THE REMEZ INEQUALITY FOR LINEAR COMBINATIONS OF SHIFTED GAUSSIANS

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Abstract. Let $\|f\|_{\mathbb{R}} := \sup_{t\in\mathbb{R}} |f(t)|$ and

$$G_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{-(t-\lambda_j)^2}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

We prove that there is an absolute constant $c_1 > 0$ such that

$$\exp(c_1(\min\{n^{1/2}s, ns^2\} + s^2)) \leq \sup_f \|f\|_{\mathbb{R}} \leq \exp(80(\min\{n^{1/2}s, ns^2\} + s^2))\,,$$

for every $s \in (0,\infty)$ and $n \geq 9$, where the supremum is taken for all $f \in G_n$ with

$$m\left(\{t \in \mathbb{R} : |f(t)| \geq 1\}\right) \leq s\,.$$

This is what we call (an essentially sharp) Remez-type inequality for the class G_n. We also prove the right higher dimensional analog of the above result.

1. Introduction

In nonlinear approximation, see [7] for instance, the classes

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\}$$

receive distinguished attention. The main results, Theorems 3.2 and 3,3, of [4] show that

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0\neq f\in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a,b-y\}}\,, \qquad y \in (a,b)\,. \tag{1.1}$$

Here, and in what follows, $\|\cdot\|_{[a,b]}$ denotes the uniform norm on $[a,b]$. (1.1) improves a result obtained earlier in [1]. Bernstein-type inequalities play a central role in approximation theory. They can be turned to inverse theorems of approximation. See, for example, the

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books by Lorentz [15] and by DeVore and Lorentz [8]. Let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n with real coefficients. Inequality (1.1) can be extended to E_n replaced by

$$\widetilde{E}_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t) e^{\lambda_j t}\,, \quad a_0, \lambda_j \in \mathbb{R}\,,\;\; P_{m_j} \in \mathcal{P}_{m_j}\,,\;\; \sum_{j=1}^{N} (m_j + 1) \leq n \right\}.$$

In fact, it is well-known that $\widetilde{E}_n$ is the uniform closure of E_n on any finite subinterval of the real number line. For a function f defined on a measurable set A let

$$\|f\|_A := \|f\|_{L_\infty A} := \|f\|_{L_\infty(A)} := \sup\{|f(x)| : x \in A\}\,,$$

and let

$$\|f\|_{L_p A} := \|f\|_{L_p(A)} := \left(\int_A |f(x)|^p \, dx \right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. In this paper we focus on the classes

$$G_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{-(t-\lambda_j)^2}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\},$$

$$\widetilde{G}_n := \left\{ f : f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,, \quad \lambda_j \in \mathbb{R}\,,\;\; P_{m_j} \in \mathcal{P}_{m_j}\,,\;\; \sum_{j=1}^{N} (m_j + 1) \leq n \right\}.$$

Note that $\widetilde{G}_n$ is the uniform closure of G_n on any finite subinterval of the real number line. Let $\mathbf{m} := (m_1, m_2, \ldots, m_k)$, where each m_j is a nonnegative integer. Let $\mathbf{j} := (j_1, j_2, \ldots, j_k)$, where each j_ν is a nonnegative integer. Let

$$\mathbf{B} := \{(j_1, j_2, \ldots, j_k) : \;\; 1 \leq j_\nu \leq m_\nu, \;\; \nu = 1, 2, \ldots, k\}\,,$$

$$\mathbf{x} := (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k\,,$$

$$\mathbf{d_j} := (d_{j_1}, d_{j_2}, \ldots, d_{j_k}) \in \mathbb{R}^k\,, \qquad \mathbf{j} \in \mathbf{B}\,,$$

$$G_{\mathbf{m}} := \left\{ f : f(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbf{B}} A_{\mathbf{j}} \exp(-\|\mathbf{x} - \mathbf{d_j}\|^2)\,, \quad A_{\mathbf{j}} \in \mathbb{R}, \;\; \mathbf{d_j} \in \mathbb{R}^k \right\},$$

where

$$\|\mathbf{x} - \mathbf{d_j}\|^2 := \sum_{\nu=1}^{k} (x_\nu - d_{j_\nu})^2\,.$$

In [12] the following fundamental result is proved.

