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(Eds.) N. K. Govil, H. N. Mhaskar, R. N. Mohapatra, Z. Nashed, and J. Szabados

Inequalities for Exponential Sums via Interpolation and Turán-Type Reverse Markov Inequalities

T. Erdélyi

Dedicated to the memory of Professor Ambikeshwar Sharma

Abstract

Interpolation was a topic in which Sharma was viewed as an almost uncontested world expert by his collaborators and many other colleagues. We survey recent results for exponential sums and linear combinations of shifted Gaussians which were obtained via interpolation. To illustrate the method exploiting the Pinkus-Smith Improvement Theorem for spans of Descartes systems, we present the proof of a Chebyshev-type inequality. Finally, in Section 6 we present three simply formulated new results concerning Turán-type reverse Markov inequalities.

1 Introduction and Notation

In his book [2] Braess writes "The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form $\sum_{j=1}^{n} a_j e^{\lambda_j t}$, where the parameters a_j and λ_j are to be determined, while n is fixed." Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j\,, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [21] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,b-\delta]} \leq c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$. Here, and in what follows, $\|\cdot\|_{[a,b]}$ denotes the uniform norm on $[a,b]$. The main result, Theorem 3.2, of [5] shows that Schmidt's inequality holds with $c(n) = 2n-1$. That is,

$$\sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}\,, \qquad y \in (a,b)\,. \tag{1.1}$$

In this Bernstein-type inequality even the pointwise factor is sharp up to a multiplicative absolute constant; the inequality

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}\,, \qquad y \in (a,b)\,,$$

is established by Theorem 3.3 in [5].

Bernstein-type inequalities play a central role in approximation theory via a method developed by Bernstein himself, which turns Bernstein-type inequalities into what are called inverse theorems of approximation; see, for example, the books by Lorentz [16] and by DeVore and Lorentz [8]. From (1.1) one can deduce in a standard fashion that if there is a sequence $(f_n)_{n=1}^{\infty}$ of exponential sums with $f_n \in E_n$ and

$$\|f - f_n\|_{[a,b]} = O(n^{-m}(\log n)^{-2})\,, \qquad n = 2, 3, \ldots\,,$$

where $m \in \mathbb{N}$ is a fixed integer, then f is m times continuously differentiable on (a,b). Let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n with real coefficients. Inequality (1.1) can be extended to E_n replaced by $\widetilde{E}_n$, where $\widetilde{E}_n$ is the collection of all functions f of the form

$$f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t) e^{\lambda_j t}\,,$$

$$a_0\,, \lambda_j \in \mathbb{R}\,, \quad P_{m_j} \in \mathcal{P}_{m_j}\,, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,.$$

In fact, it is well-known that $\widetilde{E}_n$ is the uniform closure of E_n on any finite subinterval of the real number line. For a complex-valued function f defined on a set A let

$$\|f\|_A := \|f\|_{L_\infty A} := \|f\|_{L_\infty(A)} := \sup_{x \in A}\{|f(x)|\}\,,$$

$$\|f\|_{L_p A} := \|f\|_{L_p(A)} := \left(\int_A |f(x)|^p \, dx \right)^{1/p}\,, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. We focus on the class

$$G_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{-(t-\lambda_j)^2}\,, \quad a_j\,, \lambda_j \in \mathbb{R} \right\},$$

the class $\widetilde{G}_n$, the collection of all functions f of the form

$$f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,,$$

$$\lambda_j \in \mathbb{R}\,, \quad P_{m_j} \in \mathcal{P}_{m_j}\,, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,,$$

and the class $\widetilde{G}_n^*$, the collection of all functions f of the form

$$f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,,$$

$$\lambda_j \in [-n^{1/2}, n^{1/2}]\,, \quad P_{m_j} \in \mathcal{P}_{m_j}\,, \quad \sum_{j=1}^{N} (m_j + 1) \le n\,.$$

In other words, G_n is the collection of n term linear combinations (over $\mathbb{R}$) of shifted Gaussians. Note that $\widetilde{G}_n$ is the uniform closure of G_n on any finite subinterval of the real line. Let $W(t) := \exp(-t^2)$. Combining Corollaries 1.5 and 1.8 in [9] and recalling that for the weight W the Mhaskar-Rachmanov-Saff number a_n defined by (1.4) in [9] satisfies $a_n \le c_1 n^{1/2}$ with a constant c_1 independent of n, we obtain that

$$\inf_{P \in \mathcal{P}_n} \|(P-g)W\|_{L_q(\mathbb{R})} \le c_2 n^{-m/2} \|g^{(m)} W\|_{L_q(\mathbb{R})}$$

with a constant c_2 independent of n, whenever the norm on the right-hand side is finite for some $m \in \mathbb{N}$ and $q \in [1,\infty]$. As a consequence

$$\inf_{f \in \widetilde{G}_n^*} \|f - gW\|_{L_q(\mathbb{R})} \le c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} (gW)^{(k)}(t)\|_{L_q(\mathbb{R})}$$

with a constant c_3 independent of n whenever the norms on the right-hand side are finite for each $k = 0, 1, \ldots, m$ with some $q \in [1,\infty]$. Replacing gW by g, we conclude that

$$\inf_{f \in \widetilde{G}_n^*} \|f - g\|_{L_q(\mathbb{R})} \le c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} g^{(k)}(t)\|_{L_q(\mathbb{R})} \tag{1.2}$$

with a constant c_3 independent of n whenever the norms on the right-hand side are finite for each $k = 0, 1, \ldots, m$ with some $q \in [1,\infty]$.

2 A survey of recent results

Theorems 2.1–2.5 were proved in [12].

Theorem 2.1 *There is an absolute constant c_4 such that*

$$|U_n'(0)| \le c_4 n^{1/2} \, \|U_n\|_{\mathbb{R}}$$

for all U_n *of the form* $U_n = P_n Q_n$ *with* $P_n \in \widetilde{G}_n$ *and an even* $Q_n \in \mathcal{P}_n$. *As a consequence*

$$\|P_n'\|_{\mathbb{R}} \le c_4 n^{1/2} \, \|P_n\|_{\mathbb{R}}$$

for all $P_n \in \widetilde{G}_n$.

We remark that a closer look at the proof shows that $c_4 = 5$ in the above theorem is an appropriate choice in the theorem above.

