above $C[0,1]$, $L_2[0,1]$, and $L_1[0,1]$ results, a conjecture is made, which is most likely the right "full Müntz Theorem" in $L_p[0,1]$.

The point 0 is special in the study of Müntz spaces. Replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [13] and Schwartz [23] whose works include the result that if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct, positive real numbers satisfying $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ then every function belonging to the uniform closure of $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_1}, \dots\}$ on $[a,b]$ can be extended analytically throughout the region $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}$.

We remark that the full version of Müntz's Theorem for arbitrary real exponents on an interval $[a,b]$, $0 < a < b$, is known. It states the following.

(Full Müntz's Theorem in $L_p[a,b]$, $a > 0$**).** *Let* $\{\lambda_i\}_{i=0}^{\infty}$ *be a sequence of distinct real numbers. Let* $0 < a < b$ *and* $0 < p < \infty$*. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_p[a,b]$ *if and only if*

$$\sum_{\substack{i=0\\ \lambda_i \neq 0}}^{\infty}\frac{1}{|\lambda_i|}=\infty.$$

See, for example, [23] or [10].

Extensions of Müntz's Theorem abound. For example, generalizations to complex exponents are considered in Luxemburg and Korevaar [16] and to angular regions in Anderson [1]. We showed [9] that if $\{\lambda_i\}_{i=0}^{\infty}$ is an increasing sequence of nonnegative real numbers with $\lambda_0 = 0$ then the interval $[0,1]$ in Müntz's Theorem can be replaced by an arbitrary compact set $A \subset [0,\infty)$ of positive Lebesgue measure. For further results see, for example, [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28].

2. Results

Theorem 2.1 (Full Müntz Theorem in $C[0,1]$). *Suppose $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct, positive real numbers. Then*

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty.$$

Theorem 2.2 (Full Müntz Theorem in $L_2[0,1]$). *Suppose $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/2$. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_2[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = \infty.$$

Theorem 2.3 (Full Müntz Theorem in $L_1[0,1]$). *Suppose $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than -1. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_1[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = \infty.$$

Based on the above $C[0,1]$, $L_2[0,1]$, and $L_1[0,1]$ results, it is reasonable to conjecture the following statement, half of which is straightforward by a standard method. See also the final remark of this paper.

Conjecture 2.4 (Full Müntz Theorem in $L_p[0,1]$). *Let $p \in [1,\infty]$. Suppose $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$. Then*

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_p[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \infty.$$

Some of this is touched on in Schwartz [23] without proof. It seems likely that his methods allow for the resolution of this conjecture in the case $1 \le p \le 2$.

3. Notation and Auxiliary Results

The uniform and L_p norms of a function f on a set $A \subset \mathbb{R}$ will be denoted by $\|f\|_A$ and $\|f\|_{L_p[A]}$, respectively.

Let $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct, nonnegative real numbers with $\lambda_0 = 0$. The nonnegative-valued functions x^{λ_i} are well-defined on $[0,\infty)$. The collection

$$\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

is called a (finite) Müntz system. The linear space

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ is called a (finite) Müntz space. That is, the Müntz space $M_n(\Lambda)$ is the collection of Müntz polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}, \qquad a_i \in \mathbb{R}.$$

The set

$$M(\Lambda) := \bigcup_{n=0}^{\infty} M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is called the (infinite) Müntz space associated with Λ.

One of the most basic properties of a Müntz space $M_n(\Lambda)$ is the fact that it is a Chebyshev space on every $A \subset [0,\infty)$ containing at least $n+1$ points. That is, $M(\Lambda) \subset C[A]$ and every $p \in M_n(\Lambda)$ having at least $n+1$ (distinct) zeros in A is identically 0. In fact, Müntz spaces are the "canonical" examples for Chebyshev spaces and the following properties of Müntz spaces $M_n(\Lambda)$ are well known (see, for example, [10, 12, 21]).