Theorem 1.1 (Bernstein-tye inequality for $\widetilde{G}_n$. *There is an absolute constant c_2 such that*

$$\|U_n^{(m)}\|_{L_q(\mathbb{R})} \leq (c_2^{1+1/q} nm)^{m/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0, \infty]$, and $m = 1, 2, \ldots$.

The above theorem is essentially sharp. The classical Remez inequality [17] states that if p is a polynomial of degree at most n, $s \in (0, 2)$, and

$$m(\{x \in [-1, 1] : |p(x)| \leq 1\}) \geq 2 - s\,,$$

then

$$\|p\|_{[-1,1]} \leq T_n\left(\frac{2+s}{2-s}\right)\,,$$

where T_n is the Chebyshev polynomial of degree n defined by $T_n(x) := \cos(n \arccos x)$, $x \in [-1, 1]$. This inequality is sharp and

$$T_n\left(\frac{2+s}{2-s}\right) \leq \exp(\min\{5ns^{1/2}, 2n^2 s\})\,, \qquad s \in (0, 1]\,.$$

Remez-type inequalities turn out to be very useful in various problems of approximation theory. See, for example, Borwein and Erdélyi [2], [3], and [5], Erdélyi [9], [10], and [11], Erdélyi and Nevai [13], Freud [14], and Lorentz, Golitschek, and Makovoz [16]. In [6] we proved the following result.

Theorem 1.2 (Remez-Type Inequality for E_n at 0). *Let $s \in \left(0, \frac{1}{2}\right]$. There are absolute constants $c_3 > 0$ and $c_4 > 0$ such that*

$$\exp(c_3 \min\{ns, (ns)^2\}) \leq \sup_f |f(0)| \leq \exp(c_4 \min\{ns, (ns)^2\})\,,$$

where the supremum is taken for all $f \in E_n$ with

$$m(\{x \in [-1, 1] : |f(x)| \leq 1\}) \geq 2 - s\,.$$

In fact, in [6] Theorem 1.2 above stated in a somewhat weaker form but a more accurate estimates for the values of Chebyshev polynomial appearing in the proof of the above result gives the above more complete result. In this paper we establish an essentially sharp Remez-type inequality for G_n and $\widetilde{G}_n$. We also prove the right higher dimensional analog of our main result.

2. New Results

Theorem 2.1 (Remez-Type Inequality for $\widetilde{G}_n$). *Let $s \in (0, \infty)$ and $n \geq 9$. There is an absolute constant $c_1 > 0$ such that*

$$\exp(c_1(\min\{n^{1/2}s, ns^2\} + s^2)) \leq \sup_f \|f\|_{\mathbb{R}} \leq \exp(80(\min\{n^{1/2}s, ns^2\} + s^2))\,,$$

where the supremum is taken for all $f \in \widetilde{G}_n$ with

$$m(\{t \in \mathbb{R} : |f(t)| \geq 1\}) \leq s\,.$$

Theorem 2.2 (Remez-Type Inequality for $G_{\mathbf{m}}$). *Let $s \in (0, \infty)$ and $n \geq 9$. There is an absolute constant $c_1 > 0$ such that*

$$\exp(c_1 R(m_1, m_2, \dots, m_k, s)) \leq \sup_f \|f\|_{\mathbb{R}^k} \leq \exp(80 R(m_1, m_2, \dots, m_k, s)),$$

where

$$R(m_1, m_2, \dots, m_k, s) := \sum_{j=1}^{k} (\min\{m_j^{1/2} s^{1/k}, m_j s^{2/k}\} + s^{2/k}),$$

and the supremum is taken for all $f \in G_{\mathbf{m}}$ with

$$m\left(\left\{\mathbf{x} \in \mathbb{R}^k : |f(\mathbf{x})| \geq 1\right\}\right) \leq s.$$