Theorem 2.2 *There is an absolute constant* c_5 *such that*

$$\|U_n'\|_{L_q(\mathbb{R})} \le c_5^{1+1/q} n^{1/2} \, \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$ *and* $q \in (0, \infty)$.

Theorem 2.3 *There is an absolute constant* c_6 *such that*

$$\|U_n^{(m)}\|_{L_q(\mathbb{R})} \le (c_6^{1+1/q} nm)^{m/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0, \infty]$, *and* $m = 1, 2, \ldots$.

We remark that a closer look at the proofs shows that $c_5 = 180\pi$ in Theorem 2.2 and $c_6 = 180\pi$ in Theorem 2.3 are suitable choices.

Our next theorem may be viewed as a slightly weak version of the right inverse theorem of approximation that can be coupled with the direct theorem of approximation formulated in (1.2).

Theorem 2.4 *Suppose* $q \in [1, \infty]$, m *is a positive integer,* $\varepsilon > 0$, *and* f *is a function defined on* $\mathbb{R}$. *Suppose also that*

$$\inf_{f_n \in \widetilde{G}_n} \|f_n - f\|_{L_q(\mathbb{R})} \le c_7 n^{-m/2} (\log n)^{-1-\varepsilon}, \qquad n = 2, 3, \ldots,$$

with a constant c_7 *independent of* n. *Then* f *is* m *times differentiable almost everywhere in* $\mathbb{R}$. *Also, if*

$$\inf_{f_n \in \widetilde{G}_n^*} \|f_n - f\|_{L_q(\mathbb{R})} = c_7 n^{-m/2} (\log n)^{-1-\varepsilon}, \qquad n = 2, 3, \ldots,$$

with a constant c_7 *independent of* n, *then, in addition to the fact that* f *is* m *times differentiable almost everywhere in* $\mathbb{R}$, *we also have*

$$\|(1 + |t|)^{m-j} f^{(j)}(t)\|_{L_q(\mathbb{R})} < \infty, \qquad k = 0, 1, \ldots, m.$$

Theorem 2.5 *There is an absolute constant c_8 such that*

$$\|U_n'\|_{L_q[y-\delta/2,y+\delta/2]} \leq c_8^{1+1/q} \left(\frac{n}{\delta}\right) \|U_n\|_{L_q[y-\delta,y+\delta]}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0,\infty]$, $y \in \mathbb{R}$, and $\delta \in (0, n^{1/2}]$.

In [18] H. Mhaskar writes "Professor Ward at Texas A&M University has pointed out that our results implicitly contain an inequality, known as Bernstein inequality, in terms of the number of neurons, under some conditions on the minimal separation. Professor Erdélyi at Texas A&M University has kindly sent us a manuscript in preparation, where he proves this inequality purely in terms of the number of neurons, with no further conditions. This inequality leads to the converse theorems in terms of the number of neurons, matching our direct theorem in this theory. Our direct theorem in [17] is sharp in the sense of n-widths. However, the converse theorem applies to individual functions rather than a class of functions. In particular, it appears that even if the cost of approximation is measured in terms of the number of neurons, if the degrees of approximation of a particular function by Gaussian networks decay polynomially, then a linear operator will yield the same order of magnitude in the error in approximating this function. We find this astonishing, since many people have told us based on numerical experiments that one can achieve a better degree of approximation by non-linear procedures by stacking the centers near the bad points of the target functions".

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. The collection of all linear combinations of of $e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \operatorname{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. Newman's inequality (see [3] and [19]) is an essentially sharp Markov-type inequality for $E(\Lambda_n)$ on $[0,1]$ in the case when each λ_j is non-negative.

Theorem 2.6 (Newman's Inequality) *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of nonnegative real numbers. Then

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \le \sup_{0\neq P\in E(\Lambda_n)} \frac{\|P'\|_{(-\infty,0]}}{\|P\|_{(-\infty,0]}} \le 9\sum_{j=0}^{n}\lambda_j\,.$$

An L_p version of this may be found in [3], [6], and [10].

Theorem 2.7 *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. Let $1 \le p \le \infty$. Then*

$$\|Q'\|_{L_p(-\infty,0]} \le 9\left(\sum_{j=0}^{n}\lambda_j\right)\|Q\|_{L_p(-\infty,0]}$$

for every $Q \in E(\Lambda_n)$.

The following $L_p[a,b]$ $(1 \le p \le \infty)$ analogue of Theorem 2.7 has been established in [1].

Theorem 2.8 *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers, $a, b \in \mathbb{R}$, $a < b$, and $1 \le p \le \infty$. There is a positive constant $c_9 = c_9(a,b)$ depending only on a and b such that*

$$\sup_{0\neq P\in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_p[a,b]}} \le c_9\left(n^2 + \sum_{j=0}^{n}|\lambda_j|\right)\,.$$

Theorem 2.8 was proved earlier in [4] and [10] under the additional assumptions that $\lambda_j \ge \delta j$ for each j with a constant $\delta > 0$ and with $c_9 = c_9(a,b)$ replaced by $c_9 = c_9(a,b,\delta)$ depending only on a, b, and δ. The novelty of Theorem 2.8 was the fact that $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ is an arbitrary set of real numbers; not even the non-negativity of the exponents λ_j is needed.

In [11] the following Nikolskii-Markov type inequality has been proved for $E(\Lambda_n)$ on $(-\infty,0]$.

Theorem 2.9 *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of non-negative real numbers. Suppose $0 < q \le p \le \infty$. Let μ be a non-negative integer. There are constants $c_{10} = c_{10}(p,q,\mu) > 0$*

and $c_{11} = c_{11}(p,q,\mu)$ *depending only on* p, q, *and* μ *such that for* $A := (-\infty, 0]$ *we have*

$$c_{10}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}} \leq \sup_{P\in E(\Lambda_n)} \frac{\|P^{(\mu)}\|_{L_pA}}{\|P\|_{L_qA}} \leq c_{11}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}-\frac{1}{p}},$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ *and for all* $\mu \geq 0$, *while the upper bound holds when* $\mu = 0$ *and* $0 < q \leq p \leq \infty$, *and when* $\mu \geq 1$, $p \geq 1$, *and* $0 < q \leq p \leq \infty$. *Also, there are constants* $c_{10} = c_{10}(q,\mu) > 0$ *and* $c_{11} = c_{11}(q,\mu)$ *depending only on* q *and* μ *such that*

$$c_{10}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}} \leq \sup_{P\in E(\Lambda_n)} \frac{|P^{(\mu)}(y)|}{\|P\|_{L_q(-\infty,y]}} \leq c_{11}\left(\sum_{j=0}^{n}\lambda_j\right)^{\mu+\frac{1}{q}}$$

for every $y \in \mathbb{R}$.

