IMPROVED RESULTS ON THE OSCILLATION OF THE MODULUS OF THE RUDIN-SHAPIRO POLYNOMIALS ON THE UNIT CIRCLE

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Dedicated to Paul Nevai on the occasion of his 70th birthday

Abstract. Let $R_k(t) := |P_k(e^{it})|^2$ and $S_k(t) := |Q_k(e^{it})|^2$, where P_k and Q_k are the usual Rudin-Shapiro polynomials of degree $n-1$ with $n = 2^k$. In a recent paper we combined close to sharp upper bounds for the modulus of the autocorrelation coefficients of the Rudin-Shapiro polynomials with a deep theorem of Littlewood to prove that there is an absolute constant $A > 0$ such that the equations $R_k(t) = (1+\eta)n$ and $S_k(t) = (1+\eta)n$ have at least $An^{0.5394282}$ distinct zeros in $[0, 2\pi)$ whenever η is real, $|\eta| < 2^{-8}$, and n is sufficiently large. In this paper we show that the equations $R_k(t) = (1+\eta)n$ and $S_k(t) = (1+\eta)n$ have at least $(1/2 - |\eta| - \varepsilon)n/2$ distinct zeros in $[0, 2\pi)$ for every $\eta \in (-1/2, 1/2)$, $\varepsilon > 0$, and sufficiently large $n \geq n_{\eta,\varepsilon}$.

1. Introduction

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane. Let $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle of the complex plane. The Mahler measure $M_0(f)$ is defined for bounded measurable functions f on ∂D by

$$M_0(f) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|\,dt\right).$$

It is well known, see [HL-52], for instance, that

$$M_0(f) = \lim_{q\to 0+} M_q(f),$$

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where

$$M_q(f) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{it}) \right|^q \, dt \right)^{1/q}, \qquad q > 0\,.$$

It is also well known that for a function f continuous on ∂D we have

$$M_\infty(f) := \max_{t \in [0,2\pi]} |f(e^{it})| = \lim_{q \to \infty} M_q(f)\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(f) = |c| \prod_{j=1}^{n} \max\{1, |z_j|\}$$

for every polynomial of the form

$$f(z) = c \prod_{j=1}^{n} (z - z_j)\,, \qquad c, z_j \in \mathbb{C}\,.$$

See [BE-95, p. 271] or [B-02, p. 3], for instance.

Let $\mathcal{P}_n^c$ be the set of all algebraic polynomials of degree at most n with complex coefficients. Let $\mathcal{T}_n$ be the set of all real (that is, real-valued on the real line) trigonometric polynomials of degree at most n. Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

$$\mathcal{L}_n := \left\{ f : \;\; f(z) = \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the classes

$$\mathcal{K}_n := \left\{ f : \;\; f(z) = \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \mathbb{C},\;\; |a_j| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. Observe that $\mathcal{L}_n \subset \mathcal{K}_n$ and

$$M_0(f) \le M_2(f) = \sqrt{n+1}\,, \qquad f \in \mathcal{K}_n\,.$$

Beller and Newman [BN-73] constructed unimodular polynomials $f_n \in \mathcal{K}_n$ such that $M_0(f_n) \ge \sqrt{n} - c/\log n$ with an absolute constant $c > 0$.

Section 4 of [B-02] is devoted to the study of Rudin-Shapiro polynomials. Littlewood asked if there were polynomials $f_{n_k} \in \mathcal{L}_{n_k}$ satisfying

$$c_1\sqrt{n_k + 1} \le |f_{n_k}(z)| \le c_2\sqrt{n_k + 1}\,, \qquad z \in \partial D\,,$$

with some absolute constants $c_1 > 0$ and $c_2 > 0$, see [B-02, p. 27] for a reference to this problem of Littlewood. To satisfy just the lower bound, by itself, seems very hard, and no such sequence (f_{n_k}) of Littlewood polynomials $f_{n_k} \in \mathcal{L}_{n_k}$ is known. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

$$P_0(z) := 1\,, \qquad Q_0(z) := 1\,,$$
$$P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z)\,,$$
$$Q_{k+1}(z) := P_k(z) - z^{2^k} Q_k(z)\,,$$

for $k = 0, 1, 2, \ldots$. Note that both P_k and Q_k are polynomials of degree $n-1$ with $n := 2^k$ having each of their coefficients in $\{-1, 1\}$. In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems.