We note that in both theorems the assumption $n \geq 9$ is needed only to keep the explicit absolute constant 80 on the right hand side which is already far from being optimal in our results. It is needed to ensure that

$$\lceil s \rceil^2 - s^2 \leq (s+1)^2 - s^2 \leq 3s \leq \min\{n^{1/2} s, n s^2\}, \qquad n \geq 9, \; s \geq 1.$$

3. Lemmas

Throughout this paper $\Lambda := (\lambda_i)_{i=0}^{\infty}$ denotes a sequence of real numbers satisfying

$$0 < \lambda_1 < \lambda_2 < \cdots.$$

The system

$$(1, \cosh(\lambda_1 t) - 1, \cosh(\lambda_2 t) - 1, \dots, \cosh(\lambda_n t) - 1)$$

is called a (finite) cosh system. The linear space

$$H_n(\Lambda) := \operatorname{span}\{1, \cosh(\lambda_1 t) - 1, \cosh(\lambda_2 t) - 1, \dots, \cosh(\lambda_n t) - 1\}$$

over $\mathbb{R}$ is called a (finite) cosh space. That is, the cosh space $H_n(\Lambda)$ is the collection of all possible linear combinations

$$p(t) = a_0 + \sum_{j=0}^{n} a_j (\cosh(\lambda_j t) - 1), \qquad a_j \in \mathbb{R}.$$

The set

$$H(\Lambda) := \bigcup_{n=0}^{\infty} H_n(\Lambda) = \operatorname{span}\{1, \cosh(\lambda_1 t) - 1, \cosh(\lambda_2 t) - 1, \dots\}$$

is called the (infinite) cosh space associated with Λ.

To formulate the next lemmas we introduce the following notation. Let

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n, \qquad 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n,$$

and

$$\gamma_j \leq \lambda_j, \qquad j = 1, 2, \dots, n.$$

The following lemma is stated and proved in [6] as Lemma 5.4.

Lemma 3.1. *Let $A \subset (0,\infty)$ be a compact set containing at least $n+1$ points. We have*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|p(0)|}{\|p\|_A} \leq \max_{0 \neq p \in H_n(\Gamma)} \frac{|p(0)|}{\|p\|_A} \,.$$

In fact, a closer look at the proof of Lemma 5.4 in [6] gives the following result.

Lemma 3.2. *Let $A \subset (0,\infty)$ be a compact set containing at least $2n+1$ points. Let w be a continuous function defined on A with at most n zeros in A. Then*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|p(0)|}{\|pw\|_A} \leq \max_{0 \neq p \in H_n(\Gamma)} \frac{|p(0)|}{\|pw\|_A} \,.$$

To see the proof of Lemma 3.2 one has to review only Section 3 (Chebyshev and Descartes Systems) and Section 4 (Chebyshev polynomials) of [6] to start with. Then the proof of Lemma 3.2 is pretty much along the lines of Section 5 (Comparison Lemmas) of [6]. The only change is that one has to work with the weighted Chebyshev polynomials

$$T_{n,\lambda,w} := T_n\{\lambda_1, \lambda_2, \ldots, \lambda_n; A, w\}$$

and

$$T_{n,\gamma,w} := T_n\{\gamma_1, \gamma_2, \ldots, \gamma_n; A, w\}$$

on A for $H_n(\Lambda)$ and $H_n(\Gamma)$, respectively, which have alternation characterization properties similar to Theorem 4.2 in [6].

To formulate our next lemma we introduce the notation

$$H_n(\varepsilon) := \operatorname{span}\{1, \cosh(\varepsilon t), \cosh(2\varepsilon t), \ldots, \cosh(n\varepsilon t)\} \,,$$

where $\varepsilon > 0$ is fixed. Observe that every $f \in H_n(\varepsilon)$ is of the form

$$f(t) = Q(\cosh(\varepsilon t)) \,, \qquad Q \in \mathcal{P}_n \,.$$

As a special case of Lemma 3.2 we obtain the following result.

