LITTLEWOOD-TYPE PROBLEMS ON $[0,1]$

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Abstract. A highlight of the paper states that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

for every $n \geq 2$, where $\mathcal{F}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$.

This Chebyshev-type problem is closely related to the question of how many zeros a polynomial from the above classes can have at 1. We also give essentially sharp bounds for this problem. Among others we prove that there is an absolute constant $c > 0$ such that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \leq 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ real zeros. This improves the old bound $c\sqrt{n \log n}$ given by Schur in 1933 as well as more recent related bounds of Bombieri and Vaaler, and up to the constant c this is the best possible result.

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0. Main Results

We consider the problem of minimizing the uniform norm on $[0,1]$ over polynomials $0 \neq p$ of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}\,,$$

where the modulus of the first non-zero coefficient is at least $\delta > 0$. Essentially sharp bounds are given for this problem. An interesting related result states that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

for every $n \geq 2$, where $\mathcal{F}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

This Chebyshev-type problem is closely related to the question of how many zeros a polynomial from the above classes can have at 1. We also give essentially sharp bounds for this problem.

Inter alia we prove that there is an absolute constant $c > 0$ such that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \leq 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ real zeros. This improves the old bound $c\sqrt{n \log n}$ given by Schur in 1933 as well as more recent related bounds of Bombieri and Vaaler, and up to the constant c this is the best possible result.

All the analysis rests critically on the key estimate stating that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$|f(0)|^{c_1/a} \leq \exp\left(\frac{c_2}{a}\right) \|f\|_{[1-a,1]}$$

for every $f \in \mathcal{S}$ and $a \in (0,1]$, where $\mathcal{S}$ denotes the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \leq \frac{1}{1-|z|}\,, \qquad z \in D\,.$$

1. Introduction, History, and Notation

We examine a number of problems concerning polynomials with coefficients restricted in various ways. We are particularly interested in how small such polynomials can be on the interval $[0,1]$. For example, we prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

for every $n \geq 2$, where $\mathcal{F}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$.

Littlewood considered minimization problems of this variety on the unit disk, hence, the title of the paper. His most famous, now solved, conjecture was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c \log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant.

When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view. See [2, 3, 10, 11, 20, 35].

One key to the analysis is a study of the related problem of how large an order zero these restricted polynomials can have at 1. We answer this latter question precisely for the class of polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}$$

with fixed $|a_0| \neq 0$.

Variants of these questions have attracted considerable study, though rarely have precise answers been possible to give. See in particular [1, 7, 6, 19, 39, 41]. Indeed the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n. See [22] or [11].)

We introduce the following classes of polynomials. Let

$$\mathcal{P}_n^c := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{C} \right\}$$

denote the set of algebraic polynomials of degree at most n with complex coefficients. Let

$$\mathcal{P}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R} \right\}$$

denote the set of algebraic polynomials of degree at most n with real coefficients. Let

$$\mathcal{Z}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{Z} \right\}$$

denote the set of algebraic polynomials of degree at most n with integer coefficients. Let

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1,0,1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$. Let

$$\mathcal{A}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{0,1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{0,1\}$. Finally, let

$$\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i x^i : a_i \in \{-1,1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1,1\}$.

So obviously

$$\mathcal{L}_n\,,\ \ \mathcal{A}_n \subset \mathcal{F}_n \subset \mathcal{Z}_n \subset \mathcal{P}_n \subset \mathcal{P}_n^c\,.$$

Throughout this paper the uniform norm on a set $A \subset \mathbb{R}$ is denoted by $\|.\|_A$.

In his monograph [30], Littlewood discusses the class $\mathcal{L}_n$ and its complex analogue when the coefficients are complex numbers of modulus 1. On page 25 he writes "These raise fascinating questions." It is easy to see that the L_2 norm of any polynomial of degree n with complex coefficients of modulus one on the unit circle is $\sqrt{n+1}$. (Here we have normalized so that the unit circle has length 1.) Hence the minimum supremum norm of any such polynomial on the unit circle is at least $\sqrt{n+1}$.

The Rudin-Shapiro polynomials show that there are polynomials from $\mathcal{L}_n$ with maximum modulus less than $c\sqrt{n+1}$ on the unit circle. Littlewood remarks in [30] that although it has been known for more than 50 years that $g_n(\theta) := \sum_{m=0}^{n} e^{im\log m} e^{im\theta}$ satisfies $|g_n(\theta)| < c\sqrt{n+1}$ on the real line, the existence of polynomials $p_n \in \mathcal{L}_n$ with $|p_n(z)| < c\sqrt{n+1}$ on the unit circle has only fairly recently been shown. He adds "As a matter of cold fact, many people had doubted its truth." However, it is not known whether or not there are such polynomials from $p_n \in \mathcal{L}_n$ with minimal modulus also at least $c\sqrt{n}$ on the unit circle, where $c > 0$ is an absolute constant. Littlewood conjectures that there are such polynomials.

Littlewood also makes the above conjecture in [29] as well as several others. In [28] he writes that the problem of finding polynomials of degree n with coefficients of modulus 1 and with modulus on the unit disk bounded below by $c\sqrt{n}$ "seems singularly elusive and intriguing."

Erdős conjectured that the maximum modulus of a polynomial from $\mathcal{L}_n$ is always at least $c\sqrt{n+1}$ with an absolute constant $c > 1$. Erdős offers \$100 for a solution of this problem in [E9-95]. Both Littlewood's and Erdős' conjectures are still open.

In the paper [28] Littlewood also considers $\sum_{m=0}^{n-1} \omega^{m(m+1)/2} z^m$ and shows that this polynomial has almost constant modulus (in an asymptotic sense) except on a set of size $cn^{-1/2+\delta}$. Here ω is a primitive nth root of unity. Further related results are to be found in [4, 5, 9, 12, 16, 17, 21, 25, 26, 27, 32, 34].

Carrol, Eustice, and T. Figiel [15] show that

$$\liminf \frac{\log(m(n))}{\log(n+1)} > .431\,,$$

where $m(n)$ denotes the largest value that the minimum modulus of a polynomial from $\mathcal{L}_n$ can be on the unit circle. They also prove that

$$\sup \frac{\log(m(n))}{\log(n+1)} = \lim \frac{\log(m(n))}{\log(n+1)}\,.$$

They further conjecture that $m(n)n^{-1/2}$ tends to zero (contrary to Littlewood).

The average maximum modulus is computed by Salem and Zygmund [37] who show that for all but $o(2^n)$ polynomials from $\mathcal{L}_n$ the maximum modulus on the unit disk lies between $c_1\sqrt{n\log n}$ and $c_2\sqrt{n\log n}$.

The expected L^4 norm of a polynomial $p \in \mathcal{L}_n$ is $(2n^2-n)^{1/4}$. This is due to Newman and Byrnes [33]. They also compute the L^4 norm of the Rudin-Shapiro polynomials.

In the case of complex coefficients these problems are mostly solved. A very interesting result of Kahane [24] proves the existence of polynomials of degree n with *complex* coefficients of modulus one and with minimal and maximal modulus both asymptotically $\sqrt{n+1}$ on the unit circle. See also [4].

The study of the location of zeros of these classes of polynomials begins with Bloch and Pólya [6]. They prove that the average number of real zeros of a polynomial from $\mathcal{F}_n$ is at most $c\sqrt{n}$. They also prove that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn\log\log n}{\log n}$$

real zeros. This quite weak result appears to be the first on this subject. Schur [39] and by different methods Szegő [41] and Erdős and Turán [19] improve this to $c\sqrt{n\log n}$ (see also [10]). (Their results are more general, but in this specialization not sharp.)

Our Theorem 4.1 gives the right upper bound of $c\sqrt{n}$ for the number of real zeros of polynomials from a much larger class, namely for all polynomials of the form

$$p(x)=\sum_{j=0}^{n} a_j x^j, \qquad |a_j|\le 1\,, \quad |a_0|=|a_n|=1\,, \quad a_j\in\mathbb{C}\,.$$

Schur [39] claims that Schmidt gives a version of part of this theorem. However, it does not appear in the reference he gives, namely [38], and we have not been able to trace it to any other source. Also, our method is able to give $c\sqrt{n}$ as an upper bound for the number of zeros of a polynomial $p\in\mathcal{P}_n^c$ with $|a_0|=1, |a_i|\le 1$, inside any polygon with vertices in the unit circle (of course, c depends on the polygon). This may be discussed in a later publication.