Motivated by a question of Michel Weber (Strasbourg) we proved the following two theorems in [13].

Theorem 2.10 *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of real numbers. Let $a, b \in \mathbb{R}$, $a < b$, $0 < q \leq p \leq \infty$, *and*

$$M(\Lambda_n, p, q) := \left(n^2 + \sum_{j=0}^{n}|\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}}.$$

There are constants $c_{12} = c_{12}(p,q,a,b) > 0$ *and* $c_{13} = c_{13}(p,q,a,b)$ *depending only on* p, q, a, *and* b *such that*

$$c_{12}M(\Lambda_n,p,q) \leq \sup_{P\in E(\Lambda_n)} \frac{\|P\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \leq c_{13}M(\Lambda_n,p,q).$$

Theorem 2.11 *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of real numbers. Let $a, b \in \mathbb{R}$, $a < b$, $0 < q \le p \le \infty$, *and*

$$M(\Lambda_n, p, q) := \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q} - \frac{1}{p}} .$$

There are constants $c_{14} = c_{14}(p, q, a, b) > 0$ *and* $c_{15} = c_{15}(p, q, a, b)$ *depending only on* p*,* q*,* a*, and* b *such that*

$$c_{14} M(\Lambda_n, p, q) \le \sup_{P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \le c_{15} M(\Lambda_n, p, q) \,,$$

where the lower bound holds for all $0 < q \le p \le \infty$*, while the upper bound holds when* $p \ge 1$ *and* $0 < q \le p \le \infty$.

The lower bounds in these inequalities were shown by a method in which the Pinkus-Smith Improvement Theorem plays a central role. We formulate the useful lemmas applied in the proofs of these lower bounds. To emphasize the power of the technique of interpolation, we present the short proofs of these lemmas. Then these lemmas are used to establish the Chebyshev-type inequality below for exponential sums.

Theorem 2.12 *We have*

$$|f(y)| \le \exp(\gamma(|y| + \delta)) \left(\frac{2|y|}{\delta} \right)^n \|f\|_{[-\delta,\delta]} \,, \qquad y \in \mathbb{R} \setminus [-\delta, \delta] \,,$$

for all $f \in \widetilde{E}_n$ *of the form*

$$f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t) e^{\lambda_j t} \,,$$

$$a_0 \in \mathbb{R} \,, \quad \lambda_j \in [-\gamma, \gamma] \,, \quad P_{m_j} \in \mathcal{P}_{m_j} \,, \quad \sum_{j=1}^{N} (m_j + 1) \le n \,,$$

and for all $\gamma > 0$.

3 Lemmas

Our first lemma, which can be proved by a simple compactness argument, may be viewed as a simple exercise.

Lemma 3.1 *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a,b,c \in \mathbb{R}$, $a < b$. Let $w \neq 0$ be a continuous function defined on $[a,b]$. Let $q \in (0,\infty]$. Then there exists a $0 \neq T \in E(\Delta_n)$ such that*

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}\,,$$

and there exists a $0 \neq S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}\,.$$

Our next result is an essential tool in proving our key lemmas, Lemmas 3.3 and 3.4.

Lemma 3.2 *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a,b,c \in \mathbb{R}$, $a < b < c$. Let $q \in (0,\infty]$. Let T and S be the same as in Lemma 3.1. Then T has exactly n zeros in $[a,b]$ by counting multiplicities. If $\delta_n \geq 0$, then S also has exactly n zeros in $[a,b]$ by counting multiplicities.*

The heart of the proof of our theorems is the following pair of comparison lemmas. The proofs of these are based on basic properties of Descartes systems, in particular on Descartes' Rule of Signs, and on a technique used earlier by P.W. Smith and Pinkus. Lorentz ascribes this result to Pinkus, although it was Smith [22] who published it. I learned about the method of proofs of these lemmas from Peter Borwein, who also ascribes it to Pinkus. This is the proof we present here. Section 3.2 of [3], for instance, gives an introduction to Descartes systems. Descartes' Rule of Signs is stated and proved on page 102 of [3].

Lemma 3.3 *Let*

$$\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$$

be sets of real numbers satisfying $\delta_j \le \gamma_j$ *for each* $j = 0, 1, \ldots, n$. *Let* $a, b, c \in \mathbb{R}$, $a < b \le c$. *Let* $0 \ne w$ *be a continuous function defined on* $[a, b]$. *Let* $q \in (0, \infty]$. *Then*

$$\sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} \le \sup_{P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} .$$

Under the additional assumption $\delta_n \ge 0$, *we also have*

$$\sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} \le \sup_{P \in E(\Gamma_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} .$$

Lemma 3.4 *Let*

$$\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$$

be sets of real numbers satisfying $\delta_j \le \gamma_j$ *for each* $j = 0, 1, \ldots, n$. *Let* $a, b, c \in \mathbb{R}$, $c \le a < b$. *Let* $0 \ne w$ *be a continuous function defined on* $[a, b]$. *Let* $q \in (0, \infty]$. *Then*

$$\sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} \ge \sup_{P \in E(\Gamma_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} .$$

Under the additional assumption $\gamma_0 \le 0$, *we also have*

$$\sup_{P \in E(\Delta_n)} \frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} \ge \sup_{P \in E(\Gamma_n)} \frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} .$$

4 Proofs of the Lemmas

Proof of Lemma 3.1 Since Δ_n is fixed, the proof is a standard compactness argument. We omit the details. □

To prove Lemma 3.2 we need the following two facts: (a) Every $f \in E(\Delta_n)$ has at most n real zeros by counting multiplicities. (b) If $t_1 < t_2 < \cdots < t_m$ are real numbers and $k_1, k_2, \ldots, k_m$ are positive integers such that $\sum_{j=1}^{m} k_j = n$, then there is a $f \in E(\Delta_n)$, $f \ne 0$ having a zero at t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$.

