GEORGE LORENTZ AND INEQUALITIES IN APPROXIMATION

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Abstract. George Lorentz influenced the author's research on inequalities in approximation in many ways. This is the connecting thread of this survey paper. The themes of the survey are listed at the very beginning of the Introduction.

0. Introduction

1. Bernstein-type inequalities for exponential sums.
2. Remez-type inequalities for exponential sums.
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1. Bernstein-type Inequalities for Exponential Sums

The results in this section were, in large measure, motivated by the letter of Lorentz below.

"Dear Tamás: Feb. 27, 1988

I know you are interested in Bernstein-type inequalities and I am also. In some non-linear cases one has

$$\|P'\|_X \leq \Phi(n)\|P\|_Y\,,$$

where n is the dimension of the set of the P's, and the norms are taken in different Banach spaces X and Y. For instance, inequalities of Dolzhenko and Pekarskii for rational

2000 Mathematics Subject Classifications: Primary: 41A17 functions are of this type. I have proved an inequality of this type for exponential functions $\sum_1^n a_j e^{\lambda_j x}$ or the extended exponential sums

$$\sum_{j=1}^{l} P_{k_j}(x) e^{\lambda_j x}\,, \qquad \sum_{1}^{l} (k_j+1) = n\,.$$

I enclose my arguments. Would you like to help me? Are my inequalities sharp in some sense? What are other arguments that can lead to such inequalities? Hope to see you again in April, when I will be again in Columbia.

Sincerely, George Lorentz"

In his book [41] Braess writes "The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t}\,,$$

where the parameters a_j and λ_j are to be determined, while n is fixed."

In [23] we proved the right Bernstein-type inequality for exponential sums.

Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [128] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,b-\delta]} \le c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$. Here, and in what follows, $\|\cdot\|_{[a,b]}$ denotes the uniform norm on $[a,b]$. Lorentz [103] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$, there is a constant $c(\alpha)$ depending only on α so that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$ (Xu improved this to allow $\alpha = \frac{1}{2}$), and he speculated that there may be an absolute constant c so that Schmidt's inequality holds with $c(n)$ replaced by cn. The main result, Theorem 3.2, of [23] shows that Schmidt's inequality holds with $c(n) = 2n-1$. That is,

$$\sup_{0 \ne f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{y-a, b-y\}}\,, \qquad y \in (a,b)\,. \tag{1.1}$$

In this Bernstein-type inequality even the point-wise factor is sharp up to a multiplicative absolute constant; the inequality

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a, b-y\}} \le \sup_{0 \ne f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}\,, \qquad y \in (a,b)\,,$$

is established by Theorem 3.3 in [23]. Inequality (1.1) improves an earlier result obtained in [16] where we have only $8n^2$ in the place of $2n-1$. Lorentz presented our simple elegant proof of our weaker result in [16] on pages 378 and 379 of his book [105] and remarked that the exact inequality (1.1) would appear in [17]. I do not know if Lorentz ever read my paper [23] but it looks to me he accepted it as it was presented in my book [17] with P. Borwein.

Bernstein-type inequalities play a central role in approximation theory via a machinery developed by Bernstein, which turns Bernstein-type inequalities into inverse theorems of approximation. See, for example, the books by Lorentz [104] and by DeVore and Lorentz [44]. From (1.1) one can deduce in a standard fashion that if there is a sequence $(f_n)_{n=1}^\infty$ of exponential sums with $f_n \in E_n$ that approximates f on an interval $[a,b]$ uniformly with errors

$$\|f - f_n\|_{[a,b]} = O\big(n^{-m}(\log n)^{-2}\big)\,, \qquad n = 2, 3, \ldots\,,$$

where $m \in \mathbb{N}$ is a fixed integer, then f is m times continuously differentiable on (a,b). Let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n with real coefficients. Inequality (1.1) can be extended to E_n replaced by

$$\widetilde{E}_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t) e^{\lambda_j t}\,, \quad a_0, \lambda_j \in \mathbb{R}\,, \;\; P_{m_j} \in \mathcal{P}_{m_j}\,, \;\; \sum_{j=1}^{N} (m_j + 1) \leq n \right\}.$$

In fact, it is well-known that $\widetilde{E}_n$ is the uniform closure of E_n on any finite subinterval of the real number line, see Theorem 2.3 on page 173 of [41], for instance For a function f defined on a set A let

$$\|f\|_A := \|f\|_{L_\infty A} := \|f\|_{L_\infty(A)} := \sup\{|f(x)| : x \in A\}\,,$$

and let

$$\|f\|_{L_p A} := \|f\|_{L_p(A)} := \left(\int_A |f(x)|^p \, dx \right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. In [65] we focus on the classes

$$G_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{-(t-\lambda_j)^2}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\},$$

$$\widetilde{G}_n := \left\{ f : f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,, \quad \lambda_j \in \mathbb{R}\,, \;\; P_{m_j} \in \mathcal{P}_{m_j}\,, \;\; \sum_{j=1}^{N} (m_j + 1) \leq n \right\},$$

and

$$\widetilde{G}_n^* :=$$

$$\left\{ f : f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,, \quad \lambda_j \in [-n^{1/2}, n^{1/2}]\,, \;\; P_{m_j} \in \mathcal{P}_{m_j}\,, \;\; \sum_{j=1}^{N} (m_j + 1) \leq n \right\}.$$

Note that $\widetilde{G}_n$ is the uniform closure of G_n on any finite subinterval of the real number line. Let $W(t) := \exp(-t^2)$. Combining Corollaries 1.5 and 1.8 in [45] and recalling that for the weight W the Mhaskar-Rachmanov-Saff number a_n defined by (1.4) in [45] satisfies $a_n \leq c_1 n^{1/2}$ with a constant c_1 independent of n, we obtain that

$$\inf_{P \in \mathcal{P}_n} \|(P-g)W\|_{L_q(\mathbb{R})} \leq c_2 n^{-m/2} \|g^{(m)} W\|_{L_q(\mathbb{R})}$$

with a constant c_2 independent of n, for every g for which the norm on the right-hand side is finite for some $m \in \mathbb{N}$ and $q \in [1, \infty]$. As a consequence

$$\inf_{f \in \widetilde{G}_n^*} \|f - gW\|_{L_q(\mathbb{R})} \leq c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} (gW)^{(k)}(t)\|_{L_q(\mathbb{R})}$$

with a constant c_3 independent of n for every g for which the norms on the right-hand side are finite for each $k = 0, 1, \dots, m$ with some $q \in [1, \infty]$. Replacing gW by g, we conclude that

$$\inf_{f \in \widetilde{G}_n^*} \|f - g\|_{L_q(\mathbb{R})} \leq c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} g^{(k)}(t)\|_{L_q(\mathbb{R})} \tag{1.2}$$

with a constant c_3 independent of n for every g for which the norms on the right-hand side are finite for each $k = 0, 1, \dots, m$ with some $q \in [1, \infty]$. In [65] we proved the following results.

Theorem 1.1. *There is an absolute constant c_4 such that*

$$|U_n'(0)| \leq c_4 n^{1/2} \|U_n\|_{\mathbb{R}}$$

for all U_n of the form $U_n = P_n Q_n$ with $P_n \in \widetilde{G}_n$ and an even $Q_n \in \mathcal{P}_n$. As a consequence

$$\|P_n'\|_{\mathbb{R}} \leq c_4 n^{1/2} \|P_n\|_{\mathbb{R}}$$

for all $P_n \in \widetilde{G}_n$.

We remark that a closer look at the proof shows that $c_4 = 5$ in the above theorem is an appropriate choice.

Theorem 1.2. *There is an absolute constant c_5 such that*

$$\|U_n'\|_{L_q(\mathbb{R})} \leq c_5^{1+1/q} n^{1/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$ and $q \in (0, \infty)$.

Theorem 1.3. *There is an absolute constant c_6 such that*

$$\|U_n^{(m)}\|_{L_q(\mathbb{R})} \leq (c_6^{1+1/q} nm)^{m/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0, \infty]$, and $m = 1, 2, \dots$.

We remark that a closer look at the proofs shows that $c_5 = 180\pi$ in Theorem 1.2 and $c_6 = 180\pi$ in Theorem 1.3 are appropriate choices.

Our next theorem may be viewed as a slightly weak version of the right inverse theorem of approximation that can be coupled with the direct theorem of approximation formulated in (1.2).

Theorem 1.4. *Suppose $q \in [1, \infty]$, m is a positive integer, $\varepsilon > 0$, and f is a function defined on $\mathbb{R}$. Suppose also that*

$$\inf_{f_n \in \widetilde{G}_n} \|f_n - f\|_{L_q(\mathbb{R})} \leq c_7 n^{-m/2} (\log n)^{-1-\varepsilon}, \qquad n = 2, 3, \dots,$$

with a constant c_7 independent of n. Then f is m times differentiable almost everywhere on $\mathbb{R}$. Also, if

$$\inf_{f_n \in \widetilde{G}_n^*} \|f_n - f\|_{L_q(\mathbb{R})} = c_7 n^{-m/2} (\log n)^{-1-\varepsilon}, \qquad n = 2, 3, \dots,$$

with a constant c_7 independent of n, then, in addition to the fact that f is m times differentiable almost everywhere on $\mathbb{R}$, we also have

$$\|(1 + |t|)^{m-j} f^{(j)}(t)\|_{L_q(\mathbb{R})} < \infty, \qquad j = 0, 1, \dots, m.$$

Theorem 1.5. *There is an absolute constant c_8 such that*

$$\|U_n'\|_{L_q[y-\delta/2, y+\delta/2]} \leq c_8^{1+1/q} \left(\frac{n}{\delta}\right) \|U_n\|_{L_q[y-\delta, y+\delta]}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0, \infty]$, $y \in \mathbb{R}$, and $\delta \in (0, n^{1/2}]$.

A multidimensional analogue $G_{\mathbf{m}}$ of the class G_n is introduced and studied briefly in Section 7. An element of $G_{\mathbf{m}}$ is called a Gaussian network of N neurons. In this context H. Mhaskar [110] writes "Professor Ward at Texas A&M University has pointed out that our results implicitly contain an inequality, known as Bernstein inequality, in terms of the number of neurons, under some conditions on the minimal separation. Professor Erdélyi at Texas A&M University has kindly sent us a manuscript in preparation, where he proves this inequality purely in terms of the number of neurons, with no further conditions. This inequality leads to the converse theorems in terms of the number of neurons, matching our direct theorem in this theory. Our direct theorem in [109] is sharp in the sense of n-widths. However, the converse theorem applies to individual functions rather than a class of functions. In particular, it appears that even if the cost of approximation is measured in terms of the number of neurons, if the degrees of approximation of a particular function by Gaussian networks decay polynomially, then a linear operator will yield the same order of magnitude in the error in approximating this function. We find this astonishing, since many people have told us based on numerical experiments that one can achieve a better degree of approximation by non-linear procedures by stacking the centers near the bad points of the target functions". (The concept of width is introduced and examined thoroughly in Chapter 13 of [105].)

2. Remez-type Inequalities for Exponential Sums

The classical Remez inequality [122] states that if p is a polynomial of degree at most n, $s \in (0,2)$, and

$$m\left(\{x \in [-1,1] : |p(x)| \le 1\}\right) \ge 2 - s\,,$$

then

$$\|p\|_{[-1,1]} \le T_n\left(\frac{2+s}{2-s}\right)\,,$$

where T_n defined by

$$T_n(x) := \cos(n \arccos x)\,, \qquad x \in [-1,1]\,,$$

is the Chebyshev polynomial of degree n. This inequality is sharp and

$$T_n\left(\frac{2+s}{2-s}\right) \le \exp(\min\{5ns^{1/2}, 2n^2 s\})\,, \qquad s \in (0,1]\,.$$

Remez-type inequalities turn out to be very useful in various problems of approximation theory. See, for example, Remez [122], Borwein and Erdélyi [17], [18], [25], and [27], Erdélyi [46], [51], and [52], Erdélyi and Nevai [75], Freud [83, p.122], and Lorentz, Golitschek, and Makovoz [105]. Lorentz liked my essentially sharp Remez-type inequality for trigonometric polynomials [51] and he stated in his book [105, p. 77] correctly.

Theorem 2.1. *If p is a trigonometric polynomial of degree at most n, $s \in (0, \pi/2]$, and*

$$m\left(\{t \in [-\pi,\pi) : |p(t)| \le 1\}\right) \ge 2\pi - s\,,$$

then

$$\|p\|_{[-\pi,\pi]} \le \exp(cns)$$

with an absolute constant $c > 0$ ($c = 2$ is a suitable choice).

He writes "The proof of this trigonometric Remez-type inequality needs new ideas, and is far from being a copy of the methods working in the algebraic case". My memory is that at one point Lorentz sent me an approach that seemed to reduce the proof of my Remez-type inequality for trigonometric polynomials to the Remez inequality for algebraic polynomials. The mistake in it was so subtle that at first I approved it. Later, when I observed a gap in it, Lorentz decided he would just state the result without proof. I think even today only my paper [51] contains the trigonometric Remez inequality with correct proof.

Another remarkable result in [51] is the following essentially pointwise Remez-type inequality for algebraic polynomials.

Theorem 2.2. *If p is an algebraic polynomial of degree at most n, $s \in (0,1]$, $y \in (-1,1)$, and*

$$m\left(\{x \in [-1,1] : |p(x)| \le 1\}\right) \ge 2 - s\,,$$

then

$$|p(y)| \le \exp\left(cn \min\left\{\frac{s}{\sqrt{1-y^2}}, \sqrt{s}\right\}\right)$$

with an absolute constant $c > 0$.