Bloch and Pólya [6] also prove that there are polynomials $p \in \mathcal{F}_n$ with

$$\frac{cn^{1/4}}{\sqrt{\log n}}$$

distinct real zeros of odd multiplicity. (Schur [39] claims they do it for polynomials with coefficients only from $\{-1,1\}$, but this appears to be incorrect.)

In a seminal paper Littlewood and Offord [31] prove that the number of real roots of a $p\in\mathcal{L}_n$, on average, lies between

$$\frac{c_1\log n}{\log\log\log n} \qquad \text{and} \qquad c_2\log^2 n$$

and it is proved by Boyd [13] that every $p \in \mathcal{L}_n$ has at most $c \log^2 n / \log\log n$ zeros at 1 (in the sense of multiplicity).

Kac [23] shows that the expected number of real roots of a polynomial of degree n with random uniformly distributed coefficients is asymptotically $(2/\pi)\log n$. He writes "I have also stated that the same conclusion holds if the coefficients assume only the values 1 and -1 with equal probabilities. Upon closer examination it turns out that the proof I had in mind is inapplicable... . This situation tends to emphasize the particular interest of the discrete case, which surprisingly enough turns out to be the most difficult." In a recent related paper Solomyak [40] studies the random series $\sum \pm\lambda^n$.

2. Number of Zeros at 1 of Polynomials with Restricted Coefficients

The following two theorems offer upper bounds for the number of zeros at 1 of certain classes of polynomials with restricted coefficients. The first result sharpens and generalizes results of Amoroso [1], Bombieri and Vaaler [7], and Hua [22] who give versions of this result for polynomials with integer coefficients.

Theorem 2.1. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}$$

has at most

$$c\,(n(1 - \log|a_0|))^{1/2}$$

zeros at 1.

Applying Theorem 2.1 with $q(x) := x^{-n} p(x^{-1})$ immediately gives the following.

Theorem 2.2. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}$$

has at most

$$c\,(n(1 - \log|a_n|))^{1/2}$$

zeros at 1.

The sharpness of the above theorems is shown by

Theorem 2.3. *Suppose $n \in \mathbb{N}$. Then there exists a polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{R}$$

such that p has a zero at 1 with multiplicity at least

$$\min\left\{\frac{1}{6}((n\,(1-\log|a_0|))^{1/2}-1\,,n\right\}.$$

The following two theorems can be obtained from the results above with slightly worse constants. However, we have distinct attractive proofs of Theorems 2.4 and 2.5 below and we give them also.

Theorem 2.4. *Every polynomial p of the form*

$$p(x)=\sum_{j=0}^{n}a_jx^j\,, \qquad |a_0|=1\,, \quad |a_j|\le 1\,, \quad a_j\in\mathbb{C}$$

has at most $\left\lfloor \frac{16}{7}\sqrt{n}\right\rfloor+4$ zeros at 1.

Theorem 2.5. *For every $n\in\mathbb{N}$, there exists a polynomial*

$$p_n(x)=\sum_{j=0}^{n^2-1}a_jx^j$$

such that $a_{n^2-1}=1$; $a_0,a_1,\dots,a_{n^2-2}$ are real numbers of modulus less than 1; and p_n has a zero at 1 with multiplicity at least $n-1$.

Theorem 2.5 immediately implies

Corollary 2.6. *For every $n\in\mathbb{N}$, there exists a polynomial*

$$p_n(x)=\sum_{j=0}^{n}a_jx^j\,, \qquad a_n=1\,, \quad |a_j|\le 1\,, \quad a_j\in\mathbb{R}\,,$$

and p_n has a zero at 1 with multiplicity at least $\lfloor\sqrt{n}-1\rfloor$.

The next related result is well known (in a variety of forms) but its proof is simple and we include it (see [6]).

Theorem 2.7. *There is an absolute constant $c>0$ such that for every $n\in\mathbb{N}$ there is a $p\in\mathcal{F}_n$ having at least $c\sqrt{n/\log(n+1)}$ zeros at 1.*

Theorems 2.4 and 2.7 show that the right upper bound for the number of zeros a polynomial $p\in\mathcal{F}_n$ can have at 1 is somewhere between $c_1\sqrt{n/\log(n+1)}$ and $c_2\sqrt{n}$ with absolute constants $c_1>0$ and $c_2>0$. Completely closing the gap in this problem looks quite difficult.

Our next theorem slightly generalizes Theorem 2.1 and offers an *explicit* constant.

Theorem 2.8. *If* $|a_0| \geq \exp(-L^2)$ *and* $|a_j| \leq 1$ *for each* $j = L^2+1, L^2+2, \dots, n$, *then the polynomial*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad a_j \in \mathbb{C}$$

has at most $\frac{44}{7}(L+1)\sqrt{n} + 5$ *zeros at* 1*.*

The next result is a simple observation about the maximal number of zeros a polynomial $p \in \mathcal{A}_n$ can have.

Theorem 2.9. *There is an absolute constant* $c > 0$ *such that every* $p \in \mathcal{A}_n$ *has at most* $c \log n$ *zeros at* -1*.*

Remark to Theorem 2.9. Let R_n be defined by

$$R_n(x) := \prod_{i=1}^{n} (1 + x^{a_i})\,,$$

where $a_1 := 1$ and a_{i+1} is the smallest odd integer that is greater than $\sum_{k=1}^{i} a_k$. It is tempting to speculate that R_n is the lowest degree polynomial with coefficients from $\{0,1\}$ and a zero of order n at -1. This is true for $n = 1,2,3,4,5$ but fails for $n = 6$ and hence for all larger n.

Our final result in this section shows that a polynomial $Q \in \mathcal{F}_n$ with k zeros at 1 has many other zeros on the unit circle (at certain roots of unity). A version of this may be also be deduced from results in [7].

Theorem 2.10. *Let* $p \leq n$ *be a prime. Suppose* $Q \in \mathcal{F}_n$ *and* Q *has exactly* k *zeros at* 1 *and exactly* m *zeros at a primitive* p*th root of unity. Then*

$$p(m+1) \geq k\,\frac{\log p}{\log(n+1)}\,.$$

3. The Chebyshev Problem on $[0,1]$ for Polynomials with Restricted Coefficients

If p is a polynomial of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j$$

with $a_1 = a_2 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, then we call $I(p) := a_m$ the first non-zero coefficient of p.

Our first theorem in this section shows how small the uniform norm of a polynomial $0 \neq p$ on $[0,1]$ can be under some restriction on its coefficients.

Theorem 3.1. *Let $\delta \in (0,1]$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1(n(1-\log\delta))^{1/2}\right) \le \inf_p \|p\|_{[0,1]} \le \exp\left(-c_2(n(1-\log\delta))^{1/2}\right),$$

where the infimum is taken over all polynomials p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}$$

with $|I(p)| \ge \delta \ge \exp\left(\frac{1}{2}(6-n)\right)$.

The following result is a special case of Theorem 3.1.

Theorem 3.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \le \inf_p \|p\|_{[0,1]} \le \exp\left(-c_2\sqrt{n}\right)$$

for every $n \ge 2$, where the infimum is taken over all polynomials p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}$$

with $|I(p)| = 1$.

For the class $\mathcal{F}_n$ we have

Theorem 3.3. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1\sqrt{n}\right) \le \inf_{0 \ne p \in \mathcal{F}_n} \|p\|_{[0,1]} \le \exp\left(-c_2\sqrt{n}\right)$$

for every $n \ge 2$.

Note that the lower bound in the above theorem is a special case of Theorem 3.2. The proof of the upper bound, however, requires new ideas.

The approximation rate in Theorems 3.2 and 3.3 should be compared with

$$\inf_p \|p\|_{[0,1]}^{1/n} = \frac{2^{1/n}}{4}\,,$$

where the infimum is taken for all monic $p \in \mathcal{P}_n$, and also with

$$\frac{1}{2.376\ldots} < \inf_{0 \ne p \in \mathcal{Z}_n} \|p\|_{[0,1]}^{1/n} < \frac{1+\epsilon_n}{2.3605}\,, \qquad \epsilon_n \to 0\,.$$

The first equality above is attained by the normalized Chebyshev polynomial shifted linearly to $[0,1]$ and is proved by a simple perturbation argument. The second inequality is much harder (the exact result is open) and is discussed in [10]. It is an interesting fact that the polynomials $0 \neq p \in \mathcal{Z}_n$ with the smallest uniform norm on $[0,1]$ are very different from the usual Chebyshev polynomial of degree n. For example, they have at least 52% of their zeros at either 0 or 1. Relaxation techniques do not allow for their approximate computation.