Proof of Lemma 3.2 We prove the statement for T first. Suppose to the contrary that

$$t_1 < t_2 < \cdots < t_m$$

are real numbers in $[a,b]$ such that t_j is a zero of T with multiplicity k_j for each $j=1,2,\dots,m$, $k:=\sum_{j=1}^m k_j<n$, and T has no other zeros in $[a,b]$ different from $t_1,t_2,\dots,t_m$. Let $t_{m+1}:=c$ and $k_{m+1}:=n-k\geq 1$. Choose an $0\neq R\in E(\Delta_n)$ such that R has a zero at t_j with multiplicity k_j for each $j=1,2,\dots,m+1$, and normalize so that $T(t)$ and $R(t)$ have the same sign at every $t\in[a,b]$. Let $T_\varepsilon:=T-\varepsilon R$. Note that T and R are of the form

$$T(t)=\widetilde{T}(t)\prod_{j=1}^m (t-t_j)^{k_j} \qquad \text{and} \qquad R(t)=\widetilde{R}(t)\prod_{j=1}^m (t-t_j)^{k_j}\,,$$

where both $\widetilde{T}$ and $\widetilde{R}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Hence, if $\varepsilon>0$ is sufficiently small, then $|T_\varepsilon(t)|<|T(t)|$ at every $t\in[a,b]\setminus\{t_1,t_2,\dots,t_m\}$, so

$$\|T_\varepsilon w\|_{L_q[a,b]}<\|Tw\|_{L_q[a,b]}\,.$$

This, together with $T_\varepsilon(c)=T(c)$, contradicts the maximality of T.

Now we prove the statement for S. Without loss of generality we may assume that $S'(c)>0$. Suppose to the contrary that

$$t_1<t_2<\cdots<t_m$$

are real numbers in $[a,b]$ such that t_j is a zero of S with multiplicity k_j for each $j=1,2,\dots,m$, $k:=\sum_{j=1}^m k_j<n$, and S has no other zeros in $[a,b]$ different from $t_1,t_2,\dots,t_m$. Choose a

$$0\neq Q\in\operatorname{span}\{e^{\delta_{n-k}t},e^{\delta_{n-k+1}t},\dots,e^{\delta_n t}\}\subset E(\Delta_n)\,,$$

such that Q has a zero at t_j with multiplicity k_j for each $j=1,2,\dots,m$, and normalize so that $S(t)$ and $Q(t)$ have the same sign at every $t\in[a,b]$. Note that S and Q are of the form

$$S(t)=\widetilde{S}(t)\prod_{j=1}^m (t-t_j)^{k_j} \qquad \text{and} \qquad Q(t)=\widetilde{Q}(t)\prod_{j=1}^m (t-t_j)^{k_j}\,,$$

where both $\widetilde{S}$ and $\widetilde{Q}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Let $t_{m+1}:=c$ and $k_{m+1}:=1$. Choose an

$$0\neq R\in\operatorname{span}\{e^{\delta_{n-k-1}t},e^{\delta_{n-k}t},\dots,e^{\delta_n t}\}\subset E(\Delta_n)$$

such that R has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m+1$, and normalize so that $S(t)$ and $R(t)$ have the same sign at every $t \in [a,b]$. Note that S and R are of the form

$$S(t) = \widetilde{S}(t) \prod_{j=1}^{m} (t - t_j)^{k_j} \qquad \text{and} \qquad R(t) = \widetilde{R}(t) \prod_{j=1}^{m} (t - t_j)^{k_j} \,,$$

where both $\widetilde{S}$ and $\widetilde{R}$ are continuous functions on $[a,b]$ having no zeros on $[a,b]$. Since $\delta_n \geq 0$, it is easy to see that $Q'(c)R'(c) < 0$, so the sign of $Q'(c)$ is different from the sign of $R'(c)$. Let $U := Q$ if $Q'(c) < 0$, and let $U := R$ if $R'(c) < 0$. Let $S_\varepsilon := S - \varepsilon U$. Hence, if $\varepsilon > 0$ is sufficiently small, then $|S_\varepsilon(t)| < |T(t)|$ at every $t \in [a,b] \setminus \{t_1, t_2, \dots, t_m\}$, so

$$\|S_\varepsilon w\|_{L_q[a,b]} < \|Sw\|_{L_q[a,b]} \,.$$

This, together with the inequalities $S'_\varepsilon(c) > S'(c) > 0$, contradicts the maximality of S. $\square$

Proof of Lemma 3.3 We begin with the first inequality. We may assume that $a < b < c$. The general case when $a < b \leq c$ follows by a standard continuity argument. Let $k \in \{0, 1, \dots, n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n \,, \qquad \gamma_j = \delta_j \,, \quad j \neq k \,, \quad \text{and} \quad \delta_k < \gamma_k < \delta_{k+1}$$

(let $\delta_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 3.1 and 3.2, there is a $0 \neq T \in E(\Delta_n)$ such that

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} \,,$$

where T has exactly n zeros in $[a,b]$ by counting multiplicities. Denote the distinct zeros of T in $[a,b]$ by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of T with multiplicity k_j for each $j = 1, 2, \dots, m$, and $\sum_{j=1}^m k_j = n$. Then T has no other zeros in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$. Let

$$T(t) =: \sum_{j=0}^{n} a_j e^{\delta_j t} \,, \qquad a_j \in \mathbb{R} \,.$$

Without loss of generality we may assume that $T(c) > 0$. We have $T(t) > 0$ for every $t > c$; otherwise, in addition to its n zeros in $[a,b]$ (by counting multiplicities), T would have at least one more zero in (c,∞), which is impossible. Hence

$$a_n := \lim_{t\to\infty} T(t)e^{-\delta_n t} \geq 0\,.$$

Since $E(\Delta_n)$ is the span of a Descartes system on $(-\infty,\infty)$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j}a_j > 0\,, \qquad j = 0,1,\dots,n\,.$$

Choose $R \in E(\Gamma_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j e^{\gamma_j t}\,, \qquad b_j \in \mathbb{R}\,,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1,2,\dots,m$, and normalize so that $R(c) = T(c)(> 0)$ (this $R \in E(\Gamma_n)$ is uniquely determined). Similarly to $a_n \geq 0$ we have $b_n \geq 0$. Since $E(\Gamma_n)$ is the span of a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs yields

$$(-1)^{n-j}b_j > 0\,, \qquad j = 0,1,\dots,n\,.$$

We have

$$(T-R)(t) = a_k e^{\delta_k t} - b_k e^{\gamma_k t} + \sum_{\substack{j=0\\ j\neq k}}^{n} (a_j - b_j)e^{\delta_j t}\,.$$