In fact, both Theorems 2.1 and 2.2 easily extends to the classes GTP_N and GAP_N, respectively, of generalized (trigonometric) polynomials. These classes are introduced in Section 10 where inequalities for generalized polynomials from the classes GTP_N and GAP_N are discussed.

In [27] we proved the following result.

Theorem 2.3 (Remez-Type Inequality for E_n at 0). *Let $s \in \left(0, \frac{1}{2}\right]$. There are absolute constants $c_2 > 0$ and $c_3 > 0$ such that*

$$\frac{1}{2}\exp(c_2 ns) \le \sup_f |f(0)| \le \exp(c_3 ns)\,,$$

where the supremum is taken for all $f \in E_n$ satisfying

$$m\left(\{x \in [-1,1] : |f(x)| \le 1\}\right) \ge 2 - s\,.$$

In fact, in [27] Theorem 2.2 is carelessly stated. The factor 1/2 on the left hand side is missing. More accurate estimates for the values of the Chebyshev polynomial appearing in the proof of the above result give the following more complete result.

Theorem 2.3* (Remez-Type Inequality for E_n at 0). *Let $s \in \left(0, \frac{1}{2}\right]$. There are absolute constants $c_2 > 0$ and $c_3 > 0$ such that*

$$\exp(c_2 \min\{ns, (ns)^2\}) \le \sup_f |f(0)| \le \exp(c_3 \min\{ns, (ns)^2\})\,,$$

where the supremum is taken for all $f \in E_n$ satisfying

$$m\left(\{x \in [-1,1] : |f(x)| \le 1\}\right) \ge 2 - s\,.$$

In [67] we established an essentially sharp Remez-type inequality for G_n and $\widetilde{G}_n$. We also prove the right higher dimensional analog of our main result.

Theorem 2.4 (Remez-Type Inequality for $\widetilde{G}_n$). *Let $s \in (0,\infty)$ and $n \ge 9$. There is an absolute constant $c_1 > 0$ such that*

$$\exp(c_1(\min\{n^{1/2}s, ns^2\} + s^2)) \le \sup_f \|f\|_{\mathbb{R}} \le \exp(80(\min\{n^{1/2}s, ns^2\} + s^2))\,,$$

where the supremum is taken for all $f \in \widetilde{G}_n$ satisfying

$$m\left(\{t \in \mathbb{R} : |f(t)| \ge 1\}\right) \le s\,.$$

Important results of Turán [137] are based on the following observations: *Let*

$$g(\nu) := \sum_{j=1}^{n} b_j z_j^{\nu}\,, \qquad b_j, z_j \in \mathbb{C}\,.$$

Suppose

$$\min_{1 \le j \le n} |z_j| \ge 1\,, \qquad j = 1, 2, \ldots, n\,.$$

Then

$$\max_{\nu = m+1, \ldots, m+n} |g(\nu)| \ge \left(\frac{n}{2e(m+n)}\right)^n |b_1 + b_2 + \cdots + b_n|$$

for every positive integer m.

A consequence of the preceding is the famous Turán Lemma: *if*

$$f(t) := \sum_{j=1}^{n} b_j e^{\lambda_j t}\,, \qquad b_j\,, \lambda_j \in \mathbb{C}\,, \tag{2.1}$$

and

$$\min_{1 \le j \le n} \mathrm{Re}(\lambda_j) \ge 0\,,$$

then

$$|f(0)| \le \left(\frac{2e(a+d)}{d}\right)^n \|f\|_{[a,a+d]}$$

for every $a > 0$ *and* $d > 0$.

Another consequence of this is the fact that *if*

$$p(z) := \sum_{j=1}^{n} b_j z^{\lambda_j}\,, \qquad b_j \in \mathbb{C}\,, \quad \lambda_j \in \mathbb{R}\,, \quad z = e^{i\theta}\,,$$

then

$$\max_{|z|=1} |p(z)| \le \left(\frac{4e\pi}{\delta}\right)^n \max_{\substack{|z|=1 \\ \alpha \le \arg(z) \le \alpha + \delta}} |p(z)|$$

for every $0 \le \alpha < \alpha + \delta \le 2\pi$.

Turán's inequalities above and their variants play a central role in the book of Turán [137], where many applications are also presented. The main point in these inequalities is that the exponent on the right-hand side is only the number of terms n, and so it is independent of the numbers λ_j. An inequality of the type

$$\max_{|z|=1} |p(z)| \le c(\delta)^{\lambda_n} \max_{\substack{|z|=1 \\ \alpha \le \arg(z) \le \alpha + \delta}} |p(z)|\,,$$

where $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n$ are integers and $c(\delta)$ depends only on δ, could be obtained by a simple direct argument, but it is much less useful than Turán's inequality. F. Nazarov has a seminal paper [113] devoted to Turán-type inequalities for exponential sums, and their applications to various uniqueness theorems in harmonic analysis of the uncertainty principle type. The author derives an estimate for the maximum modulus of exponential sums f of the form (2.1) on an interval $I \subset \mathbb{R}$ in terms of its maximum modulus on a measurable set $E \subset I$ of positive Lebesgue measure:

$$\sup_{t \in I} |f(t)| \leq e^{\max |\mathrm{Re} \lambda_k| m(I)} \left(\frac{Am(I)}{m(E)} \right)^{n-1} \sup_{t \in E} |f(t)| \,,$$

where A is an absolute constant.

3. Lorentz Degree of Polynomials

In 1916 S.N. Bernstein [8] observed that any polynomial p having no zeros in the interval $(-1,1)$ can be written in the form $p(x) = \sum_{j=0}^{d} a_j (1-x)^j (1+x)^{d-j}$ with all $a_j \geq 0$. Hausdorff claimed this result independently, and the Pólya-Szegő book attributes it to him [87, pp. 98–99]. The smallest natural number d for which such a representation holds is called the Lorentz degree of p and it is denoted by $d(p)$. The Lorentz degree was named after George Lorentz, who established essentially sharp Markov- and Bernstein-type inequalities on $[-1,1]$ for polynomials of Lorentz degree d, and one of his students, J.T. Scheick [127] has contributed to the topic substantially. The Lorentz degree $d(p)$ can be much larger than the ordinary degree of the polynomial p. Nevertheless, in [48] and [73], essentially sharp upper and lower bounds are given for classes of polynomials of ordinary degree n having no zeros in ellipses with axes $[-1,1]$ and $[-\varepsilon i, \varepsilon i]$, $\varepsilon \in (0,1]$. As a by-product, the proofs of Lorentz's Markov- and Bernstein-type inequalities [102] are shortened so that they are fit to print in his book [44, p. 115] with DeVore.

To formulate our main theorem from [48] we need some notations. Let φ be a positive continuous function defined on $(-1,1)$, and let

$$D(\varphi) := \{z = x + iy : \ |y| < \varphi(x), \ |x| < 1\}$$

denote the domain of the complex plane determined by it. We introduce

$$L_n(\varphi) := \{p \in \mathcal{P}_n : \ p(z) \neq 0 \,, \ z \in D(\varphi)\}$$

and

$$d_n(\varphi) := \sup_{p \in L_n(\varphi)} d(p) \,.$$

Sharpening a result in [73, Theorem 3], we proved the following in [48].

Theorem 3.1. *If*

$$1 \geq \varepsilon := \inf_{x \in (-1,1)} \frac{\varphi(x)}{\sqrt{1-x^2}} > 0 \,,$$

then

$$\frac{c_1 n}{\varepsilon^2} \leq d_n(\varphi) \leq \frac{c_2 n}{\varepsilon^2} \,,$$

where $0 < c_1 < c_2$ are absolute constants.

Let

$$p(x) := (1-x)^2 - 2(1-x)(1+x) + 4(1+x)^2 \text{ and } q(x) := (1+x) + \frac{1}{2}(1-x)\,.$$

Then $d(p) = 4$, $d(q) = 1$, and $d(pq) = 3$. As far as I know the following two questions raised in [48] are still open. Is it true that $d(pq) \geq \min\{d(p), d(q)\}$ for any polynomials p and q? Is it true that $d(pq) \geq |d(p) - d(q)|$ for any polynomials p and q?

4. Markov- and Bernstein-type inequalities for polynomials with constraints

The Markov-Bernstein inequality asserts that

$$|p'(x)| \leq \min\left\{\frac{n}{\sqrt{1-x^2}}\,,\, n^2\right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

holds for every polynomial of degree at most n with complex coefficients. Here, and in what follows, $\|p\|_A := \sup_{y \in A} |p(y)|$. Throughout his life Erdős showed a particular interest in inequalities for constrained polynomials. In a short paper in 1940 Erdős [78] has found a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(x)| \leq \min\left\{\frac{c\sqrt{n}}{(1-x^2)^2}\,,\ \frac{en}{2}\right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for every polynomial p of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1,1)$. This result motivated a number of people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov-Bernstein type inequality of Erdős has been extended later in many directions. Let $\mathcal{P}^c_{n,k}$ denote the set of all polynomials of degree at most n with *complex coefficients* and with at most k $(0 \leq k \leq n)$ zeros in the open unit disk. Let $\mathcal{P}_{n,k}$ denote the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \leq k \leq n)$ zeros in the open unit disk. Associated with $0 \leq k \leq n$ and $x \in (-1,1)$, let

$$B^*_{n,k,x} := \max\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\ n\log\left(\frac{e}{1-x^2}\right)\right\}\,, \qquad B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,,$$

and

$$M^*_{n,k} := \max\{n(k+1),\ n\log n\}\,, \qquad M_{n,k} := n(k+1)\,.$$

It is shown in [53] and [54] that

$$c_1 \min\{B^*_{n,k,x}, M^*_{n,k}\} \leq \sup_{p \in \mathcal{P}^c_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq c_2 \min\{B^*_{n,k,x}, M^*_{n,k}\}$$

for every $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. This result should be compared with the inequalities

$$c_3 \min\{B_{n,k,x}, M_{n,k}\} \leq \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq c_4 \min\{B_{n,k,x}, M_{n,k}\}$$

for every $x \in (-1,1)$, where $c_3 > 0$ and $c_4 > 0$ are absolute constants. The upper bound of this second result may be found in [14], and it may be surprising that there is a significant difference between the real and complex cases as far as Markov-Bernstein type inequalities are concerned. The lower bound of the second result is proved in [53]. It is the final piece of a long series of papers on this topic by a number of authors starting with Erdős in 1940. In addition, in [15] we established the right Markov-Bernstein type inequalities on for the classes $\mathcal{P}_{n,k}$ in $L_p[-1,1]$, $p > 0$.

Let $\mathcal{P}_n^c(r)$ be the set of all polynomials of degree at most n with *complex coefficients* and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$ $(0 < r \leq 1)$. Let $\mathcal{P}_n(r)$ be the set of all polynomials of degree at most n with The lower bound of the second result is proved in [53]. and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$ $(0 < r \leq 1)$.

Essentially sharp Markov-type inequalities for $\mathcal{P}_n^c(r)$ and $\mathcal{P}_n(r)$ on $[-1,1]$ are established in [53] and [47]. In [53] we show

$$c_1 \min\left\{\frac{n\log\left(e+n\sqrt{r}\right)}{\sqrt{r}}\,,\, n^2\right\} \leq \sup_{0 \neq p \in \mathcal{P}_n^c(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 \min\left\{\frac{n\log\left(e+n\sqrt{r}\right)}{\sqrt{r}}\,,\, n^2\right\}$$

for every $0 < r \leq 1$ with absolute constants $c_1 > 0$ and $c_2 > 0$. This result should be compared with the inequalities

$$c_3 \min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\} \leq \sup_{0 \neq p \in \mathcal{P}_n(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_4 \min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}\,, \qquad 0 < r \leq 1\,,$$

where $c_3 > 0$ and $c_4 > 0$ are absolute constants. See [47].

Lorentz included a more general version of the above result for higher derivatives allowing k $(0 \leq k \leq n)$ zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$, respectively $(0 < r \leq 1)$ In his book [105, pp. 64–73]] he followed the proof in [47], where the idea of "Lorentz representation" turns out to be crucial. Note that induction does not work here due to the lack of a Gauss-Lucas Theorem.

Let K_α be the open diamond of the complex plane with diagonals $[-1,1]$ and $[-ia,ia]$ such that the angle between $[ia,1]$ and $[1,-ia]$ is $\alpha\pi$. In [85] Halász proved that there are constants $c_1 > 0$ and $c_2 > 0$ depending only on α such that

$$c_1 n^{2-\alpha} \leq \sup_p \frac{|p'(1)|}{\|p\|_{[-1,1]}} \leq \sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 n^{2-\alpha}\,,$$

where the supremum is taken for all polynomials p of degree at most n (with either real or complex coefficients) having no zeros in K_α.

Erdős had many questions and results about polynomials with restricted coefficients. Let $\mathcal{F}_n$ denote the set of polynomials of degree at most n with coefficients from $\{-1,0,1\}$. Let $\mathcal{G}_n$ be the collection of polynomials p of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_m| = 1\,, \quad |a_j| \le 1\,,$$

where m is an unspecified nonnegative integer not greater than n. In [28] and [30] we established the right Markov-type inequalities for the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ on $[0,1]$. Namely there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \log(n+1) \le \max_{0 \ne p \in \mathcal{F}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \le c_2 n \log(n+1)$$

and

$$c_1 n^{3/2} \le \max_{0 \ne p \in \mathcal{G}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \le c_2 n^{3/2}\,.$$

Observe that the right Markov factor for $\mathcal{G}_n$ is much larger than the right Markov factor for $\mathcal{F}_n$. We also show that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \log(n+1) \le \max_{0 \ne p \in \mathcal{L}_n} \frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}} \le c_2 n \log(n+1)\,,$$

where $\mathcal{L}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1,1\}$.