Likewise, polynomials $0 \neq p \in \mathcal{F}_n$ with small uniform norm on $[0,1]$ are again quite different from polynomials $0 \neq p \in \mathcal{Z}_n$ with small uniform norm on $[0,1]$.

The story is roughly as follows. Polynomials $0 \neq p \in \mathcal{P}_n$ with leading coefficient 1 and with smallest possible uniform norm on $[0,1]$ are characterized by equioscillation and are given explicitly by the Chebyshev polynomials. In contrast, finding polynomials from $\mathcal{Z}_n$ with small uniform norm on $[0,1]$ is closely related to finding irreducible polynomials with all their roots in $[0,1]$.

The construction of non-zero polynomials from $\mathcal{F}_n$ with small uniform norm on $[0,1]$ is more or less governed by how many zeros such a polynomial can have at 1. Indeed, non-zero polynomials from $\mathcal{F}_n$ with minimal uniform norm on $[0,1]$ are forced to have close to the maximal possible number of zeros at 1.

This problem of the maximum order of a zero at 1 for a polynomial in $\mathcal{F}_n$, and closely related problems for polynomials of small height have attracted considerable attention but there is still a gap in what is known (see Theorem 2.4 and Theorem 2.7).

For the class $\mathcal{A}_n$ we have the following Chebyshev-type theorem. This result should be compared with Theorem 3.3.

Theorem 3.4. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp\left(-c_1 \log^2(n+1)\right) \leq \inf_{0 \neq p \in \mathcal{A}_n} \|p(-x)\|_{[0,1]} \leq \exp\left(-c_2 \log^2(n+1)\right)$$

for every $n \geq 2$.

Our last theorem in this section is a sharp Chebyshev-type inequality for $\mathcal{F} := \cup_{n=1}^{\infty} \mathcal{F}_n$ and $\mathcal{S}$, where $\mathcal{S}$ denotes the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \leq \frac{1}{1-|z|}, \qquad z \in D.$$

Theorem 3.5. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\exp(-c_1/a) \leq \inf_{p \in \mathcal{S},\, |p(0)|=1} \|p\|_{[1-a,1]} \leq \inf_{p \in \mathcal{F},\, |p(0)|=1} \|p\|_{[1-a,1]} \leq \exp(-c_2/a)$$

for every $a \in (0,1)$.

4. The Number of Real Zeros of Polynomials with Restricted Coefficients

Theorems 4.1 and 4.2 below give upper bounds for the number of *real* zeros of polynomials p when their coefficients are restricted in various ways.

The prototype for these theorems is given below. It was apparently first proved, at least up to the correct constant, by Schmidt in the early thirties. His complicated proof wasn't published – the first published proof is due to Schur [39]. Later new and simpler proofs and generalizations were published by Szegő [41] and Erdős and Turán [19] and others. A version of the approach of Erdős and Turán is presented in [10].

Theorem A. *Suppose*

$$p(z) := \sum_{j=0}^{n} a_j z^j \,, \qquad a_j \in \mathbb{C}$$

has m *positive real roots. Then*

$$m^2 \le 2n \log\left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}\right).$$

Theorem 4.1 improves the above bound of $c\sqrt{n \log n}$ in the cases we are interested in where the coefficients are of similar size. Up to the constant c it is the best possible result.

Theorem 4.1. *There is an absolute constant* $c > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ *zeros in* $[-1, 1]$.

There is an absolute constant $c > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_n| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ *zeros in* $\mathbb{R} \setminus (-1, 1)$.

There is an absolute constant $c > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ *real zeros.*

Theorem 4.2. *There is an absolute constant* $c > 0$ *such that every polynomial* p *of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = 1\,, \quad a_j \in \mathbb{C} \tag{4.1}$$

has at most c/a zeros in $[-1+a, 1-a]$ whenever $a \in (0,1)$.

This result is sharp up to the constant. It is possible to construct a polynomial (of degree $n \le ck^2$) of the form (4.1) with a zero of order k in the interval $(0, 1-1/k]$. This is discussed in [3].

The next theorem gives an upper bound for the number of zeros of a polynomial p lying on a subarc of the unit circle when the coefficients of p are restricted as in the first statement of Theorem 4.1.

Theorem 4.3. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad |a_0| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $cn\alpha$ zeros on a subarc I_α of length α of the unit circle if $\alpha \ge n^{-1/2}$, while it has at most $c\sqrt{n}$ zeros on I_α if $\alpha \le n^{-1/2}$. The polynomial $p(z) := z^n - 1$ ($\alpha \ge n^{-1/2}$) and Theorem 2.4 ($\alpha \le n^{-1/2}$) show that these bounds are essentially sharp.

We point out an interesting extension of Theorem 2.4 as a special case of Theorem 4.1.

Theorem 4.4. *There is an absolute constant $c > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad |a_0| = 1\,, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ zeros at a point $a \in \mathbb{C}$ with $0 < |a| \le 1$.

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has at most $c\sqrt{n}$ zeros at a point $a \in \mathbb{C}$ with $1 \le |a| < \infty$.

One should observe that Jensen's inequality implies that every function f analytic in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the growth condition

$$|f(0)| = 1\,, \qquad |f(z)| \le \frac{1}{1-|z|}\,, \qquad z \in D$$

has at most $(c/a)\log(1/a)$ zeros in the disk $D_a := \{z \in \mathbb{C} : |z| < 1-a\}$, where $0 < a < 1$ and $c > 0$ is an absolute constant.

5. Tools

The main tool in the proof of Theorem 2.1 is the following result which is of interest for its own sake.

Denote by $\mathcal{S}$ the collection of all analytic functions f on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|f(z)| \leq \frac{1}{1-|z|}\,, \qquad z \in D\,.$$

Theorem 5.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$|f(0)|^{c_1/a} \leq \exp\left(\frac{c_2}{a}\right) \|f\|_{[1-a,1]}$$

for every $f \in \mathcal{S}$ and $a \in (0,1]$.

Theorem 2.1 is proved in the next section. In the rest of this section we formulate and prove some technical lemmas used in the proof of Theorem 5.1. We need some corollaries of the

Hadamard Three Circles Theorem. *Suppose f is regular in*

$$\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}\,.$$

For $r \in [r_1, r_2]$, let

$$M(r) := \max_{|z|=r} |f(z)|\,.$$

Then

$$M(r)^{\log(r_2/r_1)} \leq M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}\,.$$

Corollary 5.2. *Let $a \in (0,1]$. Suppose f is regular inside and on the ellipse E_a with foci at $1-a$ and $1-a+\frac{1}{4}a$ and with major axis*

$$\left[1-a-\frac{9a}{64}, 1-a+\frac{25a}{64}\right]\,.$$

Let $\widetilde{E}_a$ be the ellipse with foci at $1-a$ and $1-a+\frac{1}{4}a$ and with major axis

$$\left[1-a-\frac{a}{32}, 1-a+\frac{9a}{32}\right]\,.$$

Then

$$\max_{z \in \widetilde{E}_a} |f(z)| \leq \left(\max_{z \in \left[1-a, 1-a+\frac{1}{4}a\right]} |f(z)|\right)^{1/2} \left(\max_{z \in E_a} |f(z)|\right)^{1/2}\,.$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution

$$w = \frac{a}{8}\left(\frac{z+z^{-1}}{2}\right) + \left(1-a+\frac{a}{8}\right)\,.$$

The Hadamard Three Circles Theorem is applied with $r_1 := 1, r := 2$, and $r_2 := 4$. $\square$

Corollary 5.3. *For every $f \in \mathcal{S}$ and $a \in (0,1]$*

$$\max_{z \in \widetilde{E}_a} |f(z)| \leq \left(\frac{64}{39a}\right)^{1/2} \left(\max_{z \in [1-a,1]} |f(z)|\right)^{1/2}.$$

Proof. This follows from Corollary 5.2 and the maximum principle. □

Lemma 5.4. *Suppose*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}$$

$$p(x) = (x-1)^k q(x)\,, \qquad q(x) = \sum_{j=0}^{n-k} b_j x^j\,, \quad b_j \in \mathbb{C}\,.$$

