ON THE OSCILLATION OF THE MODULUS
OF THE RUDIN-SHAPIRO POLYNOMIALS
AROUND THE MIDDLE OF THEIR RANGES

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ABSTRACT. Let either $R_k(t) := |P_k(e^{it})|^2$ or $R_k(t) := |Q_k(e^{it})|^2$, where P_k and Q_k are the usual Rudin-Shapiro polynomials of degree $n-1$ with $n = 2^k$. The graphs of R_k on the period suggest many zeros of $R_k(t) - n$ in a dense fashion on the period. Let $N(I, R_k - n)$ denote the number of zeros of $R_k - n$ in an interval $I := [\alpha, \beta] \subset [0, 2\pi]$. Improving earlier results stated only for $I := [0, 2\pi]$, in this paper we show that

$$\frac{n|I|}{8\pi} - \frac{2}{\pi}(2n \log n)^{1/2} - 1 \leq N(I, R_k - n) \leq \frac{n|I|}{\pi} + \frac{8}{\pi}(2n \log n)^{1/2}, \qquad k \geq 2,$$

for every $I := [\alpha, \beta] \subset [0, 2\pi]$, where $|I| = \beta - \alpha$ denotes the length of the interval I.

1. INTRODUCTION

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane. Let $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle of the complex plane. Littlewood polynomials are polynomials with each of their coefficients in $\{-1, 1\}$. A special sequence of Littlewood polynomials are the Rudin-Shapiro polynomials, They appear in Harold Shapiro's 1951 thesis [18] at MIT and are sometimes called just the Shapiro polynomials. They also arise independently in Golay's paper [15]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

$$P_0(z) := 1, \qquad Q_0(z) := 1,$$
$$P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z),$$
$$Q_{k+1}(z) := P_k(z) - z^{2^k} Q_k(z),$$

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for $k = 0, 1, 2, \dots$. Note that both P_k and Q_k are polynomials of degree $n - 1$ with $n := 2^k$ having each of their coefficients in $\{-1, 1\}$. In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems.

It is well known and easy to check by using the parallelogram law that

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2), \qquad z \in \partial D.$$

Hence

(1.1) $$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n, \qquad z \in \partial D.$$

It is also well known (see Section 4 of [3], for instance), that

$$Q_k(-z) = P_k^*(z) := z^{n-1} P_k(1/z), \qquad z \in \partial D,$$

and hence

(1.2) $$|Q_k(-z)| = |P_k(z)|, \qquad z \in \partial D.$$

Despite the simplicity of their definition not much is known about the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [4] and [5]. As for $k \geq 1$ both P_k and Q_k have odd degree, both P_k and Q_k have at least one real zero. The fact that for $k \geq 1$ both P_k and Q_k have exactly one real zero was proved in [4]. It has been shown in [9] that the Mahler measure (geometric mean) and the maximum modulus of the Rudin-Shapiro polynomials P_k and Q_k of degree $n - 1$ with $n := 2^k$ on the unit circle of the complex plane have the same size. That is, in addition to (1.1), the Mahler measure of the Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constants. In [10] various results on the zeros of the Rudin-Shapiro polynomials are proved and some open problems are raised. In [11] a conjecture of Saffari on the asymptotic value of the Mahler measure of the Rudin-Shapiro polynomials is proved.

For a monic polynomial

(1.3) $$P(z) = \prod_{j=1}^{n} (z - \alpha_j) = z^n + \sum_{j=0}^{n-1} a_j z^j, \qquad a_j \in \mathbb{C}, \quad a_0 \neq 0,$$

let

$$H(P) := \frac{1}{|a_0|^{1/2}} \max_{z \in \partial D} |P(z)|.$$

Let

$$\alpha_j = \rho_j e^{i\theta_j}, \qquad \rho_j > 0, \quad \theta_j \in [0, 2\pi).$$

For $I := [\alpha, \beta] \subset [0, 2\pi]$ let $N(I, P)$ denote the number of the values $j \in \{1, 2, \dots, n\}$ for which $\theta_j \in I$. In 1950 Erdős and Turán [14] proved the following result.

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Theorem 1.1. *We have*

$$\left| N(I, P) - \frac{n|I|}{2\pi} \right| \leq 16(n \log H(P))^{1/2}$$

for every monic polynomial of the form (1.3) and for every $I := [\alpha, \beta] \subset [0, 2\pi)$, where $|I| = \beta - \alpha$ denotes the length of the interval I.