For polynomials

$$p \in \mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n \quad \text{with} \quad |p(0)| = 1$$

and for $y \in [0,1)$ the Bernstein-type inequality

$$\frac{c_1 \log\left(\frac{2}{1-y}\right)}{1-y} \le \max_{\substack{p \in \mathcal{F} \\ |p(0)|=1}} \frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}} \le \frac{c_2 \log\left(\frac{2}{1-y}\right)}{1-y}$$

is also proved in [30] with absolute constants $c_1 > 0$ and $c_2 > 0$.

Let $\mathcal{P}_n^m$ be the collection of all polynomials of degree at most n with real coefficients that have at most m distinct complex zeros. In [6] we prove that

$$\max_{x \in [0,1]} |P'(x)| \le 32 \cdot 8^m n \max_{x \in [0,1]} |P(x)|$$

for every $P \in \mathcal{P}_n^m$. This is far from what we expect. We conjecture that the Markov factor $32 \cdot 8^m n$ above may be replaced by cmn with an absolute constant $c > 0$. We are not able to prove this conjecture at the moment. However, we think that our result above gives the best known Markov-type inequality for $\mathcal{P}_n^m$ on a finite interval when $m \le c \log n$.

For continuous functions p defined on the complex unit circle, and for $q \in (0,\infty)$, we define

$$\|p\|_q := \left(\int_0^{2\pi} |p(e^{it})|^q \, dt \right)^{1/q} .$$

We also define

$$\|p\|_\infty := \lim_{q \to \infty} \|p\|_q = \max_{t \in [0,2\pi]} |p(e^{it})| .$$

Based on the ideas of F. Nazarov, Queffelec and Saffari [125] showed that

$$\sup_{p \in \mathcal{L}_n} \frac{\|p'\|_q}{\|p\|_q} = \gamma_{n,q} n \,, \qquad \lim_{n \to \infty} \gamma_{n,q} = 1 \,,$$

for every $q \in (0,\infty]$, $q \neq 2$ (when $q = 2$, $\lim_{n\to\infty} \gamma_{n,q} = 3^{-1/2}$ by the Parseval Formula). It shows that Bernstein's classical inequality (extended by Arestov [2] for all $q \in (0,\infty]$) stating that

$$\|p'\|_q \leq n \|p'\|_q$$

for all polynomials of degree at most n with complex coefficients, cannot be essentially improved for the class $\mathcal{L}_n$, except for the trivial $q = 2$ case.

Let SR_n^c denote the set of all self-reciprocal polynomials $p_n \in \mathcal{P}_n^c$ satisfying

$$p_n(z) = z^n p_n(z^{-1}) .$$

Let SR_n denote the set of all real self-reciprocal polynomials of degree at most n, that is, $\mathrm{SR}_n := \mathrm{SR}_n^c \cap \mathcal{P}_n$. For a polynomial $p_n \in \mathcal{P}_n^c$ of the form

$$p_n(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C} \,, \tag{4.1}$$

$p_n \in \mathrm{SR}_n^c$ if and only if

$$a_j = a_{n-j}, \qquad j = 0, 1, \ldots, n .$$

Let ASR_n^c denote the set of all antiself-reciprocal polynomials $p_n \in \mathcal{P}_n^c$ satisfying

$$p_n(z) = -z^n p_n(z^{-1}) .$$

Let ASR_n denote the set of all real antiself-reciprocal polynomials, that is, $\mathrm{ASR}_n := \mathrm{ASR}_n^c \cap \mathcal{P}_n$. For a polynomial $p \in \mathcal{P}_n^c$ of the form (4.1), $p_n \in \mathrm{ASR}_n^c$ if and only if

$$a_j = -a_{n-j}, \quad j = 0, 1, \ldots, n .$$

Every $p_n \in \mathrm{SR}_n^c$ and $p_n \in \mathrm{ASR}_n^c$ satisfies the growth condition

$$|p_n(x)| \leq (1 + |x|^n) \|p_n\|_{[-1,1]} \,, \quad x \in \mathbb{R} \setminus [-1,1] . \tag{4.2}$$

The Markov-type (uniform) part of the following inequality is due to Kroó and Szabados [101]. For the Bernstein-type (pointwise) part, see [17].

Theorem 4.1. *There is an absolute constant $c_1 > 0$ such that*

$$|p_n'(x)| \le c_1 n \min\left\{\log n\,,\ \log\left(\frac{e}{1-x^2}\right)\right\} \|p_n\|_{[-1,1]}$$

for every $x \in (-1,1)$ and for every polynomial $p_n \in \mathcal{P}_n^c$ satisfying the growth condition (4.2), in particular for every $p_n \in \mathrm{SR}_n^c$ and for every $p_n \in \mathrm{ASR}_n^c$ $(n \ge 2)$.

It is shown in [17] that the above result is sharp for the classes SR_n and ASR_n, that is, there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \min\left\{\log n\,,\ \log\left(\frac{e}{1-x^2}\right)\right\} \le \sup_{p_n} \frac{|p_n'(x)|}{\|p_n\|_{[-1,1]}} \le c_2 n \min\left\{\log n\,,\ \log\left(\frac{e}{1-x^2}\right)\right\},$$

where the supremum is taken either for all $0 \ne p_n \in \mathrm{SR}_n$ or for all $0 \ne p_n \in \mathrm{ASR}_n$ $(n \ge 2)$. Associated with a polynomial $p_n \in \mathcal{P}_n^c$ of the form (4.1) we define the polynomial

$$p_n^*(z) = \sum_{j=0}^n \overline{a}_{n-j} z^j\,.$$

Let D and ∂D denote the open unit disk and the unit circle, respectively, of the complex plane. It is well-known and proved in [106, p. 689] that

$$\max_{z \in \partial D} \left(|p_n'(z)| + |p_n^{*\prime}(z)|\right) = n \max_{z \in \partial D} |p_n(z)|\,.$$

In particular, if $p_n \in \mathcal{P}_n^c$ is conjugate reciprocal (satisfying $p_n = p_n^*$), then

$$\max_{z \in \partial D} |p_n'(z)| \le \frac{n}{2} \max_{z \in \partial D} |p_n(z)|\,.$$

In [120] the inequality

$$\max_{z \in \partial D} |p_n'(z)| \le (n - 1/4) \max_{z \in \partial D} |p_n(z)|\,.$$

is stated for all $p_n \in \mathrm{SR}_n^c$. In this inequality the Bernstein factor $(n-1/4)$, in general, cannot be replaced by anything better than $(n-1)$, as the following example shows. Let $P \in \mathcal{P}_{4n+4}^c$ be defined by

$$P(e^{it}) = (\cos((2n+1)t) + i(\sin((2n+1)t)\sin t))e^{i(2n+2)t}\,, \qquad t \in \mathbb{R}\,.$$

Since $Q(t) := \cos((2n+1)t) \in \mathcal{T}_{2n+1}$ and $R(t) := \sin((2n+1)t)\sin t \in \mathcal{T}_{2n+2}$ are even (real) trigonometric polynomials, $P \in \mathcal{P}_{4n+4}^c$ is a self-reciprocal polynomial. Obviously $\|P\|_{\partial D} \le 1$ since

$$\cos^2((2n+1)t) + \sin^2((2n+1)t)\sin^2 t \le 1\,.$$

Also

$$iP'(e^{i\pi/2})e^{i\pi/2} = (2n+1)(-1)^{n+1}e^{i(2n+2)\pi/2} + (2n+2)(-1)^{n+1}e^{(2n+2)\pi/2},$$

hence

$$\|P'\|_{\partial D} \ge 4n+3 \ge (4n+3)\|P\|_{\partial D}\,.$$

5. Müntz-type Theorems

Müntz's classical theorem [112] characterizes sequences $\Lambda := (\lambda_j)_{j=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space $M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0,1]$. Here, $M(\Lambda)$ is the collection of all finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0,\infty)$ equipped with the uniform norm. If $A := [a,b]$ is a finite closed interval, then the notation $C[a,b] := C([a,b])$ is used.

Müntz's Theorem. *Suppose $\Lambda := (\lambda_j)_{j=0}^\infty$ is a sequence satisfying $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Then $M(\Lambda)$ is dense in $C[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty\,. \tag{5.1}$$

Extending a result of Clarkson and Erdős [42], in [25] we proved the right Müntz-type theorem on compact subsets of $[0,\infty)$ with positive Lebesgue measure.

Theorem 5.1. *If $\Lambda := (\lambda_j)_{j=0}^\infty$ is an increasing sequence of nonnegative real numbers with $\lambda_0 = 0$ and $A \subset [0,\infty)$ is a compact set with positive Lebesgue measure, then $M(\Lambda)$ is dense in $C(A)$ if and only if (5.1) holds. Moreover, if (5.1) does not hold then every function from the $C[0,1]$ closure of $H(\Lambda) := \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$ restricted to $(0, r_A)$, where*

$$r_A := \sup\{x \in A : m(A \cap [x,\infty))\} > 0\,.$$

See also [18]. This result had been expected by Erdős and others for a long time. Lorentz liked this then quite recent result too and stated it in his book [105]. In fact the key to the proof of Theorem 5.1 is the bounded Remez-type inequality for non-dense Müntz spaces, the key result in [25] and [18].

Theorem 5.2. *For every increasing sequence $\Lambda := (\lambda_j)_{j=0}^\infty$ of nonnegative real numbers with $\lambda_0 = 0$ for which (5.1) does not hold there is a constant c depending only on Λ and s (and not on A or the number of terms in p) so that*

$$\|p\|_{[0, \inf A]} \le c\, \|p\|_A$$

for every $p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every $A \subset [0,1]$ of Lebesgue measure at least $s \in (0,1)$.

Extending earlier results of Müntz, Szász, Clarkson, Erdős, L. Schwartz, P. Borwein, Erdélyi, and Operstein, in [72] we proved the result below.

Theorem 5.3 ("Full Müntz Theorem" in $L_p[0,1]$ for $p \in (0,\infty)$). *Let $p \in (0,\infty)$. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. Then $H(\Lambda)$ is dense in $L_p[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty \,. \tag{5.2}$$

Moreover, if (5.2) does not hold then every function from the $L_p[0,1]$ closure of $H(\Lambda)$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$ restricted to $(0,1)$.

In handling the non-dense case, that is, the case when (5.2) does not hold, in [72] we needed to refer to [4].

In [72] the authors were not able to include the case $p = \infty$ in their discussion. The right result when $p = \infty$ is proved in [61].

Theorem 5.4 ("Full Clarkson-Erdős-Schwartz Theorem" in $C[0,1]$). *Let $(\lambda_j)_{j=1}^{\infty}$ be a sequence of distinct positive numbers. Then $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty \,. \tag{5.3}$$

Moreover, if (5.3) does not hold then every function from the $C[0,1]$ closure of $H(\Lambda)$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$ restricted to $(0,1)$.

This result improves an earlier result by P. Borwein and Erdélyi (see [20] and [17]) stating that if (5.1) does not hold then then every function from the $C[0,1]$ closure of $H(\Lambda)$ is in $C^{\infty}(0,1)$.

In [64] we present the proof of the above "full Müntz Theorem" in $L_p[0,1]$ for $p \in (0,\infty)$ by using more elementary text book methods.

The following problems are still open.

Problem 5.5. *Characterize the compact sets $A \subset [0,\infty)$ for which "Müntz's Theorem holds", that is for which $\text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C(A)$ if and only if (5.2) holds.*

Problem 5.6. *Does Müntz's Theorem hold on every compact set $A \subset [0,\infty)$ of positive logarithmic capacity?*

Problem 5.7. *Does Müntz's Theorem hold on the ternary Cantor set?*

Problem 5.8. *Is there a compact set $A \subset [0,\infty)$ of Lebesgue measure 0 on which Müntz's Theorem holds?*

6. Remez-type inequalities and Newman's Product Problem

Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers. Let

$$R(\Lambda) := \left\{ \frac{p}{q} : p, q \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\}.$$

A surprising result of Somorjai [131] and Bak and Newman [3] is the following.

Theorem 6.1. *$R(\Lambda) \cap C[0,1]$ is always dense in $C[0,1]$.*

So division has extra usefulness. Can multiplication have this extra utility? In [115, p. 51] Newman writes "Thus we have the very sane, if very prosaic, question: P(10.6) Are the functions $(\sum a_i x^{i^2})(\sum a_j x^{j^2})$ dense in C[0,1]?" In a more general setting let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct nonnegative real numbers with $\lambda_0 = 0$. Let

$$M^k(\Lambda) := \left\{ \prod_{j=1}^{k} p_j : p_j \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\} \right\}.$$

Suppose $k \geq 2$ and $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$. Can $M^k(\Lambda)$ be dense in $C[0,1]$? In [25] we solved this Newman product problem. In fact, we observed before that it would be a reasonably simple consequence of our bounded Remez-type inequality, Theorem 5.2, for nondense Müntz spaces.

Theorem 6.2. *If $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$ $k \geq 2$, and $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $M^k(\Lambda)$ is not dense in $C(A)$.*

Remark 6.3. *$M^k(\Lambda)$ is contained (not equal to)*

$$\operatorname{span}\{x^{\lambda_{j_1}+\lambda_{j_2}+\cdots+\lambda_{j_k}} : \lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_k} \in \Lambda\}.$$

Example 6.4. *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be defined by*

$$\lambda_i := \begin{cases} 0, & j = 0 \\ 2^{j-1}, & j = 1, 2, \dots . \end{cases}$$

Then

$$\sum_{\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_k} \in \Lambda} \frac{1}{\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_k}} < \infty$$

so it follows from Müntz's Theorem that $M^k(\Lambda)$ is not dense in $C[0,1]$.