Theorem 3.1 (Existence of Chebyshev Polynomials). *Let A be a compact subset of $[0,\infty)$ containing at least $n+1$ points. Then there exists a unique (extended) Chebyshev polynomial*

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\}$$

for $M_n(\Lambda)$ on A defined by

$$T_n(x) = c\left(x^{\lambda_n} - \sum_{i=0}^{n-1} a_i x^{\lambda_i}\right),$$

where the numbers $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ are chosen to minimize

$$\left\| x^{\lambda_n} - \sum_{i=0}^{n-1} a_i x^{\lambda_i} \right\|_A$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that

$$\|T_n\|_A = 1$$

and the sign of c is determined by

$$T_n(\max A) > 0.$$

Theorem 3.2 (Alternation Characterization). *The Chebyshev polynomial*

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\} \in M_n(\Lambda)$$

is uniquely characterized by the existence of an alternation set

$$\{x_0 < x_1 < \cdots < x_n\} \subset A$$

for which

$$T_n(x_j) = (-1)^{n-j} = (-1)^{n-j}\|T_n\|_A, \qquad j = 0, 1, \dots, n.$$

We will also need the following result due to Newman [19].

Theorem 3.3 (Newman's Inequality). *The inequality*

$$\|x\,p'(x)\|_{[0,1]} \le 11\left(\sum_{i=0}^{n} \lambda_i\right)\|p\|_{[0,1]}$$

holds for every $p \in M_n(\Lambda)$.

The following inequality, proved in [9] and [10], is also needed in our proof of Theorem 2.1.

Theorem 3.4. *Suppose $\Lambda := \{\lambda_i\}_{i=1}^{\infty}$ is a sequence of nonnegative real numbers satisfying $\lambda_0 = 0$, $\lambda_i \ge 1$ for $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $\epsilon \in (0,1)$. Then there exists a constant c depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and ϵ (and not on the "length" of p) so that*

$$\|p'\|_{[0,1-\varepsilon]} \le c\|p\|_{[0,1]}$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

4. Proofs

First we use a standard approach to prove Theorem 2.2.

Proof of Theorem 2.2. We consider the approximation to x^m by elements of

$$\operatorname{span}\{x^{\lambda_0},\dots,x^{\lambda_{n-1}}\}$$

in $L_2[0,1]$ and we assume $m\neq\lambda_i$ for any i. It is well known that

$$\min_{b_i\in\mathbb{C}}\left\|x^m-\sum_{i=0}^{n-1}b_ix^{\lambda_i}\right\|_{L_2[0,1]}=\frac{1}{\sqrt{1+2m}}\prod_{i=0}^{n-1}\left|\frac{m-\lambda_i}{m+\lambda_i+1}\right|.$$

See, for example, [22] or [10]. So, for a nonnegative integer m different from any of the exponents λ_i,

$$x^m\in\overline{\operatorname{span}}\{x^{\lambda_0},x^{\lambda_1},\dots\} \tag{4.1}$$

where $\overline{\operatorname{span}}$ denotes the $L_2[0,1]$ closure of the span, if and only if

$$\limsup_n\prod_{i=0}^{n-1}\left|\frac{m-\lambda_i}{m+\lambda_i+1}\right|=0.$$

That is, (4.1) holds if and only if

$$\limsup_n\prod_{\substack{i=0\\ \lambda_i>m}}^{n-1}\left|1-\frac{2m+1}{m+\lambda_i+1}\right|\prod_{\substack{i=0\\ -1/2<\lambda_i\le m}}^{n-1}\left|1-\frac{2\lambda_i+1}{m+\lambda_i+1}\right|=0.$$

Hence (4.1) holds if and only if

$$\sum_{\substack{i=0\\ \lambda_i>m}}^{\infty}\frac{1}{2\lambda_i+1}=\infty\quad\text{or}\quad\sum_{\substack{i=0\\ -1/2<\lambda_i\le m}}^{\infty}(2\lambda_i+1)=\infty.$$

Therefore, (4.1) holds if and only if

$$\sum_{i=0}^{\infty}\frac{2\lambda_i+1}{(2\lambda_i+1)^2+1}=\infty$$

and the proof can be finished by the Weierstrass Approximation Theorem. □

The case when $\lambda_i\ge 1$ for each i in Theorem 2.1 is also standard.