Then

$$\|q\|_{[0,1]} \leq \sum_{j=0}^{n-k} |b_j| \leq (n+1)e\left(\frac{en}{k}\right)^{k-1} \leq (n+1)\left(\frac{en}{k}\right)^{k}.$$

As a consequence,

$$\|p\|_{[1-k/(9n),1]} \leq (n+1)\left(\frac{e}{9}\right)^{k}.$$

Proof. We have

$$\begin{aligned}
|b_j| &= \left|\frac{1}{j!}\frac{d^j}{dx^j}(p(x)(x-1)^{-k})\Big|_{x=0}\right| \\
&= \left|\frac{1}{j!}\sum_{m=0}^{j}\binom{j}{m}(-1)^k\frac{(k+m-1)!}{(k-1)!}p^{(j-m)}(0)\right| \\
&= \left|\sum_{m=0}^{j}\frac{(k+m-1)!}{(k-1)!m!}\frac{1}{(j-m)!}p^{(j-m)}(0)\right| = \left|\sum_{m=0}^{j}\frac{(k+m-1)!}{(k-1)!m!}\,a_{j-m}\right| \\
&\leq \binom{k+j}{k} \leq \left(\frac{e(k+j)}{k}\right)^k \leq \left(\frac{en}{k}\right)^k
\end{aligned}$$

which proves the lemma. □

To prove Theorem 2.3 our tool is the next lemma due to Halász [42].

Lemma 5.5. *For every $k \in \mathbb{N}$, there exists a polynomial $h \in \mathcal{P}_k$ such that*

$$h(0) = 1\,, \qquad h(1) = 0\,, \qquad |h(z)| < \exp\left(\frac{2}{k}\right) \quad \text{for} \quad |z| \leq 1\,.$$

To prove Theorems 2.4 and 2.8 we need Lemmas 5.6 and 5.7, respectively, below.

Lemma 5.6. *For every positive integer n, there exists an $f \in \mathcal{P}_\mu$ with*

$$\mu \le \left\lfloor \tfrac{16}{7}\sqrt{n} \right\rfloor + 4$$

such that

$$f(0) > |f(1)| + |f(2)| + \cdots + |f(n)|\,.$$

Proof. Let

$$k := \left\lfloor \tfrac{4}{7}\sqrt{n} \right\rfloor + 1$$

and

$$g(x) = \frac{1}{2}T_0(x) + T_1(x) + T_2(x) + \cdots + T_k(x)$$

where, as usual, T_i denotes the Chebyshev polynomial of degree i. We have $g(1) = k + \frac{1}{2}$, and for $0 < t \le \pi$,

$$g(\cos t) = \frac{1}{2} + \cos t + \cos 2t + \cdots + \cos kt = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} = \frac{\sin\left(k+\frac{1}{2}\right)t}{\sqrt{2(1-\cos t)}}\,,$$

whence

$$|g(x)| \le \frac{1}{\sqrt{2(1-x)}}\,, \qquad x \in [-1,1)\,.$$

Let

$$f(x) := \left(g\left(1 - \tfrac{2}{n}x\right)\right)^4\,.$$

Then $f \in \mathcal{P}_\mu$ with $\mu = 4k \le \left\lfloor \frac{16}{7}\sqrt{n} \right\rfloor + 4$ and

$$|f(1)| + |f(2)| + \cdots + |f(n)| \le \sum_{j=1}^{n} \left(\frac{4j}{n}\right)^{-2} = \frac{n^2}{16}\sum_{j=1}^{n}\frac{1}{j^2} < \frac{\pi^2}{96}n^2 < k^4 < f(0)\,,$$

and the proof is finished. $\square$

Lemma 5.7. *Let n and L be a positive integers with $1 \le L \le \sqrt{n}$. Then there exists a polynomial $f \in \mathcal{P}_\mu$ with*

$$\mu \le 4\sqrt{n} + \tfrac{9}{7}L\sqrt{n} + L + 4 \le \tfrac{44}{7}L\sqrt{n} + 4$$

such that

$$f(1) = f(2) = \cdots = f(L^2) = 0 \tag{5.1}$$

and

$$f(0) > \exp(L^2)\big(|f(L^2+1)| + |f(L^2+2)| + \cdots + |f(n)|\big)\,. \tag{5.2}$$

Proof. The required polynomial f is constructed as a product of other polynomials. First we define these factors. Let $g_0 \in \mathcal{P}_\mu$ with

$$\mu \le \left\lfloor \tfrac{16}{7}\sqrt{n} \right\rfloor + 4$$

and

$$g_0(0) > |g_0(1)| + |g_0(2)| + \cdots + |g_0(n)| \, . \tag{5.3}$$

The existence of such a polynomial g_0 is guaranteed by Lemma 5.6. For $j = 1, 2, \ldots, n$, we define

$$m_j := \lfloor \sqrt{n/j} \rfloor + 1$$

and

$$g_j(x) := T_{m_j}\left(1 - \frac{1 - \cos\frac{\pi}{2m_j}}{j}\, x\right) .$$

Then

$$g_j(0) = T_{m_j}(1) = 1 \qquad \text{and} \qquad g_j(j) = T_{m_j}\left(\cos\tfrac{\pi}{2m_j}\right) = 0 \, .$$

Also, for $0 \le x \le n$, we have

$$1 \ge 1 - \frac{1 - \cos\frac{\pi}{2m_j}}{j}\, x \ge 1 - \frac{2n\sin^2\frac{\pi}{4m_j}}{j} > 1 - \frac{2n\left(\frac{\pi}{4m_j}\right)^2}{j} > 1 - 2 = -1 \, ,$$

hence

$$|g_j(x)| \le 1 \qquad x \in [0, n] \, .$$

Now let

$$h(x) := T_{kL}\left(1 + \frac{2(L^2 - x)}{n}\right) , \qquad k := \lfloor \sqrt{n} \rfloor + 1 \, .$$

Then $h \in \mathcal{P}_{kL}$ and

$$|h(x)| \le 1 \, , \qquad L^2 \le x \le n \, .$$

Further $1 \le L \le \sqrt{n}$, $k = \lfloor \sqrt{n} \rfloor + 1$, and the concavity of the function $\cosh^{-1}(1 + 2x^2)$ on $[0, 1]$ imply

$$\begin{aligned} h(0) &= T_{kL}\left(1 + \frac{2L^2}{n}\right) = \cosh\left(kL\cosh^{-1}\left(1 + \frac{2L^2}{n}\right)\right) \\ &> \cosh\left(L\sqrt{n}\frac{L}{\sqrt{n}}\cosh^{-1}(1+2)\right) \\ &> \frac{1}{2}\exp((7/4)L^2) > \exp(L^2) \, . \end{aligned}$$

Now we define the required polynomial f by

$$f := h \prod_{j=0}^{L^2} g_j \, .$$

This f satisfies

$$f(1) = f(2) = \cdots = f(L^2) = 0 \, ,$$

$$|f(x)| \le g_0(x) \, , \qquad L^2 \le x \le n \, ,$$

and

$$f(0) = g_0(0)h(0) > \exp(L^2)g_0(0)\,.$$

These, together with (5.3) show that (5.1) and (5.2) are satisfied. The degree of the polynomial f is at most

$$\begin{aligned}
&\left(\tfrac{16}{7}\sqrt{n}+4\right) + \sum_{j=1}^{L^2} m_j + \left(\lfloor\sqrt{n}\rfloor + 1\right) L \\
&\qquad \le \left(\tfrac{16}{7}\sqrt{n}+4\right) + \left(\sqrt{n}+1\right) L + \sum_{j=1}^{L^2}\left(\sqrt{n/j}+1\right) \\
&\qquad < \left(\tfrac{16}{7}\sqrt{n}+4\right) + \left(\sqrt{n}+1\right) L + L^2 + (2L-1)\sqrt{n} \\
&\qquad \le 4L\sqrt{n} + \tfrac{9}{7}\sqrt{n} + L + 4\,.
\end{aligned}$$