Example 6.5. *Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be defined by $\lambda_j := j^2$. Then*

$$M^4(\Lambda) \subset \operatorname{span}\{x^{k^2+l^2+m^2+n^2} : k, l, m, n \in \mathbb{N}\} = \operatorname{span}\{x^n : n \in \mathbb{N}\}.$$

So in this case the non-denseness of $M^4(\Lambda)$ is not obvious at all.

7. Multivariate Approximation

In April, 1996, Lorentz sent me a letter related to a volume discussing "multivariate approximation". He speculated: "Some of the chapters may be trivial in the sense that they contain only a collection of known (important) results, others in the sense that their results mimic or are obtainable in a simple way on the univariate material. Even such "trivial" chapters are very much needed. Then there will be chapters (example: multivariate polynomial interpolation) that have very little to do with univariate results. What are these chapters?"

Knowing Lorentz's appreciation of the Remez inequality and its analog for trigonometric polynomials first proved in [51], I believe that Lorentz would like the right higher dimensional analog of Theorem 2.4. This is the only result we formulate in this section, it is proved in [67].

Let

$$\mathbf{m} := (m_1, m_2, \dots, m_k) \qquad \text{and} \qquad \mathbf{j} := (j_1, j_2, \dots, j_k)\,,$$

where each m_j and j_ν is a nonnegative integer. Let

$$\mathbf{B} := \{(j_1, j_2, \dots, j_k) : \ 1 \le j_\nu \le m_\nu, \ \nu = 1, 2, \dots, k\}\,,$$

$$\mathbf{x} := (x_1, x_2, \dots, x_k) \in \mathbb{R}^k\,,$$

$$\mathbf{d_j} := (d_{j_1}, d_{j_2}, \dots, d_{j_k}) \in \mathbb{R}^k\,, \qquad \mathbf{j} \in \mathbf{B}\,,$$

$$G_{\mathbf{m}} := \left\{ f : f(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbf{B}} A_{\mathbf{j}} \exp(-\|\mathbf{x} - \mathbf{d_j}\|)\,, \quad A_{\mathbf{j}} \in \mathbb{R}, \ \mathbf{d_j} \in \mathbb{R}^k \right\},$$

where

$$\|\mathbf{x} - \mathbf{d_j}\|^2 := \sum_{\nu=1}^{k} (x_\nu - d_{j_\nu})^2\,.$$

Theorem 7.1 (Remez-Type Inequality for $G_{\mathbf{m}}$). *Let $s \in (0, \infty)$ and $n \ge 9$. There is an absolute constant $c_1 > 0$ such that*

$$\exp(c_1 R(m_1, m_2, \dots, m_k, s)) \le \sup_f \|f\|_{\mathbb{R}^k} \le \exp(80 R(m_1, m_2, \dots, m_k, s))\,,$$

where

$$R(m_1, m_2, \dots, m_k, s) := \sum_{j=1}^{k} (\min\{m_j^{1/2} s^{1/k}, m_j s^{2/k}\} + s^{2/k})\,,$$

and the supremum is taken for all $f \in G_{\mathbf{m}}$ with

$$m\left(\left\{\mathbf{x} \in \mathbb{R}^k : |f(\mathbf{x})| \ge 1\right\}\right) \le s\,.$$

8. Newman's Inequality

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. The collection of all linear combinations of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. For a real-valued function f defined on a set A let

$$\|f\|_{L_\infty A} := \|f\|_A := \sup\{|f(x)| : x \in A\}\,,$$

and let

$$\|f\|_{L_p A} := \left(\int_A |f(x)|^p\,dx\right)^{1/p}, \qquad p > 0\,,$$

whenever the Lebesgue integral exists. Newman's inequality (see [114] and [17] is an essentially sharp Markov-type inequality for $E(\Lambda_n)$ on $[0,1]$ in the case when each λ_j is non-negative.

Theorem 8.1 (Newman's Inequality). *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. Then*

$$\frac{2}{3}\sum_{j=0}^{n} \lambda_j \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P'\|_{(-\infty,0]}}{\|P\|_{(-\infty,0]}} \leq 9 \sum_{j=0}^{n} \lambda_j\,.$$

Lorentz knew Newman's inequality and he presented the beautiful proof of Newman in the section "Markov-Type Inequalities for Müntz Polynomials" of his book [105, pp. 362–365]. When he learned from me that an extension of

P. Borwein's Markov-type inequality [12]

$$\|p'\|_{[-1,1]} \leq 9n(k+1)\|p\|_{[-1,1]} \tag{8.1}$$

to polynomials from $\mathcal{P}_{n,k}$ (the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \leq k \leq n)$ zeros in the open unit disk) can be obtained easily by using only Lorentz representation and Newman's inequality, he decided to follow my paper [49] to present a proof of (8.1) in his book [105, pp. 64–66], even though my argument gave a multiplicative absolute constant slightly worse than 9. Moreover, Lorentz [105] presents a short and simple proof of the inequality

$$\|p^{(m)}\|_{[-1,1]} \leq c(m)(n(k+1))^m \|p\|_{[-1,1]}\,, \qquad p \in \mathcal{P}_{n,k}\,, \tag{8.2}$$

for higher derivatives, where $c(m)$ is a constant depending only on m, as it is done (essentially) in [49]. Note that a simple induction does not work here due to the lack of a Gauss-Lucas type theorem.

Lorentz presents Newman's Inequality only with constant 11 rather than 9 in his book [105]. The book [17] seems to be the first one working out the details of the necessary modification and simplification of the proof of Newman's inequality with constant 9 by eliminating an application of Kolmogorov's inequality from Newman's original approach. Later we observed that the best known multiplicative constant in Newman's inequality is 8.29 given in [82].

In [35] orthonormal Müntz-Legendre polynomials were studied. As a by-product we proved an essentially sharp version of Newman's inequality in L_2. An L_p, $1 \leq p \leq \infty$, version of the upper bound in Newman's Inequality is established in [7], [22], and [55]. Here we formulate some of the main results in [55] that give the constant 8.29, which is better than 9 in [7] and [22].

Theorem 8.2. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. Let $1 \leq p \leq \infty$. Then*

$$\|Q'\|_{L_p(-\infty,0]} \leq 8.29 \left(\sum_{j=0}^{n} \lambda_j \right) \|Q\|_{L_p(-\infty,0]}$$

for every $Q \in E(\Lambda_n)$.

Theorem 8.2*. *Let $1 \leq p \leq \infty$. Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers greater than $-1/p$. Then*

$$\|xS'(x)\|_{L_p[0,1]} \leq \left(1/p + 8.29 \left(\sum_{j=0}^{n} (\lambda_j + 1/p) \right) \right) \|S\|_{L_p[0,1]}$$

for every $S \in M_n(\Lambda) := \mathrm{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

The following $L_p[a,b]$, $1 \leq p \leq \infty$, analog of Theorem 8.1 has been established in [7].

Theorem 8.3. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers, $1 \leq p \leq \infty$, $a, b \in \mathbb{R}$, and $a < b$. There is a constant $c_1 = c_1(a,b)$ depending only on a and b such that*

$$\sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_p[a,b]}} \leq c_1 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right) .$$

Theorem 8.3 was proved earlier in [21] ($p = \infty$) and [55] under the additional assumptions that $\lambda_j \geq \delta j$ for each j with a constant $\delta > 0$ and with $c_1 = c_1(a,b)$ replaced by $c_1 = c_1(a,b,\delta)$ depending only on a, b, and δ. The novelty of Theorem 8.3 was the fact that

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

is an arbitrary set of real numbers, not even the non-negativity of the exponents λ_j is needed.

In [62] the following Nikolskii-Markov type inequality has been proved for $E(\Lambda_n)$ on $(-\infty, 0]$.

Theorem 8.4. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers and $0 < q \leq p \leq \infty$. Let μ be a non-negative integer. There are constants $c_2 = c_2(p,q,\mu) > 0$ and $c_3 = c_3(p,q,\mu)$ depending only on p, q, and μ such that*

$$c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q} - \frac{1}{p}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{\|P^{(\mu)}\|_{L_p(-\infty,0]}}{\|P\|_{L_q(-\infty,0]}} \leq c_3 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q} - \frac{1}{p}} ,$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ and $\mu \geq 0$, while the upper bound holds when $\mu = 0$ and $0 < q \leq p \leq \infty$, and when $\mu \geq 1$, $p \geq 1$, and $0 < q \leq p \leq \infty$. Also, there are constants $c_2 = c_2(q,\mu) > 0$ and $c_3 = c_3(q,\mu)$ depending only on q and μ such that

$$c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q}} \leq \sup_{0 \neq P \in E(\Lambda_n)} \frac{|P^{(\mu)}(y)|}{\|P\|_{L_q(-\infty,y]}} \leq c_3 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q}}$$

for all $0 < q \le \infty$, $\mu \ge 1$, *and* $y \in \mathbb{R}$.

Motivated by a question of Michel Weber, in [66] we proved the following couple of theorems.

Theorem 8.5. *Suppose* $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, *and* $a < b$. *There are constants* $c_4 = c_4(p,q,a,b) > 0$ *and* $c_5 = c_5(p,q,a,b)$ *depending only on* p, q, a, *and* b *such that*

$$c_4 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q}-\frac{1}{p}} \le \sup_{0 \ne P \in E(\Lambda_n)} \frac{\|P\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \le c_5 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{\frac{1}{q}-\frac{1}{p}} .$$

Theorem 8.6. *Suppose* $0 < q \le p \le \infty$, $a, b \in \mathbb{R}$, *and* $a < b$. *There are constants* $c_6 = c_6(p,q,a,b) > 0$ *and* $c_7 = c_7(p,q,a,b)$ *depending only on* p, q, a, *and* b *such that*

$$c_6 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1+\frac{1}{q}-\frac{1}{p}} \le \sup_{0 \ne P \in E(\Lambda_n)} \frac{\|P'\|_{L_p[a,b]}}{\|P\|_{L_q[a,b]}} \le c_7 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right)^{1+\frac{1}{q}-\frac{1}{p}} ,$$

where the lower bound holds for all $0 < q \le p \le \infty$, *while the upper bound holds when* $p \ge 1$ *and* $0 < q \le p \le \infty$.

It has been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than that for all polynomials. He proved that

$$\sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} = \begin{cases} \frac{1}{4}(n+1)^2 , & \text{if } n \text{ is odd} \\ \frac{1}{4}n(n+2) , & \text{if } n \text{ is even} , \end{cases}$$

where the supremum is taken for all polynomials $0 \ne p$ of degree at most n that are monotone on $[-1,1]$. See [121, p. 607], for instance.

In [68] an effort is made to extend the above results of Bernstein to the classes $E(\Lambda_n)$. We proved the following couple of results.

Theorem 8.7. *Let* $n \ge 2$ *be an integer,* $b \in \mathbb{R}$. *Then there is an absolute constant* $c_1 > 0$ *such that*

$$\frac{c_1}{\log n} \sum_{j=0}^{n} \lambda_j \le \sup_P \frac{\|P'\|_{(-\infty,b]}}{\|P\|_{(-\infty,b]}} \le 9 \sum_{j=0}^{n} \lambda_j ,$$

where the supremum is taken for all $0 \ne P \in E(\Lambda_n)$ *increasing on* $(-\infty,\infty)$.

Theorem 8.8. *Let* $n \ge 2$ *be an integer. Let* $[a,b]$ *be a finite interval with length* $b-a > 0$. *There are positive constants* $c_2 = c_2(a,b)$ *and* $c_3 = c_3(a,b)$ *depending only on* a *and* b *such that*

$$c_2 \left(n^2 + \frac{1}{\log n} \sum_{j=0}^{n} |\lambda_j| \right) \le \sup_P \frac{\|P'\|_{[a,b]}}{\|P\|_{[a,b]}} \le c_3 \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right) ,$$

where the supremum is taken for all $0 \ne P \in E(\Lambda_n)$ *increasing on* $(-\infty,\infty)$.

It is expected that the factor $1/\log n$ in the above theorems can be dropped.

Most of the results in this section are fairly recent. I think Lorentz would like the results in this section and might include some of them rather than Theorems 8.3 and 8.4 in [105, p. 367] with proofs.

9. Littlewood Polynomials

The well known Littlewood Conjecture was solved by Konyagin [93] and independently by McGehee, Pigno, and B. Smith [107]. Based on these Lorentz worked out a textbook proof of the conjecture in [44].

Theorem 9.1. *Let $n_1, n_2, \dots, n_N$ be distinct integers. For some absolute constant $c > 0$,*

$$\int_0^{2\pi} \left| \sum_{k=1}^{N} e^{in_k t} \right| dt \geq c \log N \,.$$

This is an obvious consequence of

Theorem 9.2. *Let $n_1 < n_2 < \cdots < n_N$ be integers. Let $a_1, a_2, \dots, a_k$ be arbitrary complex numbers. We have*

$$\int_0^{2\pi} \left| \sum_{k=1}^{N} a_k e^{in_k t} \right| dt \geq \frac{1}{30} \sum_{k=1}^{N} \frac{|a_k|}{k} \,.$$

I read the proof of Theorem 9.2 presented in Lorentz's book [44] with special interest. This was one of my main motivations to start working on unimodular polynomials (polynomials with complex coefficients of modulus 1), and Littlewood polynomials (polynomials with coefficients from $\{-1, 1\}$), and with other classes of polynomials with various other coefficient constraints.