Proof of Theorem 2.1 assuming Theorem 2.2 and each $\lambda_i\ge 1$. We have

$$\begin{aligned}\left|x^m-\sum_{i=0}^{n}a_ix^{\lambda_i}\right|&=\left|\int_0^x\left(mt^{m-1}-\sum_{i=0}^{n}a_i\lambda_it^{\lambda_i-1}\right)dt\right|\\&\le\int_0^1\left|mt^{m-1}-\sum_{i=0}^{n}a_i\lambda_it^{\lambda_i-1}\right|dt\\&\le\left(\int_0^1\left|mt^{m-1}-\sum_{i=0}^{n}a_i\lambda_it^{\lambda_i-1}\right|^2dt\right)^{1/2}\end{aligned}$$

and

$$\left(\int_0^1 \left|t^m - \sum_{i=0}^n a_i t^{\lambda_i}\right|^2 dt\right)^{1/2} \leq \left\|t^m - \sum_{i=0}^n a_i t^{\lambda_i}\right\|_{[0,1]}$$

for every $x \in [0,1]$ and $m = 1, 2, \dots$. The assumption that $\lambda_i \geq 1$ for each i implies that

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty \quad \text{if and only if} \quad \sum_{i=1}^{\infty} \frac{2(\lambda_i - 1) + 1}{(2(\lambda_i - 1) + 1)^2 + 1} = \infty$$

and

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty \quad \text{if and only if} \quad \sum_{i=1}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = \infty.$$

If $\sum_{i=1}^{\infty} \lambda_i/(\lambda_i^2 + 1) = \infty$ then the first inequality, together with Theorem 2.2 and the Weierstrass Approximation Theorem shows that

$$\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is dense in $C[0,1]$. While if the above span is dense in $C[0,1]$ then the second inequality, together with the Weierstrass Approximation Theorem, shows that it is also dense in $L_2[0,1]$, hence Theorem 2.2 implies $\sum_{i=1}^{\infty} \lambda_i/(\lambda_i^2 + 1) = \infty$. $\square$

Note that a combination of a scaling $x \to x^\delta$ and the above argument yields a proof of Theorem 2.1 in the case when $\inf_i \lambda_i > 0$.

Yet Another Proof of Half of Theorem 2.1 when $\inf_i \lambda_i > 0$. We show that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$$

implies that $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$. This proof requires a consequence of the Hahn-Banach Theorem and the Riesz Representation Theorem. For details the reader is referred to Feinerman and Newman [15] or Rudin [22]. We will need a modification of this argument, so we briefly present it. We assume that $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct positive real numbers satisfying $\inf_i \lambda_i > 0$.

By the Hahn-Banach Theorem and the Riesz Representation Theorem

$$\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is not dense if and only if there exists a non-zero, finite (signed) Borel measure μ on $[0,1]$ with

$$\int_0^1 t^{\lambda_i}\, d\mu(t) = 0, \qquad i = 0, 1, \dots$$

where $\lambda_0 = 0$.

Suppose there is a non-zero, finite (signed) Borel measure μ on $[0,1]$ so that

$$\int_0^1 t^{\lambda_i}\, d\mu(t) = 0, \qquad i = 0, 1, \dots .$$

Let

$$f(z) := \int_0^1 t^z \, d\mu(t).$$

Then

$$g(z) := f\left(\frac{1+z}{1-z}\right)$$

is a bounded analytic function on the open unit disk satisfying

$$g\left(\frac{\lambda_n - 1}{\lambda_n + 1}\right) = 0.$$

Note that

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty$$

and the fact that $\inf_i \lambda_i > 0$ imply

$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = \infty.$$

Hence Blaschke's Theorem [22] yields that $g = 0$ on the open unit disk. Therefore $f(z) = 0$ whenever $\mathrm{Re}(z) > 0$, so

$$f(n) = \int_0^1 t^n \, d\mu(t) = 0, \qquad n = 1, 2, \ldots .$$

Note that

$$\int_0^1 t^0 \, d\mu(t) = 0$$

also holds because of the choice of μ. Now the Weierstrass Approximation Theorem yields

$$\int_0^1 u(t) \, d\mu(t) = 0$$

for every $u \in C[0,1]$, which contradicts the fact that the Borel measure on μ is non-zero. So

$$\mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

is dense in $C[0,1]$. $\square$

Proof of Theorem 2.1 when $\lambda_i \to 0$. Suppose $\Lambda := \{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct positive real numbers with $\lim_{i\to\infty} \lambda_i = 0$. We show that

$$M(\Lambda) := \mathrm{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$$

is dense in $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \lambda_i = \infty.$$

If $\sum_{n=1}^{\infty} \lambda_i = \infty$ then $\lim_{i\to\infty} \lambda_i = 0$ implies that

$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = \infty.$$

So the arguments in the previous proof yield that $M(\Lambda)$ is dense in $C[0,1]$.