It is well known and easy to check by using the parallelogram law that

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2)\,, \qquad z \in \partial D\,.$$

Hence

$$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n\,, \qquad z \in \partial D\,. \tag{1.1}$$

It is also well known (see Section 4 of [B-02], for instance), that

$$Q_k(-z) = P_k^*(z) := z^{n-1} P_k(1/z)\,, \qquad z \in \mathbb{C} \setminus 0\,,$$

and hence

$$|Q_k(-z)| = |P_k(z)|\,, \qquad z \in \partial D\,. \tag{1.2}$$

Let $K := \mathbb{R} \pmod{2\pi}$. Let $m(A)$ denote the one-dimensional Lebesgue measure of $A \subset K$. In 1980 Saffari conjectured the following.

Conjecture 1.1. *Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. We have*

$$M_q(P_k) = M_q(Q_k) \sim \frac{2^{(k+1)/2}}{(q/2+1)^{1/q}}$$

for all real exponents $q > 0$. Equivalently, we have

$$\begin{aligned}&\lim_{k\to\infty} m\left(\left\{t \in K : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha, \beta]\right\}\right)\\ =&\lim_{k\to\infty} m\left(\left\{t \in K : \left|\frac{Q_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha, \beta]\right\}\right) = 2\pi(\beta - \alpha)\end{aligned}$$

whenever $0 \le \alpha < \beta \le 1$.

This conjecture was proved for all even values of $q \le 52$ by Doche [D-05] and Doche and Habsieger [DH-04]. Recently B. Rodgers [R-16] proved Saffari's Conjecture 1.1 for all $q > 0$. See also [EZ-17]. An extension of Saffari's conjecture is Montgomery's conjecture below.

Conjecture 1.2. *Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. We have*

$$\lim_{k\to\infty} m\left(\left\{t \in K : \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right)$$
$$= \lim_{k\to\infty} m\left(\left\{t \in K : \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right) = 2m(E)$$

for any rectangle $E \subset D := \{z \in \mathbb{C} : |z| < 1\}$.

B. Rodgers [R-16] proved Montgomery's Conjecture 1.2 as well.

Despite the simplicity of their definition not much is known about the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [B-73] by Brillhart and in [BL-76] by Brillhart, Lemont, and Morton. As for $k \ge 1$ both P_k and Q_k have odd degree, both P_k and Q_k have at least one real zero. The fact that for $k \ge 1$ both P_k and Q_k have exactly one real zero was proved in [B-73]. It has been shown in [E-16c] fairly recently that the Mahler measure (M_0 norm) and the M_∞ norm of the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constants. In [E-18] various results on the zeros of Rudin-Shapiro polynomials are proved and some open problems are raised. In [E-19] a conjecture of Saffari on the asymptotic value of the Mahler measure of the Rudin-Shapiro polynomials is proved.

More on Littlewood polynomials may be found in [B-02], [E-02], [M-17], and [O-18], for example. There are many other papers on the zeros of constrained polynomials. Some of them are [BP-32], [BE-97], [BE-01], [BE-07], [BE-08a], [BE-08b], [BE-99], [BE-13], [B-97], [D-08], [E-08a], [E-08b], [E-16a], [E-16b], [L-61], [L-64], [L-66a], [L-66b], [L-68], [M-06], [Sch-32], [Sch-33], [Sz-34], and [TV-07].

2. New Results

Let $R_k(t) := |P_k(e^{it})|^2$ and $S_k(t) := |Q_k(e^{it})|^2$, where P_k and Q_k are the usual Rudin-Shapiro polynomials of degree $n-1$ with $n = 2^k$. Let $K := \mathbb{R} \pmod{2\pi}$. In [AC-18] we combined close to sharp upper bounds for the modulus of the autocorrelation coefficients of the Rudin-Shapiro polynomials with a deep theorem of Littlewood (see Theorem 1 in [L-66a]) to prove that there is an absolute constant $A > 0$ such that the equations $R_k(t) = (1+\eta)n$ and $S_k(t) = (1+\eta)n$ have at least $An^{0.5394282}$ distinct zeros in K whenever η is real, $|\eta| \le 2^{-8}$, and n is sufficiently large. In this paper we improve this result substantially.