Since $T-R$ has altogether at least $n+1$ zeros at $t_1, t_2, \dots, t_m$, and c (by counting multiplicities), it does not have any zero in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$, and c. Since

$$(e^{\delta_0 t}, e^{\delta_1 t}, \dots, e^{\delta_k t}, e^{\gamma_k t}, e^{\delta_{k+1} t}, \dots, e^{\delta_n t})$$

is a Descartes system on $(-\infty,\infty)$, Descartes' Rule of Signs implies that the sequence

$$(a_0 - b_0, a_1 - b_1, \dots, a_{k-1} - b_{k-1}, a_k, -b_k, a_{k+1} - b_{k+1}, \dots, a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k}a_k > 0$, this implies that $a_n - b_n < 0$ if $k < n$, and $-b_n < 0$ if $k = n$, so

$$(T - R)(t) < 0\,, \qquad t > c\,.$$

Since each of T, R, and $T-R$ has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m$; $\sum_{j=1}^m k_j = n$, and $T - R$ has a sign change (a zero with multiplicity 1) at c, we can deduce that each of T, R, and $T - R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0, 1, \dots, m$ with $t_0 := -\infty$ and $t_{m+1} := c$. Hence $|R(t)| \leq |T(t)|$ holds for all $t \in [a, b] \subset [a, c]$ with strict inequality at every t different from $t_1, t_2, \dots, t_m$. Combining this with $R(c) = T(c)$, we obtain

$$\frac{|R(c)|}{\|Rw\|_{L_q[a,b]}} \geq \frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P(c)|}{\|Pw\|_{L_q[a,b]}}\,.$$

Since $R \in E(\Gamma_n)$, the first conclusion of the lemma follows from this.

Now we start the proof of the second inequality of the lemma. Although it is quite similar to that of the first inequality, we present the details. We may assume that $a < b < c$ and $\delta_n > 0$. The general case when $a < b \leq c$ and $\delta_n \geq 0$ follows by a standard continuity argument. Let $k \in \{0, 1, \dots, n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n\,, \quad \gamma_j = \delta_j\,, \quad j \neq k\,, \quad \text{and} \quad \delta_k < \gamma_k < \delta_{k+1}$$

(let $\delta_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 3.1 and 3.2, there is an $0 \neq S \in E(\Delta_n)$ such that

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}\,,$$

where S has exactly n zeros in $[a, b]$ by counting multiplicities. Denote the distinct zeros of S in $[a, b]$ by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of S with multiplicity k_j for each $j = 1, 2, \dots, m$, and $\sum_{j=1}^m k_j = n$. Then S has no other zeros in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$. Let

$$S(t) =: \sum_{j=0}^n a_j e^{\delta_j t}\,, \qquad a_j \in \mathbb{R}\,.$$

Without loss of generality we may assume that $S(c) > 0$. Since $\delta_n > 0$, we have $\lim_{t\to\infty} S(t) = \infty$; otherwise, in addition to its n zeros in (a,b), S would have at least one more zero in (c,∞), which is impossible.

Because of the extremal property of S, we have $S'(c) \neq 0$. We show that $S'(c) > 0$. To see this observe that Rolle's Theorem implies that $S' \in E(\Delta_n)$ has at least $n-1$ zeros in $[t_1, t_m]$. If $S'(c) < 0$, then $S(t_m) = 0$ and $\lim_{t\to\infty} S(t) = \infty$ imply that S' has at least 2 more zeros in (t_m, ∞) (by counting multiplicities). Thus $S'(c) < 0$ would imply that S' has at least $n+1$ zeros in $[a,\infty)$, which is impossible. Hence $S'(c) > 0$, indeed. Also $a_n := \lim_{t\to\infty} S(t)e^{-\delta_n t} \geq 0$. Since $E(\Delta_n)$ is the span of a Descartes system on $(-\infty.\infty)$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j} a_j > 0\,, \qquad j = 0, 1, \dots, n\,.$$

Choose $R \in E(\Gamma_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j e^{\gamma_j t}\,, \qquad b_j \in \mathbb{R}\,,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1, 2, \dots, m$, and normalize so that $R(c) = S(c) (> 0)$ (this $R \in E(\Gamma_n)$ is uniquely determined). Similarly to $a_n \geq 0$ we have $b_n \geq 0$. Since $E(\Gamma_n)$ is the span of a Descartes system on $(-\infty, \infty)$, Descartes' Rule of Signs implies that

$$(-1)^{n-j} b_j > 0\,, \qquad j = 0, 1, \dots, n\,.$$

We have

$$(S-R)(t) = a_k e^{\delta_k t} - b_k e^{\gamma_k t} + \sum_{\substack{j=0\\ j\neq k}}^{n} (a_j - b_j) e^{\delta_j t}\,.$$

Since $S - R$ has altogether at least $n+1$ zeros at $t_1, t_2, \dots, t_m$, and c (by counting multiplicities), it does not have any zero in $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$, and c. Since

$$(e^{\delta_0 t}, e^{\delta_1 t}, \dots, e^{\delta_k t}, e^{\gamma_k t}, e^{\delta_{k+1} t}, \dots, e^{\delta_n t})$$

is a Descartes system on $(-\infty, \infty)$, Descartes' Rule of Signs implies that the sequence

$$(a_0 - b_0, a_1 - b_1, \dots, a_{k-1} - b_{k-1}, a_k, -b_k, a_{k+1} - b_{k+1}, \dots, a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k} a_k > 0$, this implies that $a_n - b_n < 0$ if $k < n$ and $-b_n < 0$ if $k = n$, so

$$(S - R)(t) < 0\,, \qquad t > c\,.$$

Since each of S, R, and $S - R$ has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m$; $\sum_{j=1}^m k_j = n$, and $S - R$ has a sign change (a zero with multiplicity 1) at c, we can deduce that each of S, R, and $S - R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0, 1, \dots, m$ with $t_0 := -\infty$ and $t_{m+1} := c$. Hence $|R(t)| \leq |S(t)|$ holds for all $t \in [a, b] \subset [a, c]$ with strict inequality at every t different from $t_1, t_2, \dots, t_m$. Combining this with $0 < S'(c) < R'(c)$ (recall that $R(c) = S(c) > 0$), we obtain

$$\frac{|R'(c)|}{\|Rw\|_{L_q[a,b]}} \geq \frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup_{P \in E(\Delta_n)} \frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}}\,.$$