This finishes the proof. $\square$

6. Proofs of the Main Results

Proof of Theorem 5.1. Let $h(z) = \frac{1}{2}(1-a)(z+z^2)$. Observe that $h(0)=0$, and there are absolute constants $c_3 > 0$ and $c_4 > 0$ such that

$$|h(e^{it})| \le 1 - c_3 t^2\,, \qquad -\pi \le t \le \pi\,,$$

and for $t \in [-c_4 a, c_4 a]$, $h(e^{it})$ lies inside the ellipse $\widetilde{E}_a$. Now let $m := \lfloor 2\pi c_4/a \rfloor + 1$. Let $\xi := \exp(2\pi i/(2m))$ be the first $2m$th root of unity, and let

$$g(z) = \prod_{j=0}^{2m-1} f(h(\xi^j z))\,.$$

Using the Maximum Principle and the properties of h, we obtain

$$\begin{aligned}
|f(0)|^{2m} = |g(0)| &\le \max_{|z|=1} |g(z)| \le \left(\max_{z\in\widetilde{E}_a} |f(z)|\right)^2 \prod_{k=1}^{m-1} \left(\frac{1}{c_3(k/m)^2}\right)^2 \\
&= \left(\max_{z\in\widetilde{E}_a} |f(z)|\right)^2 e^{c_5(m-1)} \left(\frac{m^{m-1}}{(m-1)!}\right)^4 < \left(\max_{z\in\widetilde{E}_a} |f(z)|\right)^2 e^{c_6(m-1)}\,,
\end{aligned}$$

and the theorem follows by Lemma 5.3. $\square$

Proof of Theorem 2.1. Let p be a polynomial of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_0 \ne 0\,, \quad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,.$$

Then $p \in \mathcal{S}$. Suppose p has k zeros at 1. Without loss of generality we may assume that $2\sqrt{n+1} \le k \le n$. Applying Theorem 5.1 with $a := k/(3n)$, then using Lemma 5.4, we obtain

$$|a_0|^{3c_1 n/k} \le \exp\left(\frac{9c_2 n}{k}\right) \|p\|_{[1-k/(9n),1]} \le \exp\left(\frac{9c_2 n}{k}\right)(n+1)\left(\frac{e}{9}\right)^k .$$

Taking log of both sides, and after some algebra, we obtain

$$k \log \frac{9}{e} \le \frac{3c_1 n}{k}\left(\frac{3c_2}{c_1} - \log|a_0|\right) + \log(n+1) .$$

Using $\log(n+1) \le \sqrt{n+1} \le k/2$, we can deduce that

$$k^2 \log \frac{9}{e} \le 3c_1 n\left(\frac{3c_2}{c_1} - \log|a_0|\right) + \frac{k^2}{2} .$$

Hence

$$k^2\left(\log\frac{9}{e} - \frac{1}{2}\right) \le 3c_1 n\left(\frac{3c_2}{c_1} - \log|a_0|\right) ,$$

and the result follows. □

Proof of Theorem 2.3. Let $0 \ne |a_0| \le 1$,

$$k := \left\lfloor \left(\frac{4n}{\log\left(1+|a_0|^{-1}\right)}\right)^{1/2} \right\rfloor + 1 \qquad \text{and} \qquad m := \left\lfloor \frac{n}{k} \right\rfloor .$$

Let h be a polynomial given by Lemma 5.5, that is, $h \in \mathcal{P}_k$, $h(0) = 1$, $h(1) = 0$, and if $|z| \le 1$, then $|h(z)| < \exp\left(\frac{2}{k}\right)$. Let

$$f(x) := h^m(x) =: b_0 + b_1 x + b_2 x^2 + \cdots + b_{km} x^{km} .$$

The degree of the polynomial f is $km \le n$; the multiplicity of the zero of f at 1 is at least m because of the choice of h; $f(0) = b_0 = 1$; and for $|z| \le 1$, $|f(z)| \le \exp\left(\frac{2m}{k}\right)$. The last inequality, together with the Parseval formula, implies that

$$\begin{aligned} |b_0|^2 + |b_1|^2 + \cdots + |b_{km}|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2\, dt \\ &\le \exp\left(\frac{4m}{k}\right) \le \exp\left(\frac{4n}{k^2}\right) \\ &\le \exp\left(\log\left(1+|a_0|^{-1}\right)\right) \\ &\le 1 + |a_0|^{-2} . \end{aligned}$$

Since $b_0 = 1$, it follows that each of $b_1, b_2, \dots, b_{km}$ has modulus less than $|a_0|^{-1}$.

Let $p := a_0 f$. The constant term of p is a_0; all the other coefficients of p have modulus less than 1; the multiplicity of the zero of p at 1 is at least m; and

$$\begin{aligned} m &= \left\lfloor \frac{n}{\left\lfloor (4n)^{1/2}\left(\log\left(1+|a_0|^{-1}\right)\right)^{-1/2}\right\rfloor + 1} \right\rfloor \ge \frac{1}{4}\sqrt{n\log\left(1+|a_0|^{-1}\right)} - 1 \\ &\ge \frac{1}{4\sqrt{2}}\sqrt{n\left(1-\log|a_0|\right)} - 1 \end{aligned}$$

whenever $\exp(-3n) \le |a_0| \le 1$. Note that $\exp(-3n) \le |a_0| \le 1$ implies

$$\left\lfloor (4n)^{1/2} \left(\log\left(1+|a_0|^{-1}\right)\right)^{-1/2} \right\rfloor + 1 \le 2\,(4n)^{1/2} \left(\log\left(1+|a_0|^{-1}\right)\right)^{-1/2}$$

which was used in the first inequality above.

If $|a_0| \le \exp(-3n) \le \left(\frac{1}{2}\right)^n$, then the polynomial p defined by $p(x) = a_0(x-1)^n$ is of the required form and has n zeros at 1. This finishes the proof. □

Proof of Theorem 2.4. If p has a zero at 1 of multiplicity μ, then for every polynomial $f \in \mathcal{P}_\mu^c$, we have

$$a_0 f(0) + a_1 f(1) + \cdots + a_n f(n) = 0\,. \tag{6.1}$$

Lemma 5.6 constructs a polynomial f of degree at most

$$\mu \le \left\lfloor \tfrac{16}{7}\sqrt{n} \right\rfloor + 4$$

for which

$$f(0) > |f(1)| + |f(2)| + \cdots + |f(n)|\,.$$

Equality (6.1) cannot hold with this f, so the multiplicity of the zero of p at 1 is at most the degree of f. □

Proof of Theorem 2.5. Define

$$L_n(x) := \frac{(n!)^2}{2\pi i} \int_\Gamma \frac{x^t\,dt}{\prod_{k=0}^n (t-k^2)}\,, \qquad n = 0, 1, \ldots$$

where the simple closed contour Γ surrounds the zeros of the denominator in the integrand. Then L_n is a polynomial of degree n^2 with a zero of order n at 1. (This can easily be seen by repeated differentiation and then evaluation of the above contour integral by expanding the contour to infinity.)

Also, by the residue theorem,

$$L_n(x) = 1 + \sum_{k=1}^n c_{k,n} x^{k^2}\,,$$

where

$$c_{k,n} = \frac{(-1)^n (n!)^2}{\prod_{j=0, j\ne k}^n (k^2 - j^2)} = \frac{(-1)^k 2(n!)^2}{(n-k)!(n+k)!}\,.$$

It follows that

$$c_{k,n} \le 2\,, \qquad k = 1, 2, \ldots, n$$

and $|c_{k,n}|$ is decreasing in k. Note also that $\operatorname{sign}(c_{k,n}) = (-1)^k$ for $k = 0, 1, \ldots, n$, where $c_{0,n} := 1$. Hence,

$$q_n(x) := \frac{L_n(x)}{1-x}$$

is a polynomial of degree n^2-1 with real coefficients and with a zero of order $n-1$ at 1. Also q_n has constant coefficient 1 and each of its remaining coefficients is a real number of modulus less than 1. Now let $p_n(x) := x^{n^2-1}q_n(1/x)$. $\square$

Proof of Theorem 2.7. This is a standard box principle argument. The number of different outputs of the map

$$M(P) := \left(P(1), P'(1), \dots, P^{(k-1)}(1)\right), \qquad P \in \mathcal{A}_n$$

is at most

$$\prod_{j=0}^{k-1}\left((n+1)n^j\right) \le (n+1)^{k(k+1)/2}.$$

There are 2^{n+1} different elements of $\mathcal{A}_n$. So if

$$(n+1)^{k(k+1)/2} < 2^{n+1}$$

then there are two different $P_1 \in \mathcal{A}_n$ and $P_2 \in \mathcal{A}_n$ such that

$$P_1^{(j)}(1) = P_2^{(j)}(1), \qquad j = 0, 1, \dots, k-1$$

that is $0 \ne P_1 - P_2 \in \mathcal{F}_n$ has at least k zeros at 1. Note that

$$k < \sqrt{\frac{(2\log 2)(n+1)}{\log(n+1)}} - 1 \qquad \text{implies} \qquad (n+1)^{k(k+1)/2} < 2^{n+1}$$

which finishes the proof. $\square$

Proof of Theorem 2.8. If p has a zero at 1 of multiplicity μ, then for every polynomial $f \in \mathcal{P}_\mu^c$, we have

$$a_0 f(0) + a_1 f(1) + \cdots + a_n f(n) = 0. \tag{6.2}$$

Lemma 5.7 constructs a polynomial f of degree at most

$$\mu \le 4\sqrt{n} + \tfrac{9}{7}L\sqrt{n} + L + 4 \le \tfrac{44}{7}L\sqrt{n} + 4$$

for which (5.1) and (5.2) hold. Recalling the assumptions of the theorem on the coefficients a_j of p, equality (6.2) cannot hold with this f, so the multiplicity of the zero of p at 1 is at most the degree of f. $\square$