Pichorides, who contributed essentially to the proof of the Littlewood conjecture, observed in [119] that the original Littlewood conjecture (when all the coefficients are from $\{0, 1\}$ would follow from a result on the L_1 norm of such polynomials on sets $E \subset \partial D$ of measure π. Namely if

$$\int_E \Big| \sum_{j=0}^{n} z^{k_j} \Big| \, |dz| \geq c$$

for any subset $E \subset \partial D$ of measure π with an absolute constant $c > 0$, then the original Littlewood conjecture holds. Throughout this section the measure of a set $E \subset \partial D$ is the linear Lebesgue measure of the set

$$\{t \in [-\pi, \pi) : e^{it} \in E\} \,.$$

Konyagin [92] gives a lovely probabilistic proof showing that this hypothesis fails. He does however conjecture the following: for any *fixed* set $E \subset \partial D$ of positive measure there exists a constant $c = c(E) > 0$ depending only on E such that

$$\int_E \Big| \sum_{j=0}^{n} z^{k_j} \Big| \, |dz| \geq c(E) \,.$$

In other words the sets $E_\varepsilon \subset \partial D$ of measure π in his example where

$$\int_{E_\varepsilon} \left| \sum_{j=0}^{n} z^{k_j} \right| |dz| < \varepsilon$$

must vary with $\varepsilon > 0$.

In [29] we show, among other things, that Konyagin's conjecture holds on subarcs of the unit circle ∂D.

In [84] S. Güntürk constructs certain types of near-optimal approximations of a class of analytic functions in the unit disk by power series with two distinct coefficients. More precisely, it is shown that if all the coefficients of the power series $f(z)$ are real and lie in $[-\mu, \mu]$, where $\mu < 1$, then there exists a power series $Q(z)$ with coefficients in $\{-1, +1\}$ such that $|f(z) - Q(z)| \to 0$ at the rate $\exp(C|1-z|^{-1})$ as $z \to 1$ non-tangentially inside the unit disk. Güntürk refers to P. Borwein, Erdélyi, and Kós in [34] to see that this type of decay rate is best possible. The special case $f \equiv 0$ yields a near-optimal solution to the "fair duel problem" of Konyagin, as it is described in the Introduction of [84].

In [34] we consider the problem of minimizing the uniform norm on $[0, 1]$ over polynomials $0 \neq p$ of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}\,,$$

where the modulus of the first non-zero coefficient is at least $\delta > 0$. Essentially sharp bounds are given for this problem. An interesting related result states that there are: absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp\left(-c_1\sqrt{n}\right) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{[0,1]} \leq \exp\left(-c_2\sqrt{n}\right)$$

for every $n \geq 2$, where $\mathcal{F}_n$ denotes the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

The results of [29] show that many types of polynomials cannot be small on subarcs of the unit circle in the complex plane. A typical result of [29] is the following. There are absolute constants $c_1 > 0$, $c_2 > 0$, and $c_3 > 0$ such that

$$\exp(-c_1/a) \leq \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_{L_1(A)}\,, \qquad \inf_{0 \neq p \in \mathcal{F}_n} \|p\|_A \leq \exp(-c_2/a)$$

for every subarc A of the unit circle $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ with length $0 < a < c_3$.

The lower bound results extend to the class of f of the form

$$f(z) = \sum_{j=m}^{n} a_j z^j\,, \quad a_j \in \mathbb{C}\,, \quad |a_j| \leq M\,, \quad |a_m| = 1\,,$$

with varying nonnegative integers $m \leq n$. It is also shown in [29] that functions f of the above form cannot be arbitrarily small uniformly on subarcs of the circle. However, this does not extend to sets of positive measure. It is shown that it is possible to find a polynomial of the above form that is arbitrarily small on as much of the boundary (in the sense of linear Lebesgue measure) as one likes.

The height of a polynomial

$$p_n(z) := \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \mathbb{C}\,, \quad a_n \neq 0\,,$$

is defined by

$$H(p_n) := \max\left\{\frac{|a_j|}{|a_n|}:\ \ j = 0, 1, \ldots, n\right\}.$$

An easy to formulate corollary of the results of [29] is the following.

Corollary 9.3. *Let A be a subarc of the unit circle with length $\ell(A) = a$. If (p_k) is a sequence of monic polynomials that tends to 0 in $L_1(A)$, then the sequence $H(p_k)$ of heights tends to ∞.*

In [24] We are concerned with the problem of minimizing the supremum norm, on an interval, of a nonzero polynomial of degree at most n with integer coefficients. This is an old and hard problem that cannot be exactly solved in any nontrivial cases. See the references in [24]. We examined the case of the interval $[0,1]$ in most detail. We improved the known bounds by a small but interesting amount. This allowed us to garner further information about the structure of such minimal polynomials and their factors. This was primarily a (substantial) computational exercise. We also examined some of the structure of such minimal "integer Chebyshev "polynomials. We disproved conjecture 36 (ascribed to the Chudnovsky brothers and others) in [111, p. 201]. In recent years a number of papers related to [24] were published. See [39], [86], and [138], for example, and the references in them.

In 1945 Duffin and Schaeffer proved that any power series that is bounded in a sector of the open unit disk and has coefficients from a finite subset of $\mathbb{C}$ is already a rational function. Their proof is relatively indirect. It is one purpose of [37] to give a shorter direct proof of this beautiful and surprising theorem. An easy consequence of this, for example, is that any algebraic function that has a power series expansion on the open unit disk with coefficients from a finite subset of $\mathbb{C}$ is, in fact, a rational function.

10. Inequalities for Generalized Polynomials

The function

$$f(x) := |\omega| \prod_{j=1}^{m} |x - z_j|^{r_j} \tag{10.1}$$

with $0 < r_j \in \mathbb{R}\,, z_j \in \mathbb{C}$, and $0 \neq \omega \in \mathbb{C}$ is called a generalized (algebraic) polynomial of degree $N := \sum_{j=1}^{m} r_j$. If f is a positive constant identically, its degree is defined to be 0, while if f is identically 0, its degree is defined to be -1. Let GAP_N be the set of all generalized algebraic polynomials of degree at most N. If in the representation (10.1) of f all the exponents r_j are integers, then f is the absolute value of an ordinary algebraic polynomial (of degree N). If $f \in \mathrm{GAP}_N$ is of the form (10.1) with distinct $z_j \in \mathbb{C}$, then the numbers z_j are called the zeros of f, while the exponent r_j is called the multiplicity of the zero z_j in f.

The function

$$P(x) := a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)\,, \qquad a_k, b_k \in \mathbb{R}\,, \quad a_n b_n \neq 0\,,$$

is called a real trigonometric polynomial of degree n. It is well-known that every real trigonometric polynomial P of degree (order) n can be written as

$$P(x) = \omega \prod_{j=1}^{2n} \sin((x - z_j)/2)\,,$$

where $\omega \in \mathbb{R}$, $z_j \in \mathbb{C}$, and the non-real zeros z_j of P form conjugate pairs. The function

$$P(x) := \omega \prod_{j=1}^{m} |\sin((x - z_j)/2)|^{r_j}\,, \qquad x \in \mathbb{R}\,, \tag{10.2}$$

where $0 < r_j \in \mathbb{R}$, $z_j \in \mathbb{C}$ are distinct (mod 2π), and $0 < \omega \in \mathbb{R}$, is called a generalized trigonometric polynomial of degree $N := \frac{1}{2}\sum_{j=1}^{m} r_j$. If P is a constant identically, then its degree is defined to be 0. Note that the absolute value of a real trigonometric polynomial of degree n may be viewed as a generalized trigonometric polynomial of degree n. Let GTP_N denote the set of all generalized trigonometric polynomials of degree at most N. Observe that if $P \in \mathrm{GTP}_N$ is of the form (10.2), then

$$P(x) := \omega \prod_{j=1}^{s} (\sin((x - z_j)/2) \sin((x - \overline{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}\,, \qquad x \in \mathbb{R}\,,$$

where each T_j is a real trigonometric polynomial of degree 1 being nonnegative on the real line. For a $P \in \mathrm{GTP}_N$ of the form (10.2) the numbers z_j are called the zeros of P, while the exponent r_j is called the multiplicity of the zero z_j in P.

The problem arises how to define f' for an $f \in \mathrm{GAP}_N$ and P' for a $P \in \mathrm{GTP}_N$. Observe that if $r_j \geq 1$ for each $j = 1, 2, \ldots, m$ in (10.2), then, although P' may not exist at the zeros of P, the one-sided derivatives P'_- and P'_+ exist, and their absolute values are equal. This means $|P'|$ is well defined on the real line by either $|P'_-|$ or $|P'_+|$. Similarly, if $r_j \geq 1$ for each $j = 1, 2, \ldots, m$ in (10.1), then f' is well defined on the real line. It is a simple exercise to check that if $P \in \mathrm{GTP}_N$ has only real zeros with multiplicities at least 1, then $|P'| \in \mathrm{GTP}_N$ has only real zeros as well, and at least one of any two adjacent zeros of $|P'|$ has multiplicity exactly 1. A similar comment can be made on $f \in \mathrm{GAP}_N$ having only real zeros with multiplicities at least 1.

One might expect to extend various polynomial inequalities to generalized polynomials by writing the generalized degree N in place of the ordinary degree n. However, even when it is possible, such an extension might be far from obvious. Remez-type inequalities serve as examples of a situation when a polynomial inequality easily extends to generalized polynomials.

In [76] we started with the (straightforward) extension of the algebraic and trigonometric Remez-type inequalities to GAP_N and GTP_N, and proved the following results.

Theorem 10.1. *There is an absolute constant* $0 < c_1 < 1$ *such that*

$$m\left(y \in [-1,1] : f(y) \geq \exp(-N\sqrt{s}) \max_{-1 \leq x \leq 1} f(x)\right) \geq c_1 s$$

for every $f \in \mathrm{GAP}_N$ *and* $0 < s < 2$.

Theorem 10.2. *There is an absolute constant* $0 < c_2 < 1$ *such that*

$$m\left(t \in [-\pi,\pi] : P(t) \geq \exp(-Ns) \max_{-\pi \leq \tau \leq \pi} P(\tau)\right) \geq c_2 s$$

for every $P \in \mathrm{GTP}_N$ *and* $0 < s < 2\pi$.

As a consequence, in [76] we obtained the Nikolskii-type inequalities below.

Theorem 10.3. *Let* χ *be a nonnegative, nondecreasing function defined in* $[0,\infty)$ *such that* $\chi(x)/x$ *is non-increasing in* $[0,\infty)$. *Then for* $0 < q < p \leq \infty$ *we have*

$$\|\chi(P)\|_{L_p[-\pi,\pi]} \leq (c_3(1+qN))^{1/q-1/p} \|\chi(P)\|_{L_q[-\pi,\pi]}\,, \qquad P \in \mathrm{GTP}_N\,,$$

where c_3 *is an absolute constant. If* $\chi(x) = x$, *then* $c_3 = e(4\pi)^{-1}$ *is suitable.*

Theorem 10.4. *Let* χ *be a nonnegative, nondecreasing function defined in* $[0,\infty)$ *such that* $\chi(x)/x$ *is non-increasing in* $[0,\infty)$. *Then for* $0 < q < p \leq \infty$ *we have*

$$\|\chi(f)\|_{L_p[-1,1]} \leq (c_4(1+qN))^{2/q-2/p} \|\chi(f)\|_{L_q[-1,1]}\,, \qquad f \in \mathrm{GAP}_N\,,$$

where c_4 *is an absolute constant. If* $\chi(x) = x$, *then* $c_4 = e^2(2\pi)^{-1}$ *is suitable.*

In [50] and [76] we have proved even (essentially) sharp Markov- and Bernstein-type inequalities for the classes GAP_N and GTP_N in L_p norms. In [77] we extended these results to the setting below.

Theorem 10.5. *Let* $0 < p \leq \infty$. *We have*

$$\|P'Q\|_{L_p[-\pi,\pi]} \leq c^{1+1/p}(N+M)\log(\min(N,M+1)+1)\|PQ\|_{L_p[-\pi,\pi]}$$

for any two $P \in \mathrm{GTP}_N$ *and* $Q \in \mathrm{GTP}_M$ *such that the roots of* P *and* Q *have multiplicities at least* 1. *Moreover, this inequality is sharp up to the constant* $c^{1+1/p}$.

Theorem 10.6. *Let $0 < p \le \infty$. We have*

$$\|\sqrt{1-x^2}f'(x)g(x)\|_{L_p[-1,1]} \le c^{1+1/p}(N+M)\log(\min(N,M+1)+1)\|fg\|_{L_p[-1,1]}$$

for any two $f \in \mathrm{GAP}_N$ and $g \in \mathrm{GAP}_M$ such that the roots of f and g have multiplicities at least 1. This inequality is sharp up to the constant $c^{1+1/p}$.

Theorem 10.7. *Let $0 < p \le \infty$. We have*

$$\|f'g\|_{L_p[-1,1]} \le c^{1+1/p}(N+M)^2\|fg\|_{L_p[-1,1]}$$

for any two $f \in \mathrm{GAP}_N$ and $g \in \mathrm{GAP}_M$ such that the roots of f have multiplicities at least 1. This inequality is sharp up to the factor $c^{1+1/p}$ for all $N, M \ge 1$.

In [74] we combined some of our inequalities for generalized polynomials with some other ideas and obtained the following result.