Since $R \in E(\Gamma_n)$, the second conclusion of the lemma follows from this. $\square$

Proof of Lemma 3.4 The lemma follows from Lemma 3.3 via the substitution $u = -t$. $\square$

5 Proof of the Theorem 2.12

Proof of Theorem 2.12 By a well-known and simple limiting argument we may assume that

$$f(t) = \sum_{j=0}^{n} a_j e^{\lambda_j t}, \qquad -\gamma \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \leq \gamma\,.$$

By reasons of symmetry it is sufficient to examine only the case $y > \delta$. By Lemmas 3.1 – 3.4 we may assume that

$$\lambda_j = \gamma - (n - j)\varepsilon\,, \qquad j = 0, 1, \dots, n\,,$$

for sufficiently small values of $\varepsilon > 0$, that is,

$$f(t) = e^{\gamma t} P_n(e^{-\varepsilon t})\,, \qquad P_n \in \mathcal{P}_n\,.$$

Now Chebyshev's inequality [8, Proposition 2.3, p. 101] implies that

$$\begin{aligned}
|f(y)| &= e^{\gamma y}|P_n(e^{-\varepsilon y})| \le e^{\gamma y}\left(\frac{4e^{-\varepsilon y}}{e^{\varepsilon\delta} - e^{-\varepsilon\delta}}\right)^n \|P_n(e^{-\varepsilon t})\|_{[-\delta,\delta]} \\
&\le e^{\gamma y}\left(\frac{4e^{-\varepsilon y}}{e^{\varepsilon\delta} - e^{-\varepsilon\delta}}\right)^n e^{\delta y}\|f\|_{[-\delta,\delta]} \\
&\le e^{\gamma(y+\delta)}\left(\frac{4e^{-\varepsilon y}}{e^{\varepsilon\delta} - e^{-\varepsilon\delta}}\right)^n \|f\|_{[-\delta,\delta]}\,,
\end{aligned}$$

and by taking the limit when $\varepsilon > 0$ tends to 0, the theorem follows. □

6 Turán-type reverse Markov inequalities on diamonds

Let $\varepsilon \in [0,1]$ and let D_ε be the ellipse in the complex plane with axes $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(D_\varepsilon)$ denote the collection of all polynomials of degree n with complex coefficients and with all their zeros in D_ε. Let

$$\|f\|_A := \sup_{z\in A} |f(z)|$$

for complex-valued functions defined on A. Extending a result of Turán [23], Erőd [14, III. tétel] claimed that there are absolute constants $c_1 > 0$ and c_2 such that

$$c_1(n\varepsilon + \sqrt{n}) \le \inf_{p\in\mathcal{P}_n^c(D_\varepsilon)} \frac{\|p'\|_{D_\varepsilon}}{\|p\|_{D_\varepsilon}} \le c_2(n\varepsilon + \sqrt{n})\,.$$

However, Erőd [14] presented a proof with only $c_1 n\varepsilon$ in the lower bound. It was Levenberg and Poletcky [15] who first published a correct proof of a result implying the lower bound claimed by Erőd.

Let $\varepsilon \in [0,1]$ and let S_ε be the diamond in the complex plane with diagonals $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(S_\varepsilon)$ denote the collection of all polynomials of degree n with complex coefficients and with all their zeros in S_ε.

Theorem 6.1 *There are absolute constants $c_1 > 0$ and c_2 such that*

$$c_1(n\varepsilon + \sqrt{n}) \le \inf_p \frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \le c_2(n\varepsilon + \sqrt{n})\,,$$

where the infimum is taken over all $p \in \mathcal{P}_n^c(S_\varepsilon)$ with the property

$$|p(z)| = |p(-z)|\,, \qquad z \in \mathbb{C}\,, \tag{6.1}$$

or where the infimum is taken over all real $p \in \mathcal{P}_n^c(S_\varepsilon)$.

It is an interesting question whether or not the lower bound in Theorem 6.1 holds when the infimum is taken for all $p \in \mathcal{P}_n^c(\varepsilon)$. As our next result shows this is the case at least when $\varepsilon = 1$.

Theorem 6.2 *There are absolute constants $c_1 > 0$ and c_2 such that*

$$c_1 n \le \inf_{p \in \mathcal{P}_n^c(S_1)} \frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \le c_2 n\,.$$

The following lemma is the main tool we need for the proofs of the theorems above.

Lemma 6.3 *Let $\Gamma(a,r)$ be the circle in the complex plane centered at a with radius r. Let $z_0 \in \Gamma(a,r)$. Suppose $p \in \mathcal{P}_n^c$ has at least m zeros in the disk $D(a,r)$ bounded by $\Gamma(a,r)$ and it has all its zeros in the half-plane $H(a,r,z_0)$ containing a and bounded by the line tangent to $\Gamma(a,r)$ at z_0. Then*

$$\left|\frac{p'(z_0)}{p(z_0)}\right| \ge \frac{m}{2r}\,.$$

Proof. Let $p \in \mathcal{P}_n^c$ be of the form

$$p(z) = c\prod_{k=1}^{n}(z - z_k)\,, \qquad c\,,\ z_k \in \mathbb{C}\,.$$

Then

$$r\left|\frac{p'(z_0)}{p(z_0)}\right| \quad = \quad \left|\frac{p'(z_0)(z_0 - a)}{p(z_0)}\right| = \left|\sum_{k=1}^{n} \frac{z_0 - a}{z_0 - z_k}\right|$$

$$\begin{aligned}
&= \left|\sum_{k=1}^{n}\left(1-\frac{z_k-a}{z_0-a}\right)^{-1}\right| \\
&\geq \left|\mathrm{Re}\left(\sum_{k=1}^{n}\left(1-\frac{z_k-a}{z_0-a}\right)^{-1}\right)\right| \\
&\geq \sum_{k=1}^{n}\mathrm{Re}\left(\left(1-\frac{z_k-a}{z_0-a}\right)^{-1}\right) \\
&\geq \frac{m}{2}\,,
\end{aligned}$$

since

$$\mathrm{Re}\left(\left(1-\frac{z_k-a}{z_0-a}\right)^{-1}\right) \geq \frac{1}{2}\,, \qquad z_k \in D(a,r)\,,$$

and

$$\mathrm{Re}\left(\left(1-\frac{z_k-a}{z_0-a}\right)^{-1}\right) = \mathrm{Re}\left(\frac{z_0-z_k}{z_0-a}\right) \geq 0\,, \qquad z_k \in H(a,r,z_0)\,.$$