Theorem 2.1. *The equations $R_k(t) = n$ and $S_k(t) = n$ have at least $n/4 + 1$ distinct zeros in K. Moreover, with the notation $t_j := 2\pi j/n$, there are at least $n/2 + 2$ values of $j \in \{0, 1 \ldots, n-1\}$ for which the interval $[t_j, t_{j+1}]$ has at least one zero of the equation $R_k(t) = n$, and there are at least $n/2 + 2$ values of $j \in \{0, 1 \ldots, n-1\}$ for which the interval $[t_j, t_{j+1}]$ has at least one zero of the equation $S_k(t) = n$,*

Theorem 2.2. *The equations $R_k(t) = (1+\eta)n$ and $S_k(t) = (1+\eta)n$ have at least $(1/2 - |\eta| - \varepsilon)n/2$ distinct zeros in K for every $\eta \in (-1/2, 1/2)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$,*

3. Lemma

In the proof of Theorem 2.1 we need the lemma below stated and proved as Lemma 3.1 in [E-16].

Lemma 3.1. *Let $n \geq 2$ be an integer, $n := 2^k$, and let*

$$z_j := e^{it_j}\,, \quad t_j := \frac{2\pi j}{n}\,, \quad j \in \mathbb{Z}\,.$$

We have

$$\begin{aligned} P_k(z_j) &= 2P_{k-2}(z_j)\,, \qquad j = 2u\,,\ u \in \mathbb{Z}\,, \\ P_k(z_j) &= (-1)^{(j-1)/2} 2i\, Q_{k-2}(z_j)\,, \qquad j = 2u+1\,,\ u \in \mathbb{Z}\,, \end{aligned}$$

where i is the imaginary unit.

4. Proofs

Proof of Theorem 2.1. We prove the statement for R_k. The proof remains the same for S_k by replacing R_k by S_k. Let $k \geq 2$ be an integer. For the sake of brevity let

$$A_j := R_{k-2}(t_j) - n/4\,, \qquad j = 0, 1, \ldots, n\,.$$

Using the notation of Lemma 3.1 we study the $(n+1)$-tuple $\langle A_0, A_1, \ldots, A_n \rangle$. Observe that R_{k-2} is a real trigonometric polynomial of degree $n/4 - 1 = 2^k/4 - 1$, and hence $R_{k-2}(t) - n/4$ has at most $n/2 - 2$ zeros in K. Therefore the Intermediate Value Theorem yields that the number of sign changes in the $(n+1)$-tuple $\langle A_0, A_1, \ldots, A_n \rangle$ is at most $n/2 - 2$. Thus there are integers

$$0 \leq j_1 < j_2 < \cdots < j_m \leq n-1$$

with $m \geq n - (n/2 - 2) = n/2 + 2$ such that

$$A_{j_\nu} A_{j_\nu + 1} \geq 0\,, \qquad \nu = 0, 1, \ldots, m\,. \tag{4.1}$$

Using Lemma 3.1 we have either

$$\begin{aligned}
16A_{j_\nu}A_{j_\nu+1} =&(4(R_{k-2}(t_{j_\nu})-n/4))(4(R_{k-2}(t_{j_\nu+1})-n/4)) \\
=&(4|P_{k-2}(e^{it_{j_\nu}})|^2-n)(4|P_{k-2}(e^{it_{j_\nu+1}})|^2-n) \\
=&(|P_k(e^{it_{j_\nu}})|^2-n)(|Q_k(e^{it_{j_\nu+1}})|^2-n) \\
=&(|P_k(e^{it_{j_\nu}})|^2-n)(n-|P_k(e^{it_{j_\nu+1}})|^2)\,,
\end{aligned} \tag{4.2}$$

or

$$\begin{aligned}
16A_{j_\nu}A_{j_\nu+1} =&(4(R_{k-2}(t_{j_\nu})-n/4))(4(R_{k-2}(t_{j_\nu+1})-n/4)) \\
=&(4|P_{k-2}(e^{it_{j_\nu}})|^2-n)(4|P_{k-2}(e^{it_{j_\nu+1}})|^2-n) \\
=&(|Q_k(e^{it_{j_\nu}})|^2-n)(|P_k(e^{it_{j_\nu+1}})|^2-n) \\
=&(|n-|P_k(e^{it_{j_\nu}})|^2)(|P_k(e^{it_{j_\nu+1}})|^2-n)\,.
\end{aligned} \tag{4.3}$$