Lemma 3.3. *Let $A \subset (0,\infty)$ be a compact set containing at least $2n+1$ points. Let $0 < n\varepsilon \leq \lambda_1$. Let $w \in H_n(\varepsilon)$. Then*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|(pw)(0)|}{\|pw\|_A} \leq \max_{0 \neq p \in H_{2n}(\varepsilon)} \frac{|p(0)|}{\|p\|_A} \,.$$

We also need the result below from [6].

Lemma 3.4. *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $s \in \left(0, \frac{1}{2}\right]$. Assume that $A \subset [0,1]$ is a compact set with Lebesgue measure $m(A) \geq 1 - s$. Then*

$$|f(0)| \leq \exp\left(\min\{10ns, 8(ns)^2\}\right) \|f\|_A$$

for all $f \in H_n(\varepsilon)$.

In fact, the above result is not stated explicitly in [6]. However, it follows from its proof that if $\varepsilon \in \left(0, \frac{1}{2}\right)$, $s \in \left(0, \frac{1}{2}\right]$, and $\|f\|_A \leq 1$, then

$$\begin{aligned}
|f(0)| &\leq T_n\left(\left(2 + \frac{2(\cosh(\varepsilon s) - 1)}{\cosh(\varepsilon) - 1}\right) \Big/ \left(2 - \frac{2(\cosh(\varepsilon s) - 1)}{\cosh(\varepsilon) - 1}\right)\right) \\
&\leq \exp\left(\min\left\{5n\left(\frac{2(\cosh(\varepsilon s) - 1)}{\cosh(\varepsilon) - 1}\right)^{1/2}, 2n^2\frac{2(\cosh(\varepsilon s) - 1)}{\cosh(\varepsilon) - 1}\right\}\right) \\
&\leq \exp\left(\min\left\{5n\left(\frac{4(\varepsilon s)^2}{\cosh(\varepsilon) - 1}\right)^{1/2}, 2n^2\frac{4(\varepsilon s)^2}{\cosh(\varepsilon) - 1}\right\}\right) \\
&\leq \exp\left(\min\left\{5n\left(\frac{4(\varepsilon s)^2}{\varepsilon^2}\right)^{1/2}, 2n^2\frac{4(\varepsilon s)^2}{\varepsilon^2}\right\}\right) \\
&\leq \exp\left(\min\{10ns, 8(ns)^2\}\right) . \qquad \square
\end{aligned}$$

Combining Lemmas 3.3 and 3.4 we are led to the following result.

Lemma 3.5. *Let $s \in \left(0, \frac{1}{2}\right]$. Assume that $A \subset [0,1]$ is a compact set with Lebesgue measure $m(A) \geq 1 - s$. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. Let $0 < n\varepsilon \leq \lambda_1$. Let $w \in H_n(\varepsilon)$. Then, we have*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|(pw)(0)|}{\|pw\|_A} \leq \max_{0 \neq p \in H_{2n}(\varepsilon)} \frac{|p(0)|}{\|p\|_A} \leq \exp\left(\min\{20ns, 32(ns)^2\}\right) .$$

Note that for every polynomial $w \in \mathcal{P}_n$ there is a sequence (w_k) with $w_k \in H_n(1/k)$ such that

$$\lim_{k \to \infty} \|w_k - w\|_{[0,1]} = 0 .$$

This follows easily from

$$t = \lim_{\varepsilon \to 0+} (2\varepsilon)^{-1}(e^{\varepsilon t} - e^{-\varepsilon t}) .$$

Hence the result below is an immediate consequence of Lemma 3.5.

Lemma 3.6. *Let $0 \neq w \in \mathcal{P}_n$. Let $s \in (0, \frac{1}{2}]$. Assume that $A \subset [0,1]$ is a compact set with Lebesgue measure $m(A) \geq 1 - s$. Then*

$$\max_{0 \neq p \in H_n(\Lambda)} \frac{|(pw)(0)|}{\|pw\|_A} \leq \exp\left(\min\{20ns, 32(ns)^2\}\right) .$$

It is worthwhile to transform the above lemma linearly from the interval $[0,1]$ to the interval $[0, \delta]$.