In [19] K. Soundararajan proved that the constant 16 in the above result may be replaced by $8/\pi$. Another improvement of the Erdős-Turán theorem may be found in [8], for example. Rudin-Shapiro polynomials play a key role in [2] as well as in [13] to prove the existence of flat Littlewood polynomials, a recent breakthrough result. More on Rudin-Shapiro polynomials may be found in [6,7,17].

2. New Results

Let either $R_k(t) := |P_k(e^{it})|^2$ or $R_k(t) := |Q_k(e^{it})|^2$, and $n := 2^k$. In [1] we combined close to sharp upper bounds for the modulus of the autocorrelation coefficients of the Rudin-Shapiro polynomials with a deep theorem of Littlewood (see Theorem 1 in [16]) to prove that there is an absolute constant $A > 0$ such that the equation $R_k(t) = (1+\eta)n$ with $n := 2^k$ has at least $An^{0.5394282}$ distinct solutions in $[0, 2\pi)$ whenever η is real, $|\eta| \leq 2^{-8}$, and n is sufficiently large. In this paper we improve this result substantially.

Theorem 2.1. *Let $k \geq 0$ and $n := 2^k$ be integers and let $N(I, R_k - n)$ denote the number of zeros of $R_k(t) - n$ in an interval $I := [\alpha, \beta] \subset [0, 2\pi)$. We have*

$$\frac{n|I|}{8\pi} - \frac{2}{\pi}(2n \log n)^{1/2} - 1 \leq N(I, R_k - n) \leq \frac{n|I|}{\pi} + \frac{8}{\pi}(2n \log n)^{1/2}, \qquad k \geq 2,$$

for every $I := [\alpha, \beta] \subset [0, 2\pi]$, where $|I| = \beta - \alpha$ denotes the length of the interval I.

This extends the main result in [12] from the case $I := [0, 2\pi)$ to the case $I = [\alpha, \beta] \subset [0, 2\pi]$. In our proof of Theorem 2.1 we combine ideas used in [12] and a classical result of Erdős and Turán [14] with a constant improved recently by Soundararajan [19].

3. Lemmas

In the proof of Theorem 2.1 we need the lemma below stated and proved as Lemma 3.1 in [9].

Lemma 3.1. *Let $k \geq 2$ and $n := 2^k$ be integers, and let*

$$z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{n}, \quad j \in \mathbb{Z}.$$

We have

$$P_k(z_j) = 2P_{k-2}(z_j), \qquad j = 2u, \ u \in \mathbb{Z},$$
$$P_k(z_j) = (-1)^{(j-1)/2} 2i\, Q_{k-2}(z_j), \qquad j = 2u + 1, \ u \in \mathbb{Z},$$

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where i is the imaginary unit.

For a trigonometric polynomial of the form

$$(3.1) \qquad T(\theta) = \pm 2\cos(m\theta) + \sum_{j=-m+1}^{m-1} a_j e^{ij\theta}, \qquad a_j \in \mathbb{C},$$

let

$$H(T) := \max_{\theta \in \mathbb{R}} |T(\theta)|.$$

For $I := [\alpha, \beta] \subset [0, 2\pi]$ let $N(I, T)$ denote the number of zeros of T in I counted with multiplicities.

Lemma 3.2. *We have*

$$N(I, T) - \frac{m|I|}{\pi} \leq \frac{8}{\pi}(2m \log H(T))^{1/2}$$

for every trigonometric polynomial T of the form (3.1) and for every $I := [\alpha, \beta] \subset [0, 2\pi]$, where $|I| := \beta - \alpha$.

Proof. This follows from the Erdős-Turán inequality (Theorem 1.1) with 16 replaced by Soundararajan's constant $8/\pi$. $\square$

Lemma 3.3. *Let $k \geq 0$ and $n := 2^k$ be integers. We have*

$$N(I, R_k - n) - \frac{n|I|}{\pi} \leq \frac{8}{\pi}(2n \log n)^{1/2}$$

for every $I := [\alpha, \beta] \subset [0, 2\pi]$, where $|I| := \beta - \alpha$.

Proof. Observe that $R_k - n$ is of the form (3.1) with $m := n - 1$. It follows from (1.1) that

$$H(R_k - n) = \max_{\theta \in \mathbb{R}} |R_k(\theta) - n| \leq n,$$

and the lemma follows from Lemma 3.2 $\square$

Replacing n by $n/4$ we get the following corollary.

Lemma 3.4. *Let $k \geq 2$ and $n := 2^k$ be integers. We have*

$$N(I, R_{k-2} - n/4) - \frac{n|I|}{4\pi} \leq \frac{4}{\pi}(2n \log n)^{1/2}$$

for every $I := [\alpha, \beta] \subset [0, 2\pi]$, where $|I| := \beta - \alpha$.