Theorem 10.8. *For all Jacobi weight functions $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha \ge -1/2$ and $\beta \ge -1/2$, the inequalities*

$$\max_{x\in[-1,1]} \frac{p_n^2(w,x)}{\sum_{k=0}^n p_k^2(w,x)} \le \frac{4\left(2+\sqrt{\alpha^2+\beta^2}\right)}{2n+\alpha+\beta+2}$$

and

$$\max_{x\in[-1,1]} \sqrt{1-x^2}w(x)p_n^2(w,x) \le \frac{2e\left(2+\sqrt{\alpha^2+\beta^2}\right)}{\pi}$$

hold for $n = 0, 1, \dots,$ where $p_n(w,x)$ denote the orthonormal polynomials of degree n associated with the weight w on $[-1,1]$.

11. Markov- and Bernstein-type inequalities for rational functions

We denote by $\mathcal{P}_n^r$ and $\mathcal{P}_n^c$ the sets of all algebraic polynomials of degree at most n with real or complex coefficients, respectively. The sets of all trigonometric polynomials of degree at most n with real or complex coefficients, respectively, are denoted by $\mathcal{T}_n^r$ and $\mathcal{T}_n^c$. We will use the notation

$$\|f\|_A = \sup_{z\in A}|f(z)|$$

for continuous functions f defined on A. Let

$$D := \{z \in \mathbb{C} : |z| \le 1\}, \qquad \partial D := \{z \in \mathbb{C} : |z| = 1\}, \qquad K := \mathbb{R}\,(\mathrm{mod} 2\pi)\,.$$

The classical inequalities of Bernstein [8] state that

$$\begin{aligned}
&|p'(z_0)| \le n\|p\|_{\partial D}, && p \in \mathcal{P}_n^c, \quad z_0 \in \partial D,\\
&|t'(\theta_0)| \le n\|t\|_K, && t \in \mathcal{T}_n^c, \quad \theta_0 \in K,\\
&|p'(x_0)| \le \frac{n}{\sqrt{1-x_0^2}}\|p\|_{[-1,1]}, && p \in \mathcal{P}_n^c, \quad x_0 \in (-1,1).
\end{aligned}$$

Proofs of the above inequalities may be found in almost every book on approximation theory, see [104], for instance. An extensive study of Markov- and Bernstein-type inequalities is presented in [120],[121], and [17].

In [19] we study the rational function spaces:

$$\mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \partial D) := \left\{ \frac{p_n(z)}{\prod_{j=1}^{n} (z - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on ∂D with $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus \partial D$;

$$\mathcal{T}_n^c(a_1, a_2, \ldots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} \sin((\theta - a_j)/2)} : t_n \in \mathcal{T}_n^c \right\}$$

on K with $\{a_1, a_2, \ldots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$;

$$\mathcal{P}_n^c(a_1, a_2, \ldots, a_n; [-1, 1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^{n} (x - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $[-1, 1]$ with $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus [-1, 1]$;

$$\mathcal{P}_n^c(a_1, a_2, \ldots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^{n} (x - a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $\mathbb{R}$ with $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$, and

$$\mathcal{P}_n^r(a_1, a_2, \ldots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^{n} |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $\mathbb{R}$ with $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

The spaces

$$\mathcal{T}_n^r(a_1, a_2, \ldots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} |\sin((\theta - a_j)/2)|} : t_n \in \mathcal{T}_n^r \right\}$$

on K with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$ and

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1,1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^{n} |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $[-1,1]$ with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1,1]$ have been studied in [36] and [17], and the sharp Bernstein-Szegő type inequalities

$$f'(\theta_0)^2 + \widetilde{B}_n(\theta_0)^2\, f(\theta_0)^2 \le \widetilde{B}(\theta_0)^2 \|f\|_K^2, \qquad \theta_0 \in K\,,$$

for every $f \in \mathcal{T}_n^r(a_1, a_2, \dots, a_{2n}; K)$ with

$$(11.1) \qquad (a_1, a_2, \dots, a_{2n}) \subset \mathbb{C} \setminus \mathbb{R}, \qquad \mathrm{Im}(a_j) > 0, \quad j = 1, 2, \dots, 2n\,,$$

and

$$(11.2) \qquad (1 - x_0^2) f'(x_0)^2 + B_n(x_0)^2 f(x_0)^2 \le B_n(x_0)^2 \|f\|_{[-1,1]}^2, \qquad x_0 \in (-1,1)\,,$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1,1])$ with

$$\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1,1]$$

have been proved, where

$$\widetilde{B}_n(\theta) := \frac{1}{2} \sum_{j=1}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta}|^2}\,, \qquad \theta \in K$$

and

$$B_n(x) := \mathrm{Re} \left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{a_j - x} \right), \qquad x \in [-1,1]$$

with the choice of $\sqrt{a_j^2 - 1}$ is determined by

$$\left| a_j - \sqrt{a_j^2 - 1} \right| < 1.$$

These inequalities give sharp upper bound for $|f'(\theta_0)|$ and $|f'(x_0)|$ only at n points in K and $[-1,1]$, respectively. In [19] we establish Bernstein-type inequalities for the spaces

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n, \partial D) \quad \text{and} \quad \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)$$

which are sharp at every $z \in \partial D$ and $\theta \in K$, respectively. An essentially sharp Bernstein-type inequality is also established for the space

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_;[-1,1])\,.$$

A Bernstein-type inequality of Russak [118] is extended to the spaces

$$\mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R})\,,$$

and a Bernstein-Szegő type inequality is established for the spaces

$$\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R})\,.$$

For a polynomial

$$q_n(z) = c\prod_{j=1}^n (z - a_j), \qquad 0 \neq c \in \mathbb{C}, \qquad a_j \in \mathbb{C}$$

we define

$$q_n^*(z) = \overline{c}\prod_{j=1}^n (1 - \overline{a}_j z) = z^n \overline{q}_n(z^{-1})\,.$$

It is well-known, and simple to check, that

$$|q_n(z)| = |q_n^*(z)|, \qquad z \in \partial D\,.$$

We also define the Blaschke products

$$S_n(z) := \prod_{j=1}^n \frac{1 - \overline{a}_j z}{z - a_j}$$

associated with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$, and

$$\widetilde{S}_n(z) := \prod_{j=1}^n \frac{z - \overline{a}_j}{z - a_j}$$

associated with $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

In [19] we proved the following five theorems. The first one is called the "Borwein-Erdélyi inequality" in [130].

Theorem 11.1. *Let $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \partial D$. Then*

$$|f'(z_0)| \leq \max\left\{ \sum_{\substack{j=1\\|a_j|>1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1\\|a_j|<1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \partial D)$ and $z_0 \in \partial D$. If the first sum is not less than the second sum for a fixed $z_0 \in \partial D$, then equality holds for $f = c\,S_n^+$, $c \in \mathbb{C}$, where S_n^+ is the Blaschke product associated with those a_j for which $|a_j| > 1$. If the first sum is not greater than the second sum for a fixed $z_0 \in \partial D$, then equality holds for $f = c\,S_n^-$, $c \in \mathbb{C}$, where S_n^- is the Blaschke product associated with those a_j for which $|a_j| < 1$.

Theorem 11.2. *Let* $\{a_1, a_2, \dots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(\theta_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)<0}}^{2n} \frac{|e^{ia_j}|^2 - 1}{|e^{ia_j} - e^{i\theta_0}|^2}, \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)>0}}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta_0}|^2} \right\} \|f\|_K$$

for every $f \in \mathcal{T}_n^c(a_1, a_2, \dots, a_{2n}; K)$ *and* $\theta_0 \in K$. *If the first sum is not less than the second sum for a fixed* $\theta_0 \in K$, *then equality holds for* $f(\theta) = cS_{2n}^+(e^{i\theta})$, $c \in \mathbb{C}$. *If the first sum is not greater than the second sum for a fixed* $\theta_0 \in K$, *then equality holds for* $f(\theta) = cS_{2n}^-(e^{i\theta})$, $c \in \mathbb{C}$. S_{2n}^+ *and* S_{2n}^- *associated with* $\{e^{ia_1}, e^{ia_2}, \dots, e^{ia_{2n}}\}$ *are defined as in Theorem 1.*

Theorem 11.3.. *Let* $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus [-1, 1]$ *and*

$$c_j := a_j - \sqrt{a_j^2 - 1}, \qquad |c_j| < 1$$

with the choice of root in $\sqrt{a_j^2 - 1}$ *determined by* $|c_j| < 1$. *Then*

$$|f'(x_0)| \leq \frac{1}{\sqrt{1 - x_0^2}} \max \left\{ \sum_{j=1}^{n} \frac{|c_j|^{-2} - 1}{|c_j^{-1} - z_0|^2}, \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|c_j - z_0|^2} \right\} \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; [-1, 1])$ *and* $x_0 \in (-1, 1)$, *where* z_0 *is defined by*

$$z_0 := x_0 + i\sqrt{1 - x_0^2}, \qquad x_0 \in (-1, 1).$$

Note that

$$B_n(x_0) = \mathrm{Re} \left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{a_j - x_0} \right) = \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|c_j - z_0|^2}, \qquad x_0 \in (-1, 1).$$

Our next result extends an inequality established by Russak [118] to wider families of rational functions.

Theorem 11.4. *Let* $\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(x_0)| \leq \max \left\{ \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)>0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2}, \sum_{\substack{j=1 \\ \mathrm{Im}(a_j)<0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2} \right\} \|f\|_{\mathbb{R}}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \dots, a_n; \mathbb{R})$ and $x_0 \in \mathbb{R}$. If the first sum is not less than the second sum for a fixed $x_0 \in \mathbb{R}$, then equality holds for $f = c\tilde{S}_n^+$, $c \in \mathbb{C}$, where $\tilde{S}_n^+$ is the Blaschke product associated with the poles a_j lying in the upper half-plane

$$H^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

If the first sum is not greater than the second sum for a fixed $x_0 \in \mathbb{R}$, then equality holds for $f = c\tilde{S}_n^-$, $c \in \mathbb{C}$, where $\tilde{S}_n^-$ is the Blaschke product associated with the poles a_j lying in the lower half-plane

$$H^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}.$$

Our last result in [19] is a Bernstein-Szegő type inequality for $\mathcal{P}_n^r(a_1, a_2, \dots, a_{2n}; \mathbb{R})$. It follows from the Bernstein-Szegő type inequality (11.2) for $\mathcal{P}_n^r(a_1, a_2, \dots, a_n; [-1, 1])$.

Theorem 11.5. *Let*

$$\{a_1, a_2, \dots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}, \qquad \mathrm{Im}(a_j) > 0, \quad j = 1, 2, \dots, n.$$

Then

$$f'(x_0)^2 + \widehat{B}_n(x_0)^2 f(x_0)^2 \le \widehat{B}_n(x_0)^2 \|f\|_{\mathbb{R}}^2, \qquad x_0 \in \mathbb{R},$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R})$, where

$$\widehat{B}_n(x) := \sum_{j=1}^{n} \frac{\mathrm{Im}(a_j)}{|x - a_j|^2}, \qquad x \in \mathbb{R}.$$

We remark that equality holds in Theorem 11.5 if and only if x_0 is a maximum point of f (i.e. $f(x_0) = \pm\|f\|_{\mathbb{R}}$) or f is a "Chebyshev polynomial" for the space $\mathcal{P}_n^r(a_1, a_2, \dots, a_n; \mathbb{R})$ which can be explicitly expressed by using the results of [36] and [17].

Note that Bernstein's classical inequalities are contained in Theorem 11.1, 11.2, and 11.3 as limiting cases, by taking

$$\{a_1^{(k)}, a_2^{(k)}, \dots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

in Theorems 11.1 and 11.3 so that $\lim_{k\to\infty} |a_j^{(k)}| = \infty$ for each $j = 1, 2, \dots, n$, and by taking

$$\{a_1^{(k)}, a_2^{(k)}, \dots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

in Theorem 11.2 so that $a_{n+j}^{(k)} = \overline{a}_j^{(k)}$ and $\lim_{k\to\infty} |\mathrm{Im}(a_j^{(k)})| = \infty$ for each $j = 1, 2, \dots, n$. Further results can be obtained as limiting cases by fixing $a_1, a_2, \dots, a_m$, $1 \le m \le n$, in Theorems 11.1 and 11.3, and by taking

$$\{a_1, a_2, \dots, a_m, a_{m+1}^{(k)}, a_{m+2}^{(k)}, \dots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

so that $\lim_{k\to\infty} |a_j^{(k)}| = \infty$ for each $j = m+1, m+2, \dots, n$. One may also fix the poles $a_1, a_2, \dots, a_m$, $a_{n+1}, a_{n+2}, \dots, a_{n+m}$, $1 \le m \le n$, in Theorem 11.2 and take

$$\{a_1, \dots, a_m, a_{m+1}^{(k)}, \dots, a_n^{(k)}, a_{n+1}, \dots, a_{n+m}, a_{n+m+1}^{(k)}, \dots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

so that $a_{n+j}^{(k)} = \overline{a}_j^{(k)}$ and $\lim_{k\to\infty} |\mathrm{Im}(a_j^{(k)})| = \infty$ for each $j = m+1, m+2, \dots, n$.

12. Nikolskii-type inequalities for shift-invariant function spaces

The well known results of Nikolskii assert that the essentially sharp inequality

$$\|h_n\|_{L_q[-1,1]} \leq c(p,q)n^{2/p-2/q}\|h_n\|_{L_p[-1,1]}$$

holds for all algebraic polynomials h_n of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$, while the essentially sharp inequality

$$\|t_n\|_{L_q[-\pi,\pi]} \leq c(p,q)n^{1/p-1/q}\|t_n\|_{L_p[-\pi,\pi]}$$

holds for all trigonometric polynomials t_n of degree at most n with complex coefficients and for all $0 < p < q \leq \infty$. The subject started with two famous papers [116] and [134]. There are quite a few related papers in the literature.