Lemma 3.7. *Let $0 \neq w \in \mathcal{P}_n$, $\delta > 0$, and $s \in (0, \delta/2]$. Assume that $A \subset [0, \delta]$ is a compact set with Lebesgue measure $m(A) \geq \delta - s$. Then*

$$\max_{0 \neq f \in H_n(\Lambda)} \frac{|(fw)(0)|}{\|fw\|_A} \leq \exp\left(\min\{20ns/\delta, 32(ns/\delta)^2\}\right) .$$

Applying Lemma 3.7 with $\delta := n^{1/2}$ and $w(t) := (1 - t^2/n)^n \in \mathcal{P}_{2n}$, and using the inequality

$$0 \leq w(t) = (1 - t^2/n)^n \leq \exp(-t^2) , \qquad t \in [-n^{1/2}, n^{1/2}] ,$$

we obtain the following result.

Lemma 3.8. *Assume that $s \in (0, \frac{1}{2}n^{1/2}]$ and $A \subset [0, n^{1/2}]$ is a compact set with Lebesgue measure $m(A) \geq n^{1/2} - s$. Then*

$$\max_{0 \neq h \in H_n(\Lambda)} \frac{|f(0)|}{\|f(t)\exp(-t^2)\|_A} \leq \exp(\min\{30n^{1/2}s, 64ns^2\}) .$$

Applying Lemma 3.7 with n replaced by $N := 8\lceil s^2 \rceil$, $w(t) := (1 - t^2/(4s^2))^{\lceil 4s^2 \rceil}$, and $\delta := 2s$, and using the inequality

$$0 \leq w(t) = (1 - t^2/(4s^2))^{\lceil 4s^2 \rceil} \leq \exp(-t^2) , \qquad t \in [-2s, 2s] ,$$

we obtain the following result.

Lemma 3.9. *Assume that $s > \frac{1}{2}n^{1/2}$ and $A \subset [0, 2s]$ is a compact set with Lebesgue measure $m(A) \geq s$. Then*

$$\max_{0 \neq f \in H_n(\Lambda)} \frac{|f(0)|}{\|f(t)\exp(-t^2)\|_A} \leq \max_{0 \neq f \in H_N(\Lambda)} \frac{|f(0)|}{\|f(t)\exp(-t^2)\|_A} \leq \exp(80(s+1)^2) .$$

To prove the lower bound of the theorem we need the following well-known result (see pages 246 and 247 of [2], for example).

Lemma 3.10. *We have*

$$\|p(t)\exp(-t^2/2)\|_{\mathbb{R}} \leq \|p(t)\exp(-t^2/2)\|_{[-4n^{1/2}, 4n^{1/2}]}$$

for every $p \in \mathcal{P}_n$.

4. Proof of Theorems 2.1 and 2.2

Using the lemmas in the previous section we can easily prove Theorem 2.1.

Proof of Theorem 2.1. It is sufficient to prove the theorem only in the case when $f \in G_n$, the case when $f \in \widetilde{G}_n$ follows from it by approximation. First we prove the upper bound. Let $s \in (0, \infty)$. Assume that $f \in G_n$ and

$$m(\{t \in \mathbb{R} : |f(t)| \geq 1\}) \leq s\,.$$

Since $f \in G_n$ and $y \in \mathbb{R}$ imply that $f_y \in G_n$, where $f_y(t) := f(y+t)$, it is sufficient to prove that

$$|f(0)| \leq \exp(80(\min\{n^{1/2}s, ns^2\} + s^2))\,.$$

We have

$$g(t) := \frac{1}{2}(f(t) + f(-t)) = h(t)\exp(-t^2) \in G_{2n}\,, \qquad h \in H_n(\Lambda)\,,$$

where, as before,

$$H_n(\Lambda) = \text{span}\{(1\,, \cosh(\lambda_1 t) - 1\,, \cosh(\lambda_2 t) - 1\,, \dots\,, \cosh(\lambda_n t) - 1)\}$$