Proof of Theorem 2.9. Suppose $P \in \mathcal{A}_n$ has m zeros at -1. Then $(1+x)^m$ divides P and the quotient is a polynomial with integer coefficients. On evaluating P at 1 we see that $n+1 \ge P(1) \ge 2^m$ and the result follows. $\square$

Proof of Theorem 2.10. Let

$$\xi_j := \exp\left(\frac{2\pi i j}{p}\right), \qquad j = 1, 2, \dots, p-1.$$

Let $Q \in \mathcal{F}_n$ be of the form

$$Q(x) = (x-1)^k R(x)\,,$$

where R is a polynomial of degree at most $n-k$ with integer coefficients. Then, for every integer $m \leq k$, we have

$$Q^{(m)}(x) = (x-1)^{k-m} S(x)\,,$$

where S is a polynomial of degree at most $n-k$ with integer coefficients. Hence

$$K := \prod_{j=1}^{p-1} Q^{(m)}(\xi_j) = \prod_{j=1}^{p-1} (\xi_j - 1)^{k-m} \prod_{j=1}^{p-1} S(\xi_j) =: p^{k-m} N\,,$$

where both K and N are integers by the fundamental theorem of symmetric polynomials. Further

$$|K| \leq \prod_{j=1}^{p-1} (n+1)n^m \leq (n+1)^{(p-1)(m+1)}\,.$$

Hence $K \neq 0$ implies

$$p^{k-m} \leq (n+1)^{(p-1)(m+1)}\,,$$

that is,

$$k - m \leq \frac{(p-1)(m+1)\log(n+1)}{\log p}\,,$$

and the result follows. $\square$

Proof of Theorem 3.1. First we prove the lower bound. If P is one of those polynomials over which the infimum is taken, then

$$P(x) = x^m Q(x)$$

with an integer $0 \leq m \leq n$ and with a polynomial Q of the form

$$Q(x) = \sum_{j=0}^{n-m} b_j x^j\,, \qquad |b_j| \leq 1\,, \quad b_j \in \mathbb{C}\,, \quad |Q(0)| = |b_0| \geq \delta\,.$$

Applying Theorem 5.1 with

$$0 < a := \left(\frac{1 - \log|b_0|}{n}\right)^{1/2} \leq \left(\frac{2-(6-n)}{2n}\right)^{1/2} < \frac{1}{\sqrt{2}}\,,$$

we obtain

$$\|Q\|_{[1-a,1]} \geq |b_0|^{c_1/a} \exp\left(\frac{-c_2}{a}\right) \geq \exp\left(-c_3(n(1-\log\delta))^{1/2}\right)$$

with an absolute constant $c_3 > 0$. Now observe that $x \in [1-a, 1]$ implies

$$x^m \geq x^n \geq (1-a)^n \geq \exp\left(-\frac{na}{1-a}\right) \geq \exp\left(-c_4(n(1-\log\delta))^{1/2}\right)$$

with an absolute constant $c_4 > 0$, and the lower bound is proved.

Next we prove the upper bound. Suppose $n \in \mathbb{N}$ and $b_0 := \delta$ satisfy

$$\exp\left(\frac{1}{2}(6-n)\right) \leq |b_0| \leq 1\,.$$

Let $\nu := \lfloor n/5 \rfloor$. This implies

$$\exp(-3\nu) \leq \exp\left(\frac{1}{2}(6-n)\right) \leq |b_0| \leq 1\,.$$

Then by Theorem 2.3 (recall the cases distinguished in its proof), there exists a polynomial S_ν of the form

$$S_\nu(x) = \sum_{j=0}^{\nu} b_j x^j\,, \qquad |b_j| \leq 1 \quad b_j \in \mathbb{C}$$

such that S_ν has a zero at 1 with multiplicity at least

$$k \geq \frac{1}{6}(\nu\,(1-\log|b_0|))^{1/2} - 1\,.$$

Let

$$S_\nu(x) := (1-x)^k Q_{\nu-k}(x)\,, \qquad Q_{\nu-k} \in \mathcal{P}^c_{\nu-k}\,.$$

Let $P_n \in \mathcal{P}^c_n$ be defined by

$$P_n(x) := x^{4\nu} S_\nu(x)\,.$$

Note that $I(P_n) = b_0 = \delta$, so P_n is one of those polynomials over which the infimum in the theorem is taken. Now Lemma 5.4 implies that

$$\begin{aligned}
\|P_n\|_{[0,1]} &\leq \|x^{4\nu}(1-x)^k\|_{[0,1]} \|Q_{\nu-k}\|_{[0,1]} \\
&\leq \left(\frac{4\nu}{4\nu+k}\right)^{4\nu} \left(\frac{k}{4\nu+k}\right)^{k} (\nu+1) \left(\frac{e\nu}{k}\right)^{k} \\
&\leq (\nu+1)\left(\frac{e}{4}\right)^k = \exp(-k\log(4/e) + \log(\nu+1)) \\
&\leq \exp\left(-c_5(n(1-\log|b_0|))^{1/2}\right) = \exp\left(-c_5(n(1-\log\delta))^{1/2}\right)
\end{aligned}$$

with an absolute constant $c_5 > 0$, and the upper bound of the theorem follows. □

As we have already remarked, Theorem 3.2 is an immediate consequence of Theorem 3.1.

Proof of Theorem 3.3. The lower bound of the theorem follows from Theorem 3.2. To prove the upper bound of the theorem we argue as follows.

Without loss of generality we may assume that $n \in \mathbb{N}$ is sufficiently large. Let $k := \lfloor \frac{1}{2}\sqrt{n} \rfloor$. Let

$$1 - k/(2n) =: y_0 < y_1 < \cdots < y_k := 1$$

be $k+1$ equidistant points. We use a counting argument to find a polynomial $f \in \mathcal{F}_{n-1}$ with the property

$$|f(y_j)| \le 2^{1-\sqrt{n}}\,, \qquad j = 0, 1, \dots, k\,. \tag{6.3}$$

if n is sufficiently large. Indeed, we can divide the $(k+1)$–dimensional cube

$$Q := \{(x_0, x_1, \dots, x_k) : \ \ x_j \in [0, n+1)\,, \ \ j = 0, 1, \dots, k\}$$

into $(m(n+1))^{k+1}$ subcubes by defining

$$Q_{i_0, i_1, \dots, i_k} := \left\{ (x_0, x_1, \dots, x_k) : \ \ x_j \in \left[\frac{i_j}{m}, \frac{i_j+1}{m} \right) \,, \ \ j = 0, 1, \dots, k \right\} \,,$$

where $(i_0, i_1, \dots, i_k)$ are $(k+1)$–tuples of integers with $0 \le i_j \le m(n+1) - 1$ for each $j = 0, 1, \dots, k$. Note that if $P \in \mathcal{A}_{n-1}$, then

$$M(P) := (P(y_0), P(y_1), \dots, P(y_k)) \in Q\,.$$

Also, there are exactly 2^n elements of $\mathcal{A}_{n-1}$. Therefore, if

$$(m(n+1))^{k+1} < 2^n$$

holds, then there exist two different $P_1 \in \mathcal{A}_n$ and $P_2 \in \mathcal{A}_n$, and a subcube $Q_{i_0, i_1, \dots, i_k}$ such that

$$M(P_1) = (P_1(y_0), P_1(y_1), \dots, P_1(y_k)) \in Q_{i_0, i_1, \dots, i_k}$$

and

$$M(P_2) = (P_2(y_0), P_2(y_1), \dots, P_2(y_k)) \in Q_{i_0, i_1, \dots, i_k}\,.$$

Hence, for $0 \ne f := P_1 - P_2 \in \mathcal{F}_n$, we have

$$|f(y_j)| \le m^{-1}\,, \qquad j = 0, 1, \dots, k\,.$$

Now choose $m := \lfloor 2^{-\sqrt{n}} \rfloor$. This, together with $k := \lfloor \frac{1}{2}\sqrt{n} \rfloor$, yields that the inequality $(m(n+1))^k < 2^n$ holds provided n is sufficiently large, and (6.1) follows.