If $\eta := \sum_{i=1}^{\infty} \lambda_i < \infty$ then by Theorem 3.3 (Newman's Inequality),

$$\|xp'(x)\|_{[0,1]} \le 11\eta \|p\|_{[0,1]}$$

holds for every $p \in M(\Lambda)$. This implies that $M(\Lambda)$ fails to be dense in $C[0,1]$. $\square$

Proof of Theorem 2.1 when

$$\{\lambda_i : i \in \mathbb{N}\} = \{\alpha_i : i \in \mathbb{N}\} \cup \{\beta_i : i \in \mathbb{N}\}$$

with

$$\lim_{i\to\infty} \alpha_i = 0 \quad \textit{and} \quad \lim_{i\to\infty} \beta_i = \infty$$

holds. In this case we need to show that $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \frac{1}{\beta_i} = \infty. \tag{4.3}$$

If (4.2) holds then the already examined cases yield the denseness of

$$\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

in $C[0,1]$. Now assume that (4.2) does not hold, so

$$\sum_{i=1}^{\infty} \alpha_i < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty. \tag{4.3}$$

For notational convenience, let

$$\begin{aligned} T_{n,\alpha} &:= T_n\{1, x^{\alpha_1}, \dots, x^{\alpha_n} : [0,1]\} \\ T_{n,\beta} &:= T_n\{1, x^{\beta_1}, \dots, x^{\beta_n} : [0,1]\} \\ T_{2n,\alpha,\beta} &:= T_{2n}\{1, x^{\alpha_1}, \dots, x^{\alpha_n}, x^{\beta_1}, \dots, x^{\beta_n} : [0,1]\} \end{aligned}$$

(we use the notation introduced in Section 3). It follows from Theorem 3.3 (Newman's Inequality) and the Mean Value Theorem that for every $\epsilon > 0$ there exists a $k_1(\epsilon) \in \mathbb{N}$ depending only on $\{\alpha_i\}_{i=1}^{\infty}$ and ϵ (and not on n) so that $T_{n,\alpha}$ has at most $k_1(\epsilon)$ zeros in $[\epsilon, 1)$ and at least $n - k_1(\epsilon)$ zeros in $(0, \epsilon)$.

Similarly, Theorem 3.4 and the Mean Value Theorem imply that for every $\epsilon > 0$ there exists a $k_2(\epsilon) \in \mathbb{N}$ depending only on $\{\beta_i\}_{i=1}^{\infty}$ and ϵ (and not on n) so that $T_{n,\beta}$ has at most $k_2(\epsilon)$ zeros in $(0, 1-\epsilon]$ and at least $n - k_2(\epsilon)$ zeros $(1-\epsilon, 1)$.

Now, counting the zeros of $T_{n,\alpha}-T_{2n,\alpha,\beta}$ and $T_{n,\beta}-T_{2n,\alpha,\beta}$, we can deduce that for every $\epsilon>0$ there exists a $k(\epsilon)\in\mathbb{N}$ depending only on $\{\lambda_i\}_{i=1}^{\infty}$ and ϵ (and not on n) so that $T_{2n,\alpha,\beta}$ has at most $k(\epsilon)$ zeros in $[\epsilon,1-\epsilon]$.

Let $\epsilon:=1/4$ and $k:=k(1/4)$. Pick $k+4$ points

$$\frac{1}{4}<\eta_0<\eta_1<\cdots<\eta_{k+3}<\frac{3}{4}$$

and a function $f\in C[0,1]$ so that $f(x)=0$ for all $x\in[0,1/4]\cup[3/4,1]$, while

$$f(\eta_i):=2\cdot(-1)^i,\qquad i=0,1,\dots,k+3.$$

Assume that there exists a $p\in\operatorname{span}\{1,x^{\lambda_1},x^{\lambda_2},\dots\}$ so that

$$\|f-p\|_{[0,1]}<1.$$

Then $p-T_{2n,\alpha,\beta}$ has at least $2n+1$ zeros in$(0,1)$. However, for sufficiently large n,

$$p-T_{2n,\alpha,\beta}\in\operatorname{span}\{1,x^{\lambda_1},\dots,x^{\lambda_{2n}}\}$$

which can have at most $2n$ zeros in $[0,\infty)$. This contradiction shows that

$$\operatorname{span}\{1,x^{\lambda_1},x^{\lambda_2},\dots\}$$

is not dense in $C[0,1]$. □

Proof of Theorem 2.1 when $\{\lambda_i\}_{i=1}^{\infty}$ has a cluster point in $(0,\infty)$. In this case the already examined case that $\inf_i\lambda_i>0$ implies that

$$\operatorname{span}\{1,x^{\lambda_1},x^{\lambda_2},\dots,\}$$

is dense in $C[0,1]$. □

By this the proof of Theorem 2.1 is finished (each of the possible cases has been considered).