Let V_n be a vector space of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. We say that V_n is shift invariant (on $\mathbb{R}$) if $f \in V_n$ implies that $f_a \in V_n$ for every $a \in \mathbb{R}$, where $f_a(x) := f(x-a)$ on $\mathbb{R}$. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of distinct COMPLEX numbers. The collection of all linear combinations of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{C}$ will be denoted by

$$E(\Lambda_n) := \operatorname{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums of $n+1$ terms. Examples of shift invariant spaces of dimension $n+1$ include $E(\Lambda_n)$. In [66] Theorem 8.5 is proved. Using the L_∞ norm on a fixed subinterval $[a+\delta, b-\delta] \subset [a,b]$ in the numerator in Theorem 8.5, we proved the following essentially sharp result in [27].

Theorem 12.1. *If $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ is a set of distinct real numbers, then the inequality*

$$\|f\|_{L_\infty[a+\delta,b-\delta]} \leq e8^{1/p}\left(\frac{n+1}{\delta}\right)^{1/p}\|f\|_{L_p[a,b]}$$

holds for every $f \in E(\Lambda_n)$, $p > 0$, and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$.

The key to this result is the Remez-type inequality of Theorem 2.3*. Having real exponents λ_j in the above theorems is essential in the proof using some Descartes system methods. In [32] we prove an analogous result for complex exponents λ_j, in which case Descartes system methods cannot help us in the proof.

Theorem 12.2. *Let $V_n \subset C[a,b]$ be a shift invariant vector space of complex-valued functions defined on $\mathbb{R}$ of dimension $n+1$ over $\mathbb{C}$. Let $p \in (0,2]$. Then*

$$\|f\|_{L_\infty[a+\delta,b-\delta]} \leq 2^{2/p^2}\left(\frac{n+1}{\delta}\right)^{1/p}\|f\|_{L_p[a,b]}$$

for every $f \in V_n$ and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$.

Problem 12.3. *Is it possible to extend a version of Theorem 12.2 to ALL $p > 0$?*

Let $\varepsilon \in [0,1]$ and let D_ε be the ellipse of the complex plane with large axis $[-1,1]$ and small axis $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(D_\varepsilon)$ denote the collection of all polynomials of degree n with complex coefficients and with all their zeros in D_ε. Extending a result of Turán [136], Erőd [81, III. tétel] proved that

$$c_1(n\varepsilon + \sqrt{n}) \le \inf_p \frac{\|p'\|_{D_\varepsilon}}{\|p\|_{D_\varepsilon}} \le c_2(n\varepsilon + \sqrt{n})\,,$$

where the infimum is taken for all $p \in \mathcal{P}_n^c(D_\varepsilon)$ Recently Levenberg and Poletcky [95] rediscovered this beautiful result.

Let $\varepsilon \in [0,1]$ and let S_ε be the diamond of the complex plane with diagonals $[-1,1]$ and $[-i\varepsilon, i\varepsilon]$. Let $\mathcal{P}_n^c(S_\varepsilon)$ denote the collection of all polynomials of degree n with complex coefficients and with all their zeros in S_ε. Let

$$\|f\|_A := \sup_{z \in A} |f(z)|$$

for complex-valued functions defined on A.

In [69] the following result is proved.

Theorem 13.1. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1(n\varepsilon + \sqrt{n}) \le \inf_p \frac{\|p'\|_{S_\varepsilon}}{\|p\|_{S_\varepsilon}} \le c_2(n\varepsilon + \sqrt{n})\,,$$

where the infimum is taken for all $p \in \mathcal{P}_n^c(S_\varepsilon)$ with the property

$$|p(z)| = |p(-z)|\,, \qquad z \in \mathbb{C}\,,$$

or where the infimum is taken for all real $p \in \mathcal{P}_n^c(S_\varepsilon)$.

It is an interesting question whether or not the lower bound in Theorem 13.1 holds for all $p \in \mathcal{P}_n^c(\varepsilon)$. As our next result in [69] shows this is the case at least when $\varepsilon = 1$.

Theorem 13.2. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 n \le \inf_p \frac{\|p'\|_{S_1}}{\|p\|_{S_1}} \le c_2 n\,,$$

where the infimum is taken for all (complex) $p \in \mathcal{P}_n^c(S_1)$.

Motivated by the author's initial results in this section, Sz. Révész [123] established the right order Turán -type converse Markov inequalities on convex domains of the complex plane. His main theorem contains the results in this section as special cases. Révész' proof is also elementary, but rather subtle.

Theorem 13.3. *Let $K \subset \mathbb{C}$ be a bounded convex domain. Then for every $p \in \mathcal{P}_n^c$ having no zeros in K we have*

$$\frac{\|p'\|_K}{\|p\|_K} \geq c(K)n \qquad \text{with} \qquad c(K) = 0.0003\frac{w(K)}{d(K)^2}\,,$$

where $d(K)$ is the diameter of K and

$$w(K) := \min_{\gamma \in [-\pi,\pi]} \left(\max_{z \in K} \operatorname{Re}(ze^{-i\gamma}) - \min_{z \in K} \operatorname{Re}(ze^{-i\gamma}) \right)$$

is the minimal width of K.

In particular, the lower bound in Theorem 13.1 holds for all $p \in \mathcal{P}_n^c(\varepsilon)$.

14. Ultraflat sequences of Unimodular Polynomials

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. The class

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k,\, a_k \in \mathbb{C}\,,\, |a_k| = 1 \right\}$$

is often called the collection of all (complex) unimodular polynomials of degree n. The class

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\}$$

is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi \sum_{k=0}^{n} |a_k|^2 = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z \in \partial D} |P_n(z)|\,.$$

An old problem (or rather an old theme) is the following.

Problem 14.1 (Littlewood's Flatness Problem). *How close can a $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1}\,, \qquad z \in \partial D? \tag{14.1}$$

Obviously (14.1) is impossible if $n \geq 1$. So one must look for less than (14.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [98] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that

$$(n+1)^{-1/2}|P_n(e^{it})|$$

converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 14.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}\,, \qquad z \in \partial D\,.$$

Definition 14.2*. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of ultraflat sequences of unimodular polynomials seemed very unlikely in in view of a 1957 conjecture of P. Erdős (Problem 22 in [79]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{14.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [91], Kahane [90] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)\,.$$

(Kahane's paper contained a slight error though which was corrected in [125].) Thus the Erdős conjecture (14.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given by Beck [5]. For an account of some of the work done till the mid 1960's, see Littlewood's book [99] and [125].

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ be (ε_n)-ultraflat. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,,\;\; R_n(t) = |P_n(e^{it})|\,, \qquad t \in \mathbb{R}\,. \tag{14.3}$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable on $\mathbb{R}$. This property of α_n is assumed in the rest of this section. The structure of ultraflat sequences of unimodular polynomials is studied in [56], [58], [59], and [60] where several conjectures of Saffari [126] (see also [124] and [125]) are proved.

Theorem 14.3 (Uniform Distribution Theorem for the Angular Speed). *Suppose (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (14.3), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$m(\{t \in [0, 2\pi] : 0 \leq \alpha_n'(t) \leq nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0,1]$, where

$$\lim_{n\to\infty} \max_{x \in [0,1]} |\gamma_n(x)| = 0\,.$$

Theorem 14.4 (Negligibility Theorem for the Higher Derivatives). *Suppose* (P_n) *is an ultraflat sequence of polynomials* $P_n \in \mathcal{K}_n$*. Then, with the notation (14.3), for every integer* $r \geq 2$*, we have*

$$\max_{0 \leq t \leq 2\pi} |\alpha_n^{(r)}(t)| \leq \gamma_{n,r} n^r$$

with suitable constants $\gamma_{n,r} > 0$ *converging to* 0 *for every fixed* $r = 2, 3, \ldots$.

Theorem 14.5 (The Moments of the Angular Speed). *Let* $q > 0$*. Suppose* (P_n) *is an ultraflat sequence of polynomials* $P_n \in \mathcal{K}_n$*. Then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} |\alpha_n'(t)|^q \, dt = \frac{n^q}{q+1} + \delta_{n,q} n^q \,,$$

and as a limit case,

$$\max_{0 \leq t \leq 2\pi} |\alpha_n'(t)| = n + \delta_n n \,.$$

with suitable constants $\delta_{n,q}$ *and* δ_n *converging to* 0 *as* $n \to \infty$ *for every fixed* q.

An immediate consequence of Theorem 14.5 is the remarkable fact that for large values of $n \in \mathbb{N}$, the $L_q(\partial D)$ Bernstein factors

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt}$$

of the elements of ultraflat sequences of polynomials $P_n \in \mathcal{K}_n$ are essentially independent of the polynomials. More precisely Theorem 14.5 and (14.3) impliy the following result.

Theorem 14.6 (The Bernstein Factors). *Let* (P_n) *be a* ε_n*-ultraflat sequence of polynomials* $P_n \in \mathcal{K}_n$*. For* $q > 0$ *we have*

$$\frac{\int_0^{2\pi} |P_n'(e^{it})|^q \, dt}{\int_0^{2\pi} |P_n(e^{it})|^q \, dt} = \frac{n^q}{q+1} + o_{n,q} n^q \,,$$

and as a limit case,

$$\frac{\max_{0 \leq t \leq 2\pi} |P_n'(e^{it})|}{\max_{0 \leq t \leq 2\pi} |P_n(e^{it})|} = n + o_n n \,.$$

with suitable constants $o_{n,q}$ *and* o_n *converging to* 0 *as* $n \to \infty$ *for every fixed* q.

In [59] an extension of Saffari's uniform distribution conjecture to higher derivatives is also proved.

Theorem 14.7. *Suppose* (P_n) *be a* ε_n*-ultraflat sequence of polynomials* $P_n \in \mathcal{K}_n$*. Then*

$$m\left\{t \in [0, 2\pi] : |P_n^{(r)}(e^{it})| \leq n^{r+1/2} x^r \right\} = 2\pi x + o_{r,n}(x)$$

for every $x \in [0,1]$*, where* $\lim_{n \to \infty} o_{r,n}(x) = 0$ *for every fixed* $r = 1, 2, \ldots$ *and* $x \in [0,1]$.

For every fixed $r = 1, 2, \ldots$, the convergence of $o_{n,r}(x)$ is uniform on $[0,1]$ by Dini's Theorem.

For continuous functions f defined on $[0, 2\pi]$, and for $q \in (0, \infty)$, we define

$$\|f\|_q := \left(\int_0^{2\pi} |f(t)|^q \, dt \right)^{1/q} .$$

We also define

$$\|f\|_\infty := \lim_{q \to \infty} \|f\|_q = \max_{t \in [0, 2\pi]} |f(t)| .$$

In [63], based on the results in [59], we resolved yet another conjecture of Saffari and Queffelec, see (1.30) in [125].

Theorem 14.8. *Let $q \in (0, \infty)$. If (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$, and $q \in (0, \infty)$, then for $f_n(t) := \mathrm{Re}(P_n(e^{it}))$ we have*

$$\|f_n\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{1/2}$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{(q+1)\Gamma\left(\frac{q}{2}+1\right)\sqrt{\pi}} \right)^{1/q} n^{3/2} ,$$

where Γ denotes the usual gamma function, and the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$.

In [60] we proved Saffari's "near-orthogonality conjecture" below.

Theorem 14.9. *Assume that (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then*

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = o(n) , \qquad P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k .$$

Here, as usual, $o(n)$ denotes a quantity for which $\lim_{n \to \infty} o(n)/n = 0$. The statement remains true if the ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, $0 < n_1 < n_2 < \cdots$.

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k , \qquad a_k \in \mathbb{C} ,$$

then its conjugate polynomial is defined by

$$Q_n^*(z) := z^n \overline{Q}_n(1/z) := \sum_{k=0}^{n} \overline{a}_{n-k} z^k .$$

In terms of the above definition Theorem 14.9 may be rewritten as

Theorem 14.10. *Assume that (P_n) is an ultraflat sequence of polynomials $P_n \in \mathcal{K}_n$. Then*
$$\frac{1}{2\pi}\int_{\partial D} |P_n(z) - P_n^*(z)|^2\,|dz| = 2n + o(n)\,.$$

We remark that straightforward modifications in the proofs of each of the above theorems yield that they remain true if the ultraflat sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of polynomials

$$P_{n_k} \in \mathcal{K}_{n_k}\,, \qquad 0 < n_1 < n_2 < \cdots\,.$$

In [31, Theorem 3] the following inequality has been observed.

Theorem 14.11. *Let P be a conjugate reciprocal unimodular polynomial of degree n. Then*
$$\max_{z\in\partial D} |P(z)| \geq (1+\varepsilon)\sqrt{n+1}$$
with $\varepsilon := \sqrt{4/3} - 1$.