$\square$

Proof of Theorem 6.1 The upper bound can be obtained by considering

$$p_n(z) := (z^2-1)^{\lfloor n/2 \rfloor}(z-1)^{n-2\lfloor n/2 \rfloor}\,.$$

We omit the simple calculation. To prove the lower bound we consider three cases.
Case 1: Property (6.1) holds and $\varepsilon \in [n^{-1/2}, 1]$. Choose a point z_0 on the boundary of S_ε such that

$$|p(z_0)| = \|p\|_{S_\varepsilon}\,. \tag{6.2}$$

Property (6.1) implies that

$$|p(-z_0)| = \|p\|_{S_\varepsilon}\,. \tag{6.3}$$

Without loss of generality we may assume that $z_0 \in [i\varepsilon, 1]$. A simple calculation shows that there are disks $D_1 := D_1(\varepsilon, c, z_0)$ and $D_2 := D_2(\varepsilon, c, -z_0)$ in the complex plane such that D_1 has radius $r = c\varepsilon^{-1}$

and is tangent to $[i\varepsilon, 1]$ at z_0, D_2 has radius $r = c\varepsilon^{-1}$ and is tangent to $[-1, -i\varepsilon]$ at $-z_0$, and $S_\varepsilon \subset D_1 \cup D_2$ for every sufficiently large absolute constant $c > 0$. Since $p \in \mathcal{P}_n^c$ has each of its zeros in S_ε, either p has at least $n/2$ zeros in D_1 or p has at least $n/2$ zeros in D_2. In the first case Lemma 6.3 and (6.2) imply

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(z_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{n}{4r} = \frac{1}{4c}\, n\varepsilon\,.$$

In the other case Lemma 6.3 and (6.3) imply

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(-z_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(-z_0)}{p(-z_0)}\right| \geq \frac{n}{4r} = \frac{1}{4c}\, n\varepsilon\,.$$

Case 2: $p \in \mathcal{P}_n^c(\varepsilon)$ is real and $\varepsilon \in [n^{-1/2}, 1]$. Choose a point z_0 on the boundary of S_ε such that

$$|p(z_0)| = \|p\|_{S_\varepsilon}\,. \tag{6.4}$$

Without loss of generality we may assume that $z_0 \in [i\varepsilon, 1]$. Since $p \in \mathcal{P}_n^c(\varepsilon)$ is real, we have

$$|p(\overline{z}_0)| = \|p\|_{S_\varepsilon}\,. \tag{6.5}$$

Let $D_1 := D_1(\varepsilon, c, z_0)$ and $D_2 := D_2(\varepsilon, c, \overline{z}_0)$ be disks of the complex plane such that D_1 has radius $r = c\varepsilon^{-1}$ and is tangent to $[i\varepsilon, 1]$ at z_0 from below, D_2 has radius $r = c\varepsilon^{-1}$ and is tangent to $[-1, -i\varepsilon]$ at $\overline{z}_0$ from above. Denote the boundary of D_1 by Γ_1 and the boundary of D_2 by Γ_2. A simple calculation shows that if the absolute constant $c > 0$ is sufficiently large, then Γ_1 intersects the boundary of S_ε only at $a_1 \in [-1, i\varepsilon]$ and $b_1 \in [-i\varepsilon, 1]$, while Γ_2 intersects the boundary of S_ε only at $a_2 \in [-1, -i\varepsilon]$ and $b_2 \in [i\varepsilon, 1]$. Also, if the absolute constant $c > 0$ is sufficiently large, then

$$|a_1 - i\varepsilon| \leq \frac{1}{64}, \quad |a_2 + i\varepsilon| \leq \frac{1}{64}, \quad |b_1 - 1| \leq \frac{1}{64}, \quad |b_2 - 1| \leq \frac{1}{64}\,. \tag{6.6}$$

In the sequel let the absolute constant $c > 0$ be so large that inequalities (6.6) hold. If $p \in \mathcal{P}_n^c(\varepsilon)$ has at least αn zeros in D_1, then by using Lemma 6.3 and (6.4), we deduce

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(z_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{\alpha n}{2r} = \frac{\alpha}{2c}\, n\varepsilon\,.$$

If $p \in \mathcal{P}_n^c(\varepsilon)$ has at least αn zeros in D_2, then by using Lemma 6.3 and (6.5), we deduce

$$\frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \geq \frac{|p'(\overline{z}_0)|}{\|p\|_{S_\varepsilon}} = \left|\frac{p'(\overline{z}_0)}{p(\overline{z}_0)}\right| \geq \frac{\alpha n}{2r} = \frac{\alpha}{2c}\, n\varepsilon\,.$$

Hence we may assume that $p \in \mathcal{P}_n^c(\varepsilon)$ has at least $(1-\alpha)n$ zeros in $S_\varepsilon \setminus D_1$ and it has at least $(1-\alpha)n$ zeros in $S_\varepsilon \setminus D_2$. Combining this with (6.6), we obtain that $p \in \mathcal{P}_n^c(\varepsilon)$ has at least $(1-2\alpha)n$ zeros in the disk centered at 1 with radius $1/32$. However, we show that this situation cannot occur if the absolute constant $\alpha > 0$ is sufficiently small. Indeed, let $p \in \mathcal{P}_n^c(\varepsilon)$ be of the form $p = fg$ with

$$f(z) = \prod_{j=1}^{n_1} (z - u_j) \qquad \text{and} \qquad g(z) = \prod_{j=1}^{n_2} (z - v_j)\,,$$

where

$$u_j \in \mathbb{C}\,, \qquad j = 1, 2, \ldots, n_1\,, \quad n_1 \leq 2\alpha n\,, \tag{6.7}$$

and

$$|v_j - 1| \leq \frac{1}{32}\,, \qquad j = 1, 2, \ldots, n_2\,, \quad n_2 \geq (1-2\alpha)n\,. \tag{6.8}$$