Observe that $\|p^{(k+1)}\|_{[0,1]} \le n^{k+2}$ for every $p \in \mathcal{F}_{n-1}$, in particular, for $f \in \mathcal{F}_{n-1}$ satisfying (6.3).

Let $y \in [y_0, 1]$ be an arbitrary point different from each y_j. By a well-known formula for divided differences,

$$\frac{f(y)}{\prod_{j=0}^{k} (y - y_j)} + \sum_{i=0}^{k} \frac{f(y_i)}{(y_i - y) \prod_{j=0, j \ne i}^{k} (y_i - y_j)} = \frac{1}{(k+1)!}\, f^{(k+1)}(\xi)$$

for some $\xi \in [y_0, 1]$.

Combining (6.3) and our two observations above, we obtain

$$
\begin{aligned}
&|f(y)| \\
&\leq \frac{1}{(k+1)!}\left|f^{(k+1)}(\xi)\right|\left|\prod_{j=0}^{k}(y-y_j)\right| + \sum_{i=0}^{k}|f(y_i)|\left|\frac{\prod_{j=0}^{k}(y-y_j)}{(y_i-y)\prod_{j=0,j\neq i}^{k}(y_i-y_j)}\right| \\
&\leq \frac{1}{(k+1)!}\,n^{k+2}\frac{(k+1)!}{(2n)^{k+1}} + 2^{1-\sqrt{n}}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!} \\
&\leq 2^{-(k+1)}n + 2^{1-\sqrt{n}}\,2^k \leq 2^{-(1/2)\sqrt{n}}n + 2^{1-\sqrt{n}}\,2^{(1/2)\sqrt{n}} \\
&\leq \exp\left(-c_3\sqrt{n}\right)
\end{aligned}
$$

with an absolute constant $c_3 > 0$. Since $y \in [y_0, 1]$ is arbitrary, we have proved that

$$\|f\|_{[y_0,1]} \leq \exp\left(-c_3\sqrt{n}\right)\,,$$

where $y_0 = 1-k/(2n) \leq 1-\frac{1}{4}n^{-1/2}$ for sufficiently large n. Now for $g(x) := x^n f(x)$, we have $g \in \mathcal{F}_{2n}$ and

$$\|g\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

with an absolute constant $c_2 > 0$.

The proof of the theorem is now finished. □

Proof of Theorem 3.4. First we prove the lower bound. Suppose $0 \neq p(-x) \in \mathcal{A}_n$ has exactly k zeros at 1. Then, using Theorem 2.9 and Markov's Inequality, we obtain

$$
\begin{aligned}
\|P\|_{[0,1]} &\geq (2n)^{-2k}|P^{(k)}(1)| \geq (2n)^{-c\log n} \\
&\geq \exp\left(-c(\log n)\log(2n)\right)
\end{aligned}
$$

(note that $|P^{(k)}(1)|$ is a positive integer, hence at least 1), and the lower bound of the theorem follows. The upper bound follows from the following example. Let

$$Q_m(x) := x^{3^m}\prod_{k=0}^{m}\left(1+x^{3^k}\right).$$

Then $Q_m \in \mathcal{A}_{3^{m+1}}$, and for $x \in [0,1]$,

$$
\begin{aligned}
0 \leq -Q_m(-x) = x^{3^m}\prod_{k=0}^{m}\left(1-x^{3^k}\right) &\leq \left(x^{3^m}(1-x)^{m+1}\right)\prod_{k=0}^{m}\left(\sum_{j=0}^{3^k-1}x^j\right) \\
&\leq \left(\frac{m+1}{3^m+m+1}\right)^{m+1}\left(1-\frac{m+1}{3^m+m+1}\right)^{3^m}\prod_{k=0}^{m}3^k \\
&\leq \left(\frac{m+1}{3^m+m+1}\right)^{m+1}3^{m(m+1)/2} \leq \exp\left(-c(m+1)^2\right)
\end{aligned}
$$

with an absolute constant $c > 0$, and the upper bound of the theorem follows. □

Proof of Theorem 3.5. The lower bound follows from Theorem 5.1 immediately. The upper bound is a simple corollary of Theorem 3.3 with the choice $n := \lfloor (c/a)^2 \rfloor$ with a suitable absolute constant $c > 0$. Indeed, without loss of generality we may assume that $a \in (0, 1/2]$ and $n := \lfloor (c/a)^2 \rfloor$ is an integer. The upper bound in Theorem 3.3 implies that there is a $0 \neq p \in \mathcal{F}_n$ such that

$$\|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right) \leq \exp\left(-c_2 c/a\right) . \tag{6.4}$$

Then p is of the form

$$p(x) = x^k q(x)\,, \qquad q \in \mathcal{F}_n\,, \quad q(0) = 1\,, \quad 0 \leq k \leq n\,. \tag{6.5}$$

Here

$$x^k \geq x^n \geq (1-a)^{(c/a)^2} \geq \exp\left(-(c_2 c/(2a)\right)\,, \qquad x \in [1-a, 1] \tag{6.6}$$

with a suitable absolute constant $c > 0$. Now by (6.4), (6.5), and (6.6), we have $q \in \mathcal{F}$, $q(0) = 1$, and

$$\|q\|_{[1-a,1]} \leq \exp\left(-(c_2 c/(2a)\right) .$$

□

Proof of Theorem 4.2. By using a substitution $q(x) = p(-x)$, it is sufficient to estimate the number of zeros only in $(0, 1-a]$. Let p be a polynomial of the form in the theorem. If $a > 1/2$, then p has no zero in $[0, 1-a]$. So we assume that $a \leq 1/2$.First we estimate the number of zeros only in $[1-(5/4)a, 1-a]$. Denote the number of the zeros of p in $[1-(5/4)a, 1-a]$ by m. Let y be a point in $[1-(5/4)a, 1-a]$ at which

$$|p(y)| \geq \exp\left(-\frac{c_3}{a}\right)\,,$$

where $c_3 > 0$ is an absolute constant. The existence of such a point y is guaranteed by Theorem 5.1 (one needs to combine it with a linear transformation). By using a well-known formula for divided differences at the m zeros of p in $[1-(5/4)a, 1-a]$ and at y, there exists a $\xi \in [1-(5/4)a, 1-a]$ such that

$$|p^{(m)}(\xi)| \geq m!\,\left(\frac{a}{4}\right)^{-m} |p(y)| \geq m!\,\left(\frac{a}{4}\right)^{-m} \exp\left(-\frac{c_3}{a}\right) .$$

Estimating $|p^{(m)}(\xi)|$ by the Cauchy integral formula on the circle centered at ξ with radius $a/2$ (note that $\xi + a/2 \leq 1 - a/2$), we obtain

$$|p^{(m)}(\xi)| \leq m!\,\frac{2(a/2)\pi}{2\pi}\,\frac{2}{a}\,\left(\frac{a}{2}\right)^{-m-1} .$$

Combining the previous two inequalities, we get

$$2^m \leq \frac{2}{a}\,\exp\left(\frac{c_3}{a}\right) \leq \exp\left(\frac{c_4}{a}\right)\,,$$

which gives $m \le c_5/a$. Now counting the zeros of p in

$$[1-(5/4)a, 1-a]\,, \quad [1-(5/4)^2a, 1-(5/4)a]\,, \quad \dots\,, \quad [1-(5/4)^k a, 1-(5/4)^{k-1}a]\,,$$

where k is the smallest positive integer for which $(5/4)^k a \ge 1/2$, and applying the already proved estimate with a, $(5/4)a, \dots, (5/4)^{k-1}a$, we get the bound of the theorem for the number of zeros in $[1/2, 1-a]$. Since p has no zeros in $[0, 1/2)$, the proof is finished. $\square$