Proof of Theorem 2.3. Assume that

$$\operatorname{span}\{x^{\lambda_0},x^{\lambda_1},\dots\}$$

is dense in $L_1[0,1]$. Let m be a fixed nonnegative integer. Let $\epsilon>0$. Choose a

$$p\in\operatorname{span}\{x^{\lambda_0},x^{\lambda_1},\dots\}$$

so that

$$\|x^m-p(x)\|_{[0,1]}<\epsilon.$$

Now let

$$q(x):=\int_0^x p(t)\,dt\in\operatorname{span}\{x^{\lambda_0+1},x^{\lambda_1+1},\dots\}.$$

Then

$$\left\| \frac{x^{m+1}}{m+1} - q(x) \right\|_{[0,1]} < \epsilon.$$

So the Weierstrass Approximation Theorem yields that

$$\operatorname{span}\{1, x^{\lambda_0+1}, x^{\lambda_1+1}, \dots\}$$

is dense in $C[0,1]$. Now Theorem 2.1 implies that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2 + 1} = \infty.$$

Now assume that

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i+1)^2 + 1} = \infty. \tag{4.4}$$

By the Hahn-Banach Theorem and the Riesz Representation Theorem

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is not dense in $L_1[0,1]$ if and only if there exists a $0 \neq h \in L_\infty[0,1]$ satisfying

$$\int_0^1 t^{\lambda_i} h(t)\, dt = 0, \qquad i = 0, 1, \dots .$$

Suppose there exists a $0 \neq h \in L_\infty[0,1]$ so that

$$\int_0^1 t^{\lambda_i} h(t)\, dt = 0, \qquad i = 0, 1, \dots .$$

Let

$$f(z) := \int_0^1 t^z h(t)\, dt.$$

Then

$$g(z) := f\left(\frac{1+z}{1-z} - 1\right)$$

is a bounded analytic function on the open unit disk satisfying

$$g\left(\frac{\lambda_n}{\lambda_n + 2}\right) = 0.$$

Note that (4.4) implies

$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n}{\lambda_n + 2}\right|\right) = \infty.$$

Hence Blaschke's Theorem [22] yields that $g = 0$ on the open unit disk. Therefore $f(z) = 0$ whenever $\mathrm{Re}(z) > -1$, so

$$f(n) = \int_0^1 t^n h(t)\,dt = 0, \qquad n = 0, 1, \dots .$$

Now the Weierstrass Approximation Theorem yields

$$\int_0^1 u(t)h(t)\,dt = 0$$

for every $u \in C[0, 1]$, which contradicts the fact that $0 \neq h$. So

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_1[0, 1]$. $\square$

Remark to Conjecture 2.4. The method used in the previous proof can be used to show that if $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$ satisfying

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \infty$$

then

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is dense in $L_p[0, 1]$. It is the other half of the conjecture we are not able to prove completely so far. Half of this other half is also proved in [10]. Namely if $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$ converging to $-1/p$ and

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} < \infty$$

then

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is not dense in $L_p[0, 1]$.

Another case of the open half follows from Theorem 2.3 (Full Müntz Theorem in $L_1[0, 1]$) and Hölder's Inequality. Namely if $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$ converging to ∞ and

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} < \infty$$

then

$$\mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is not dense in $L_1[0, 1]$, so it is not dense in $L_p[0, 1]$.

The harder direction of Theorem 2.1 also follows from the following bounded Bernstein-type inequality. This states that if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of distinct positive numbers satisfying

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} < \infty$$

then for every $\epsilon > 0$, there is a constant c_ϵ so that

$$|p'(x)| \leq \frac{c_\epsilon}{x} \|p\|_{[0,1]}, \qquad x \in (0, 1-\epsilon]$$

for every

$$p \in \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}.$$

The proof of this is rather complicated. We present a proof in [10].

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Department of Mathematics, Simon Fraser University, Burnaby B.C., Canada V5A 1S6