Let I be the subinterval of $[-1, i\varepsilon]$ with endpoint -1 and length $1/32$. Let $y_0 \in I$ be chosen so that $|f(y_0)| = \|f\|_I$. We show that $|p(z_0)| < |p(y_0)|$, a contradiction. Indeed, by Chebyshev's inequality [8, Theorem 6.1, p. 75] and (6.7) we have

$$|f(y_0)| \geq \left(\frac{1}{128}\right)^{n_1} \geq \left(\frac{1}{128}\right)^{2\alpha n}\,,$$

hence

$$\left|\frac{f(y_0)}{f(z_0)}\right| \geq \left(\frac{1}{256}\right)^{2\alpha n}\,. \tag{6.9}$$

Also, (6.8) implies

$$\left|\frac{g(y_0)}{g(z_0)}\right| \geq \frac{\left(\frac{31}{16}\right)^{n_2}}{\left(\sqrt{2}+\frac{1}{32}\right)^{n_2}} \geq \left(\frac{31}{24}\right)^{(1-2\alpha)n}\,. \tag{6.10}$$

By (6.9) and (6.10),

$$\left|\frac{p(y_0)}{p(z_0)}\right| = \left|\frac{f(y_0)}{f(z_0)}\right| \left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\left(\frac{1}{256}\right)^{2\alpha} \left(\frac{31}{24}\right)^{(1-2\alpha)}\right)^n > 1\,,$$

if $\alpha > 0$ is a sufficiently small absolute constant. This finishes the proof in this case.
Case 3: $\varepsilon \in [0, n^{-1/2}]$. The lower bound of the theorem follows now from a result of Erőd [14, III. tétel] proved by Levenberg and Poletcky [15]. $\square$

Proof of Theorem 6.2 Choose a point $z_0 \in S_1$ such that $|p(z_0)| = \|p\|_{S_1}$. Without loss of generality we may assume that $z_0 \in \left[1, \frac{1}{2}(1+i)\right]$. A simple calculation shows that there is an absolute constant $r > 0$ such that the circle $\Gamma := \Gamma(r, z_0)$ with radius r that is tangent to $[1, i]$ at z_0 and intersects the boundary of S_1 only at $a \in [-1, i]$ and $b \in [-i, 1]$. Moreover, if the $r > 0$ is sufficiently large, then

$$|a - i| \leq \frac{\sqrt{2}}{64} \qquad \text{and} \qquad |b - 1| \leq \frac{\sqrt{2}}{64}\,. \tag{6.11}$$

We denote the disk with boundary Γ by $D := D(r, z_0)$. If $p \in \mathcal{P}_n^c(1)$ has at least αn zeros in D, then by Lemma 6.3 we deduce

$$\frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \geq \frac{|p'(z_0)|}{\|p\|_{S_1}} = \left|\frac{p'(z_0)}{p(z_0)}\right| \geq \frac{\alpha n}{2r}\,.$$

Hence we may assume that $p \in \mathcal{P}_n^c(1)$ has at most αn zeros in D, and hence that $p \in \mathcal{P}_n^c(1)$ has at least $(1-\alpha)n$ zeros in $S_1 \setminus D$. However, we show that this situation cannot occur if the absolute constant $\alpha > 0$ is sufficiently small. Indeed, let $p \in \mathcal{P}_n^c(1)$ be of the form $p = fg$ with

$$f(z) = \prod_{j=1}^{n_1} (z - u_j) \qquad \text{and} \qquad g(z) = \prod_{j=1}^{n_2} (z - v_j)\,,$$

where

$$u_j \in \mathbb{C}\,, \qquad j = 1, 2, \ldots, n_1\,, \quad n_1 \leq \alpha n\,, \tag{6.12}$$

and

$$v_j \in S_1 \setminus D\,, \qquad j = 1, 2, \ldots, n_2\,, \quad n_2 \geq (1-\alpha)n\,. \tag{6.13}$$

Let I be the subinterval of $[-1,-i]$ with endpoint -1 and length $\sqrt{2}/4$. Let $y_0 \in I$ be chosen so that $|f(y_0)| = \|f\|_I$. We show that $|p(z_0)| < |p(y_0)|$, a contradiction. Indeed, by Chebyshev's inequality [8, Theorem 6.1, p. 75] and (6.12) we have

$$|f(y_0)| \geq \left(\frac{\sqrt{2}}{16}\right)^{n_1} \geq \left(\frac{\sqrt{2}}{16}\right)^{\alpha n},$$

hence

$$\left|\frac{f(y_0)}{f(z_0)}\right| \geq \left(\frac{\sqrt{2}}{32}\right)^{\alpha n}. \tag{6.14}$$

Also, (6.11) and (6.13) imply

$$\begin{aligned} \left|\frac{g(y_0)}{g(z_0)}\right| &\geq \frac{\left(\sqrt{2}\left(\left(1-\frac{1}{64}\right)^2+\left(\frac{1}{4}\right)^2\right)^{1/2}\right)^{n_2}}{\left(\sqrt{2}\left(1+\left(\frac{1}{64}\right)2\right)^{1/2}\right)^{n_2}} \\ &\geq \left(\frac{66}{65}\right)^{n_2/2} \geq \left(\frac{66}{65}\right)^{(1/2-\alpha)n}. \end{aligned} \tag{6.15}$$

By (6.14) and (6.15)

$$\left|\frac{p(y_0)}{p(z_0)}\right| = \left|\frac{f(y_0)}{f(z_0)}\right| \left|\frac{g(y_0)}{g(z_0)}\right| \geq \left(\left(\frac{\sqrt{2}}{32}\right)^{\alpha}\left(\frac{66}{65}\right)^{(1/2-\alpha)}\right)^{n} > 1$$

if $\alpha > 0$ is a sufficiently small absolute constant. □

Motivated by the initial results in this section, Sz. Révész [20] established the right order Turán -type converse Markov inequalities on convex domains of the complex plane. His main theorem contains the results in this section as special cases. Révész's proof is also elementary, but rather subtle. It is expected to appear in the Journal of Approximation Theory soon.

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Department of Mathematics
Texas A&M University
College Station, Texas, 77843, USA
Email address: terdelyi@math.tamu.edu