Combining (4.1), (4.2), and (4.3), we can deduce that

$$(|P_k(e^{it_{j_\nu}})|^2-n)(|P_k(e^{it_{j_\nu+1}})|^2-n)=-16A_{j_\nu}A_{j_\nu+1}\leq 0\,,\qquad \nu=0,1,\ldots,m\,.$$

Hence the Intermediate Value Theorem implies that $R_k(t)-n=|P_k(e^{it})|^2-n$ has at least one zero in each of the intervals

$$[t_{j_\nu},t_{j_\nu+1}]\,,\qquad \nu=0,1,\ldots,m\,.$$

Recalling that $m\geq n/2+2$ we conclude that $R_k(t)-n=|P_k(e^{it})|^2-n$ has at least $m/2=n/4+1$ distinct zeros in K. $\square$

Proof of Theorem 2.2. We prove the statement for R_k. The proof remains the same for S_k by replacing R_k by S_k. The statement for R_k follows from the proof of Theorem 2.1 combined with B. Rodgers's resolution of Saffari's Conjecture 1.1. Assume that $R_k(t)=|P_k(e^{it})|^2$, the proof in the case $R_k(t)=|Q_k(e^{it})|^2$ is the same. Also, we may assume that $\eta>0$, the case $\eta=0$ is contained in Theorem 2.1. We use the notation in the proof of Theorem 2.1. Recall that each of the intervals

$$[t_{j_\nu},t_{j_\nu+1}]\,,\qquad \nu=0,1,\ldots,m\,,$$

has at least one zero of R_k-n. On the other hand, by Saffari's Conjecture proved by Rodgers [R-16] we have

$$m(\{t\in K:|R_k(t)-n|\leq|\eta|n\})<2\pi(1+\varepsilon)|\eta|$$

for every $\eta\in(-1/2,1/2)$, $\varepsilon>0$, and sufficiently large $k\geq k_{\eta,\varepsilon}$. Hence, with the notation

$$B_\eta:=\{t\in K:|R_k(t)-n|\leq|\eta|n\}\,,$$

there are at least $m-(1+\varepsilon)|\eta|n$ distinct values of $\nu \in \{1,2,\dots,m\}$ such that

$$[t_{j_\nu}, t_{j_\nu+1}] \setminus B_\eta \neq \emptyset$$

for every $\eta \in (-1/2,1/2)$, and sufficiently large $k \geq k_{\eta,\varepsilon}$. Hence by the Intermediate Value Theorem there are at least $m-(1+\varepsilon)|\eta|n$ distinct values of $\nu \in \{1,2,\dots,m\}$ for which $|R_k(t)-n| = |\eta|n$ has a zero in $(t_{j_\nu}, t_{j_\nu+1})$ for every $\eta \in (-1/2,1/2)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$. Now observe that (1.1) and (1.2) imply that

$$|P_k(z)|^2 + |P_k(-z)|^2 = 2n\,, \qquad z \in \partial D\,,$$

that is,

$$R_k(t) - n = n - R_k(t+\pi)\,, \qquad t \in K\,.$$

Hence for every $\eta \in (-1/2,1/2)$ the number of distinct zeros of $|R_k(t)-n| = |\eta|n$ in K is exactly twice the number of distinct zeros of $R_k(t) = (1+\eta)n$ in K. We conclude that there are at least

$$\frac{1}{2}(m-(1+\varepsilon)\eta n) \geq (1/2 - |\eta| - \varepsilon)n/2$$

distinct values of $\nu \in \{1,2,\dots,m\}$ for which $R_k(t) - n = \eta n$ has a zero in $(t_{j_\nu}, t_{j_\nu+1})$ for every $\eta \in (-1/2,1/2)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$. $\square$

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