Proof of Theorem 4.1. By using a substitution $q(x) = x^n p(x^{-1})$, it is sufficient to prove only the first statement; the other two follow from it. Also, by Theorem 4.2 and by the substitution $q(x) = p(-x)$, it is sufficient to prove that p has at most $c\sqrt{n}$ zeros in $[1-n^{-1/2}, 1]$. Let p be a polynomial of the form in the theorem. Denote the number of the zeros of p in $[1-n^{-1/2}, 1]$ by m. Let $y \in [1-n^{-1/2}, 1]$ be a point at which

$$|p(y)| \ge \exp\left(-c_6 n^{1/2}\right)\,,$$

where $c_6 > 0$ is an absolute constant. The existence of such a point y is guaranteed by Theorem 5.1. By using a well-known formula for divided differences at the m zeros of p in $[1-n^{-1/2}, 1]$ and at y, there exists a $\xi \in [1-n^{-1/2}, 1]$ such that

$$|p^{(m)}(\xi)| \ge m!\left(n^{-1/2}\right)^{-m} |p(y)| \ge m!\, n^{m/2} \exp\left(-c_6 n^{1/2}\right)\,.$$

Estimating $|p^{(m)}(\xi)|$ by the Cauchy integral formula on the circle centered at ξ with radius $2n^{-1/2}$, we obtain

$$\begin{aligned}|p^{(m)}(\xi)| &\le m!\,\frac{2n^{-1/2}\pi}{2\pi}\, n \left(1+2n^{-1/2}\right)^n \left(2n^{-1/2}\right)^{-m-1}\\ &\le m!\, n^{m/2} \exp\left(c_7 n^{1/2}\right) 2^{-m-1}\,.\end{aligned}$$

Combining the previous two inequalities, we get

$$2^m \le \exp\left(c_8 n^{1/2}\right)\,,$$

which gives $m \le c_9 n^{1/2}$. This finishes the proof. $\square$

Proof of Theorem 4.3. The proof is a straightforward modification of that of Theorem 4.1. We omit the details. $\square$

Proof of Theorem 4.4. The proof follows easily from Theorem 4.1 by studying $q(x) := p(ax)$. $\square$

7. Remarks

There is an obvious interval dependence in the problem of finding non-zero polynomials from $\mathcal{F}_n$ with minimal uniform norm. It is quite easy to argue that on any interval $[0, \delta]$ with $\delta < 1/2$ the only polynomials from $\mathcal{F}_n$ with minimal uniform norm are $\pm x^n$. On $[0, 1/2]$ all of $\pm x^n$ and $\pm(x^n - x^{n-1})$ are extremals. On any interval $[0, \delta]$ with $\delta > 1/2$ the polynomials $\pm(x^n - x^{n-1})$ have smaller supremum norm than the supremum norm of x^n, so the nature of the extremals change at $1/2$. Also, on any interval $[1, \gamma]$ with $\gamma > 1$, $\inf_f \|f\|_{[1,\gamma]} > 0$, where the infimum is taken over all $0 \ne f \in \cup_{n=1}^{\infty} \mathcal{F}_n$.

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References

1. F. Amoroso, *Sur le diamètre transfini entier d'un intervalle réel*, Ann. Inst. Fourier, Grenoble **40** (1990), 885–911.
2. B. Aparicio, *New bounds on the minimal Diophantine deviation from zero on* $[0,1]$ *and* $[0,1/4]$, Actus Sextas Jour. Mat. Hisp.-Lusitanas (1979), 289–291.
3. F. Beaucoup, P. Borwein, D. Boyd and C. Pinner, *Multiple Roots of [-1,1] Power Series*, Jour. London Math. Soc.(to appear).
4. J. Beck, *Flat polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. **23** (1991), 269–277.
5. A.T. Bharucha-Reid and M. Sambandham, *Random polynomials*, Academic Press, Orlando, 1986.
6. A. Bloch and G. Pólya, *On the roots of certain algebraic equations*, Proc. London Math. Soc **33** (1932), 102–114.
7. E. Bombieri and J. Vaaler, *Polynomials with low height and prescribed vanishing*, in Analytic Number Theory and Diophantine Problems, Birkhauser (1987), 53–73.
8. P. Borwein and T. Erdélyi, *Markov and Bernstein type inequalities for polynomials with restricted coefficients*, Ramanujan J. **1** (1997), 309–323.
9. P. Borwein and T. Erdélyi, *The integer Chebyshev problem*, Math. Computat. **65** (1996), 661–681.
10. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995.
11. P. Borwein and C. Ingalls, *The Prouhet, Tarry, Escott problem*, Ens. Math. **40** (1994), 3–27.
12. J. Bourgain, *Sul le minimum d'une somme de cosinus*, Acta Arith. **45** (1986), 381–389.
13. D. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients*, Math. Comput. **66** (1997), 1697–1703.
14. J.S. Byrnes and D.J. Newman, *Null Steering Employing Polynomials with Restricted Coefficients*, IEEE Trans. Antennas and Propagation **36** (1988), 301–303.
15. F.W. Carrol, D. Eustice and T. Figiel, *The minimum modulus of polynomials with coefficients of modulus one*, Jour. London Math. Soc. **16** (1977), 76–82.
16. J. Clunie, *On the minimum modulus of a polynomial on the unit circle*, Quart. J. Math. **10** (1959), 95–98.
17. P.J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960), 191–212.
18. P. Erdős, *Some old and new problems in approximation theory: research problems 95-1*, Constr. Approx. **11** (1995), 419–421.
19. P. Erdős and P. Turán, *On the distribution of roots of polynomials*, Annals of Math. **57** (1950), 105–119.
20. Le Baron O. Ferguson, *Approximation by Polynomials with Integral Coefficients*, Amer. Math. Soc., Rhode Island, 1980.

21. G.T. Fielding, *The expected value of the integral around the unit circle of a certain class of polynomials*, Bull. London Math. Soc. **2** (1970), 301–306.

22. L.K. Hua, *Introduction to Number Theory*, Springer-Verlag, Berlin Heidelberg, New York, 1982.

23. M. Kac, *On the average number of real roots of a random algebraic equation, II*, Proc. London Math. Soc. **50** (1948), 390–408.

24. J-P. Kahane, *Some Random Series of Functions*, vol. 5, Cambridge Studies in Advanced Mathematics, Cambridge, 1985; Second Edition.

25. J-P. Kahane, *Sur les polynômes á coefficients unimodulaires*, Bull. London Math. Soc **12** (1980), 321–342.

26. S. Konjagin, *On a problem of Littlewood*, Izv. A. N. SSSR, ser. mat. **45, 2** (1981), 243–265.

27. T.W. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224.

28. J.E. Littlewood, *On the mean value of certain trigonometric polynomials*, Jour. London Math. Soc. **36** (1961), 307–334.

29. J.E. Littlewood, *On polynomials $\sum^n \pm z^m$ and $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta i}$*, Jour. London Math. Soc. **41** (1966), 367–376.

30. J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

31. J.E. Littlewood and A.C. Offord, *On the number of real roots of a random algebraic equation, II*, Proc. Cam. Phil. Soc. **35** (1939), 133-148.

32. K. Mahler, *On two extremal properties of polynomials*, Illinois J. Math. **7** (1963), 681–701.

33. D.J. Newman and J.S. Byrnes, *The L^4 norm of a polynomial with coefficients* ± 1, MAA Monthly **97** (1990), 42–45.

34. D.J. Newman and A. Giroux, *properties on the unit circle of polynomials with unimodular coefficients*, in Recent Advances in Fourier Analysis and its Applications J.S. Byrnes and J.F. Byrnes, Eds.), Kluwer, 1990, pp. 79–81..

35. A. Odlyzko and B. Poonen, *Zeros of polynomials with 0,1 coefficients*, Ens. Math. **39** (1993), 317–348.

36. G. Pólya and G. Szegő, *Problems and Theorems in Analysis, Volume I*, Springer-Verlag, New York, 1972.

37. R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math **91** (1954), 254–301.

38. E. Schmidt, *Über algebraische Gleichungen vom Pólya-Bloch-Typos*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1932), 321.

39. I. Schur, *Untersuchungen über algebraische Gleichungen.*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1933), 403–428.

40. B. Solomyak, *On the random series $\sum \pm\lambda^n$ (an Erdős problem)*, Annals of Math. **142**, 611–625.

41. G. Szegő, *Bemerkungen zu einem Satz von E. Schmidt uber algebraische Gleichungen.*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1934), 86–98.

42. P. Turán, *On a New Method of Analysis and its Applications*, Wiley, New York, 1984.

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