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4. Proof of Theorem 2.1

Proof of Theorem 2.1. Let $k \geq 0$ and $n := 2^k$ be integers, and let $I := [\alpha, \beta] \subset [0, 2\pi]$. Assume that $R_k(t) = |P_k(e^{it})|^2$. The case $R_k(t) = |Q_k(e^{it})|^2$ follows from it by (1.2). The upper bound of the theorem follows from Lemma 3.3. We now prove the lower bound of the theorem, which is more subtle. Without loss of generality we may assume that

$$|I| \geq \frac{4\pi}{n},$$

otherwise there is nothing to prove. For the sake of brevity let

$$A_j := R_{k-2}(t_j) - n/4, \qquad j \in \mathbb{Z}.$$

Let $t_j := 2\pi j/n$ be the same as in Lemma 3.1. We define the integers h and M by

$$t_h < \alpha \leq t_{h+1} < t_{h+M+1} \leq \beta < t_{h+M+2}.$$

Observe that

$$(4.1) \qquad M \geq \frac{n|I|}{2\pi} - 2.$$

We study the M-tuple $\langle A_{h+1}, A_{h+2}, \ldots, A_{h+M} \rangle$. Lemma 3.4 implies that $R_{k-2}(t) - n/4$ has at most

$$(4.2) \qquad \frac{n|I|}{4\pi} + \frac{4}{\pi}(2n \log n)^{1/2}$$

zeros in I. Therefore the Intermediate Value Theorem yields that the number of sign changes in the M-tuple $\langle A_{h+1}, A_{h+2}, \ldots, A_{h+M} \rangle$ is at most as large as the value in (4.2). Hence (4.1) and (4.2) imply that there are integers

$$h + 1 \leq j_1 < j_2 < \cdots < j_N \leq h + M$$

with

$$(4.3) \qquad N \geq \frac{n|I|}{2\pi} - 2 - \frac{n|I|}{4\pi} - \frac{4}{\pi}(2n \log n)^{1/2} = \frac{n|I|}{4\pi} - \frac{4}{\pi}(2n \log n)^{1/2} - 2$$

such that

$$(4.4) \qquad A_{j_\nu} A_{j_\nu+1} \geq 0, \qquad \nu = 1, 2, \ldots, N.$$

Using Lemma 3.1 we have either

$$
\begin{aligned}
(4.5) \qquad 16 A_{j_\nu} A_{j_\nu+1} &= (4(R_{k-2}(t_{j_\nu}) - n/4))(4(R_{k-2}(t_{j_\nu+1}) - n/4)) \\
&= (4|P_{k-2}(e^{it_{j_\nu}})|^2 - n)(4|P_{k-2}(e^{it_{j_\nu+1}})|^2 - n) \\
&= (|P_k(e^{it_{j_\nu}})|^2 - n)(|Q_k(e^{it_{j_\nu+1}})|^2 - n) \\
&= (|P_k(e^{it_{j_\nu}})|^2 - n)(n - |P_k(e^{it_{j_\nu+1}})|^2),
\end{aligned}
$$

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or

$$(4.6) \qquad 16A_{j_\nu}A_{j_\nu+1} = (4(R_{k-2}(t_{j_\nu}) - n/4))(4(R_{k-2}(t_{j_\nu+1}) - n/4))$$
$$= (4|P_{k-2}(e^{it_{j_\nu}})|^2 - n)(4|P_{k-2}(e^{it_{j_\nu+1}})|^2 - n)$$
$$= (|Q_k(e^{it_{j_\nu}})|^2 - n)(|P_k(e^{it_{j_\nu+1}})|^2 - n)$$
$$= (n - |P_k(e^{it_{j_\nu}})|^2)(|P_k(e^{it_{j_\nu+1}})|^2 - n).$$

Combining (4.4), (4.5), and (4.6), we can deduce that

$$(|P_k(e^{it_{j_\nu}})|^2 - n)(|P_k(e^{it_{j_\nu+1}})|^2 - n) = -16A_{j_\nu}A_{j_\nu+1} \leq 0, \qquad \nu = 1, 2, \dots, N.$$

Hence the Intermediate Value Theorem implies that $R_k(t) - n = |P_k(e^{it})|^2 - n$ has at least one zero in each of the intervals

$$[t_{j_\nu}, t_{j_\nu+1}], \qquad \nu = 1, 2, \dots, N.$$

Recalling (4.3) we conclude that $R_k(t) - n = |P_k(e^{it})|^2 - n$ has at least

$$N/2 \geq \frac{n|I|}{8\pi} - \frac{2}{\pi}(2n \log n)^{1/2} - 1$$

distinct zeros in I. $\square$

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