with some $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. We have $|f(0)| = |g(0)|$ and

$$m(\{t \in [0, \infty) : |g(t)| \geq 1\}) \leq s\,.$$

Let

$$A := \{t \in [0, \infty) : |g(t)| \leq 1\}\,.$$

Lemmas 3.8 and 3.9 yield that

$$\begin{aligned} |f(0)| = & |g(0)| \leq \frac{|g(0)|}{\|g\|_A} = \max_{0 \neq h \in H_n(\Lambda)} \frac{|h(0)|}{\|h(t)\exp(-t^2)\|_A} \\ & \leq \exp(80(\min\{n^{1/2}s, ns^2\} + s^2))\,, \end{aligned}$$

which finishes the proof of the upper bound of the theorem. Now we prove the lower bound of the theorem. Let $T_n(x) = \cos(n \arccos x)\,,\ x \in [-1, 1]\,,$ be the Chebyshev polynomial of degree n. To prove the lower bound of the theorem let $N := 20(n + s^2)$ and let α and β be chosen so that $\alpha s^2 + \beta = 1$ and $\alpha N + \beta = -1$, that is $s^2 < N/20$ and

$$\alpha := \frac{-2}{N - s^2} \qquad \text{and} \qquad \beta := 1 + \frac{2s^2}{N - s^2}\,.$$

Let

$$V_n(t) := T_n(\alpha t^2 + \beta)\exp(-t^2/2)\,, \tag{4.1}$$

and

$$U_n(t) := V_n(t)\exp(-t^2/2) = T_n(\alpha t^2 + \beta)\exp(-t^2)\,. \tag{4.2}$$

Clearly $U_n \in \widetilde{G}_{2n}$,

$$|V_n(t)| \le 1\,, \qquad t \in \left[-\sqrt{N}, \sqrt{N}\right] \setminus [-s,s]\,, \tag{4.3}$$

and

$$\begin{aligned} |V_n(t)| &\le |T_n(\beta)| = T_n\left(1 + \frac{2s^2}{N-s^2}\right) \le T_n\left(1 + 3s^2/N\right) \\ &\le \exp\left(5n\,\frac{s}{\sqrt{N}}\right) \le \exp(N/4)\,, \qquad t \in [-s,s]\,. \end{aligned} \tag{4.4}$$

Using Lemma 3.10, (4.3), and (4.4), we obtain

$$\begin{aligned} |U_n(t)| &= |V_n(t)\exp(-t^2/2)| \le \left(\max_{|x|\le\sqrt{N}} |V_n(x)|\right)\exp(-t^2/2) \\ &\le \exp(N/4)\exp(-N/2) \le \exp(-N/4)\,, \qquad t \in \mathbb{R} \setminus \left[-\sqrt{N}, \sqrt{N}\right]\,. \end{aligned} \tag{4.5}$$

Combining (4.1), (4.2), (4.3), and (4.5), we have

$$|U_n(t)| \le \exp(-s^2/2)\,, \qquad t \in \mathbb{R} \setminus [-s,s]\,. \tag{4.6}$$

Also

$$\begin{aligned} |U_n(0)| &= |V_n(0)| = |T_n(\beta)| = T_n\left(1 + \frac{2s^2}{N-s^2}\right) \ge T_n\left(1 + 2s^2/N\right) \\ &\ge \exp\left(c_5 \min\left\{n\left(2s^2/N\right)^{1/2},\, n^2(2s^2/N)\right\}\right) \\ &\ge \begin{cases} \exp(c_6 \min\{n^{1/2}s, ns^2\})\,, & s \in (0, n^{1/2}]\,, \\ \exp(c_6 n)\,, & s > n^{1/2}\,, \end{cases} \end{aligned} \tag{4.7}$$

with suitable absolute constants $c_5 > 0$ and $c_6 > 0$. This together with (4.6) and $U_n \in \widetilde{G}_{2n}$ gives the lower bound of the theorem. □

Proof of Theorem 2.2. The theorem follows from Theorem 2.1 by a rather straightforward induction on the dimension k. □

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