In [126] another "near orthogonality" relation has been conjectured. Namely it was suspected that if (P_{n_m}) is an ultraflat sequence of polynomials $P_{n_m} \in \mathcal{K}_{n_m}$, then

$$\sum_{k=0}^{n} a_{k,n}\overline{a}_{n-k,n} = o(n)\,, \qquad P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k\,, \qquad n = n_m\,, \quad m = 1, 2, \ldots\,,$$

However, it was Saffari himself, who showed with Queffelec [125], that this could not be any farther away from being true. They constructed an ultraflat sequence (P_{n_m}) of plain-reciprocal polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ such that

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k\,, \qquad a_{k,n} = a_{n-k,n}\,, \qquad k = 0, 1, 2, \ldots, n\,,$$

and hence
$$\sum_{k=0}^{n} a_{k,n}\overline{a}_{n-k,n} = n+1\,, \qquad n = n_m\,, \quad m = 1, 2, \ldots\,.$$

15. Zeros of polynomials with restricted coefficients

Let $0 \leq n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T_N(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the already mentioned Littlewood's Conjecture with a textbook proof in [44]. It is not difficult to prove the statement in general even in the case $n_1 = 0$. One possible way is to use the identity
$$\sum_{j=1}^{n_N} T_N((2j-1)\pi/n_N) = 0\,.$$

See [94], for example. Another way is to use Theorem 2 of [108]. So there is certainly no shortage of possible approaches to prove the starting observation of this section even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication B. Conrey asked how fast the number of zeros of the above sums in a period tend to ∞ as a function N. In [43] the authors observed that for an odd prime p the Fekete polynomial $f_p(z) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$ (the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Conrey's question in general does not appear to be easy.

Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [99, problem 22] poses the following research problem, which appears to still be open: "If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m \theta)$? Possibly $N-1$, or not much less." Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half century. In a recent paper [38] we showed that this is false. There exist cosine polynomials $\sum_{m=1}^{N} \cos(n_m \theta)$ with the n_m integral and all different so that the number of its real zeros in the period is $O(N^{5/6} \log N)$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N} \cos(n_m \theta)$ always has many zeros in the period. In [33] we prove the following.

Theorem 15.1. *Suppose the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite and the set $\{j \in \mathbb{N} : a_j \neq 0\}$ is infinite. Let*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt)\,.$$

Then $\lim_{n \to \infty} \mathcal{N}(T_n) = \infty$., where $\mathcal{N}(T_n)$ denotes the number of sign changes of T_n in the period $[-\pi, \pi)$.

The book [13] deals with a number of related topics. Littlewood [96], [97], [98], [99], [100] was interested in many closely related problems. See also [31].

The study of the location of zeros of polynomials from

$$\mathcal{F}_n := \left\{ p : p(x) = \sum_{i=0}^{n} a_i x^i\,, \quad a_i \in \{-1, 0, 1\} \right\}$$

begins with Bloch and Pólya [10]. They prove that the average number of real zeros of a polynomial from $\mathcal{F}_n$ is at most $c\sqrt{n}$. They also prove that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn \log\log n}{\log n}$$

real zeros. This result, which appears to be the first on this subject, shows that polynomials from $\mathcal{F}_n$ do not behave like unrestricted polynomials. Schur [129] and by different methods Szegő [133] and Erdős and Turán [80] improve the above bound to $c\sqrt{n \log n}$ (see also [17]).

In [34] we give the right upper bound of $c\sqrt{n}$ for the number of real zeros of polynomials from a large class, namely the class of all polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad |a_j| \le 1\,, \quad |a_0| = |a_n| = 1\,, \quad a_j \in \mathbb{C}\,.$$

In [26] we extend this result by proving the following theorems.

Theorem 15.2. *Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| = 1\,, \quad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle, where the constant c depends only on the polygon.

Theorem 15.3. *There is an absolute constant c such that*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| = |a_n| = 1\,, \quad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most $c(n\alpha + \sqrt{n})$ zeros in the strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| \le \alpha\}$, and in the sector $\{z \in \mathbb{C} : |\arg(z)| \le \alpha\}$.

Theorem 15.4. *Let $\alpha \in (0,1)$. Every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_0| = 1\,, \quad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most c/α zeros inside any polygon with vertices on the circle $\{z \in \mathbb{C} : |z| = 1 - \alpha\}$, where the constant c depends only on the number of the vertices of the polygon.

The paper [26] containing Theorems 15.2, 15.3, and 15.4 appeared sooner than [34]. The book [17] contains only a few related weeker results. Our Theorem 2.1 in [34] is a simple consequence of Theorem 15.2, and it sharpens and generalizes results of Amoroso [1], Bombieri and Vaaler [11], and Hua [88] who gave upper bounds for the multiplicity of a zero that a polynomial with integer coefficients may have at 1.

The sharpness of Theorem 15.2 can be seen by the theorem below proved in [34].

Theorem 15.5. *For every $n \in \mathbb{N}$, there exists a polynomial p_n of the form given in Theorem 15.2 with real coefficients so that p_n has a zero at 1 with multiplicity at least $\lfloor\sqrt{n}\rfloor - 1$.*

When $0 < \alpha \le n^{-1/2}$, the sharpness of Theorem 15.3 is shown by the polynomials

$$q_n(z) := p_n(z) + z^{2n+1} p_n(z^{-1})\,,$$

where p_n are the polynomials in Theorem 15.5. Namely the polynomials q_n of degree $2n+1$ are of the required form with $\lfloor\sqrt{n}\rfloor - 1 \ge c(n\alpha + \sqrt{n})$ zeros at 1. When $n^{-1/2} \le \alpha \le 1$, the sharpness of Theorem 15.3 is shown by the polynomials $q_n(z) := z^n - 1$.

The next theorem proved in [26] shows the sharpness of Theorem 15.4.

Theorem 15.6. *For every $\alpha \in (0,1)$, there exists a polynomial p_n of the form given in Theorem 15.4 with real coefficients so that p_n has a zero at $1-\alpha$ with multiplicity at least $\lfloor 1/\alpha \rfloor - 1$. (It can also be arranged that $n \le 1/\alpha^2 + 2$.)*

Let $D(z_0, r) := \{z \in \mathbb{C} : \ |z - z_0| < r\}$ denote the open disk of the complex plane centered at $z_0 \in \mathbb{C}$ with radius $r > 0$. As a remark to Theorem 15.5 we point out that a more or less straightforward application of Jensen's formula gives that there is an absolute constant $c > 0$ such that every polynomial p of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j \,, \qquad |a_0| = 1\,, \quad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

has at most $(c/\alpha)\log(1/\alpha)$ zeros in the disk $D(0, 1-\alpha)$, $\alpha \in (0,1)$.k An example in [57], using only real coefficients, suggested by Nazarov, shows that this upper bound for the number of zeros in the disk $D(0, 1-\alpha)$ is, up to the absolute constant $c > 0$, best possible. So, in particular, the constant in Theorem 15.4 cannot be made independent of the number of vertices of the polygon.

Also, it is shown in [57] that there are polynomials $p \in \mathcal{K}_n$ with no zeros in the annulus

$$\{z \in \mathbb{C} : 1 - c\log n/n < |z| < 1 + c\log n/n\}\,,$$

where c is an absolute constant. It is conjectured that every $p \in \mathcal{K}_n$ has a zero in the annulus $\{z \in \mathbb{C} : 1 - c/n < |z| < 1 + c/n\}$, where $c > 0$ is an absolute constant.

The class $\mathcal{F}_n$ and various related classes have been studied from several points of view. Littlewood's monograph [99] contains a number of interesting, challenging, and still open problems about polynomials with coefficients from $\{-1, 1\}$. The distribution of zeros of polynomials with coefficients from $\{0, 1\}$ is studied in [117] by Odlyzko and Poonen.

Bloch and Pólya [10] also prove that there are polynomials $p \in \mathcal{F}_n$ with

$$\frac{cn^{1/4}}{\sqrt{\log n}} \tag{15.1}$$

distinct real zeros of odd multiplicity. (Schur [129] claims they do it for polynomials with coefficients only from $\{-1, 1\}$, but this appears to be incorrect.)

In a seminal paper Littlewood and Offord [100] prove that the number of real roots of a $p \in \mathcal{L}_n$, on average, lies between

$$\frac{c_1 \log n}{\log\log\log n} \qquad \text{and} \qquad c_2 \log^2 n$$

and it is proved by Boyd [40] that every $p \in \mathcal{L}_n$ has at most $c\log^2 n/\log\log n$ zeros at 1 (in the sense of multiplicity). It is conjectured that every $p \in \mathcal{L}_n$ has at most $c\log n$ zeros at 1 with an absolute constant $c > 0$.

Kac [89] shows that the expected number of real roots of a polynomial of degree n with random uniformly distributed coefficients is asymptotically $(2/\pi)\log n$. He writes "I have also stated that the same conclusion holds if the coefficients assume only the values 1 and -1 with equal probabilities. Upon closer examination it turns out that the proof I had in mind is inapplicable.... This situation tends to emphasize the particular interest of the discrete case, which surprisingly enough turns out to be the most difficult." In a related paper Solomyak [132] studies the random series $\sum \pm \lambda^n$."

In [71] we improve the lower bound (15.1) in the result of Bloch and Pólya to $cn^{1/4}$. Moreover we allow a much more general coefficient constraint in our main result. Our approach is quite different from that of Bloch and Pólya.

Theorem 15.7. *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that for every*

$$\{a_0, a_1, \dots, a_n\} \subset [1, M], \qquad 1 \le M \le \exp(c_1 n^{1/4}),$$

there are

$$b_0, b_1, \dots, b_n \in \{-1, 0, 1\}$$

such that

$$P(z) = \sum_{j=0}^{n} b_j a_j z^j$$

has at least $c_2 n^{1/4}$ distinct sign changes in $(0,1)$.

Let ∂D denote the unit circle of the complex plane. Let

$$\|P\| := \max_{z \in \partial D} |P(z)|.$$

A classical result of Erdős and Turán [80] is the following.

Theorem (Erdős–Turán). *If the zeros of*

$$P(z) := \sum_{j=0}^{n} a_j z^j, \qquad a_j \in \mathbb{C}, \quad a_0 a_n \neq 0,$$

are denoted by

$$z_j = r_j \exp(i\varphi_j), \qquad r_j > 0, \quad \varphi_j \in [0, 2\pi), \quad j = 1, 2, \dots, n,$$

then for every $0 \le \alpha < \beta \le 2\pi$ we have

$$\Big| \sum_{j \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \Big| < 16 \sqrt{n \log R},$$

where

$$R := |a_0 a_n|^{-1/2} \|P\|$$

and

$$I(\alpha, \beta) := \{j : \alpha \le \varphi_j \le \beta\}.$$

Note that some books quote this result with

$$R := |a_0 a_n|^{-1/2} (|a_0| + |a_1| + \dots + |a_n|)$$

in place of $R := |a_0 a_n|^{-1/2} \|P\|$. In fact, the weaker result is an obvious corollary of the stronger one by observing that $\|P\| \le |a_0| + |a_1| + \dots + |a_n|$. In [70] we offer a subtle improvement of the above Erdős–Turán Theorem.

Theorem 15.8. *If the zeros of*

$$P(z) := \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,, \quad a_0 a_n \neq 0\,,$$

are denoted by

$$z_j = r_j \exp(i\varphi_j)\,, \qquad r_j > 0\,, \quad \varphi_j \in [0, 2\pi)\,, \quad j = 1, 2, \dots, n\,,$$

then for every $0 \le \alpha < \beta \le 2\pi$ *we have*

$$\sum_{j \in I_1(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi}\, n \le 16\sqrt{n \log R_1}\,,$$

and

$$\sum_{j \in I_2(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi}\, n \le 16\sqrt{n \log R_2}\,,$$

where

$$R_1 := |a_n|^{-1}\|P\|\,, \qquad R_2 := |a_0|^{-1}\|P\|\,,$$

and

$$I_1(\alpha,\beta) := \{j : \alpha \le \varphi_j \le \beta,\, r_j \ge 1\}\,, \qquad I_2(\alpha,\beta) := \{j : \alpha \le \varphi_j \le \beta,\, r_j \le 1\}\,.$$

This result is closely related to a recent paper of V. Totik and P. Varjú [135]. In fact, it may as well be derived from part (ii) of Theorem 1.1 in [135]. However, here we do not rely on this recent result. Our approach is based on the interesting observation that the Erdős–Turán Theorem above improves itself.

A straightforward consequence of the above theorem is the following.

Corollary 15.9. *If the modulus of a monic polynomial* P *of degree* n *(with complex coefficients) on the unit circle of the complex plane is at most* $1 + o(1)$ *uniformly, then the multiplicity of each zero of* P *on the unit circle is* $o(n^{1/2})$.

References

1. F. Amoroso, *Sur le diamètre transfini entier d'un intervalle réel*, Ann. Inst. Fourier, Grenoble **40** (1990), 885–911.

2. V.V. Arestov, *On integral inequalities for trigonometric polynomials and their derivatives*, Izv. **45** (1981), 3–22.

3. J. Bak and D. J. Newman, *Rational combinations of* $x^{\lambda_k}, \lambda_k \ge 0$ *are always dense in* $C[0,1]$, J. Approx. Theory **23** (1978), 155–157.

4. J. Bastero, *ℓ_q-subspaces of stable p-Banach spaces*, Archiv. Math. **40** (1983), 538–544.

5. J. Beck, *"Flat" polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. (1991), 269–277.

6. D. Benko and T. Erdélyi, *Markov inequality for polynomials of degree n with m distinct zeros*, J. Approx. Theory **122** (2003), 241–248.

7. D. Benko, T. Erdélyi, and J. Szabados,, *The full Markov-Newman inequality for Müntz polynomials on positive intervals*, Proc. Amer. Math. Soc. **131** (2003), 2385–2391.

8. S.N. Bernstein, *Sur la représentation des polynomes positifs, in*, "Collected Papers of S.N. Bernstein," Vol. 1, "Constructive Theory of Functions (1905-1930)," pp. 251-252, Academy of Sciences, USSR, 1952 [in Russian; the original paper was published in 1915].

9. F. Beaucoup, P. Borwein, D. Boyd, and C. Pinner, *Multiple roots of* $[-1,1]$ *power series*, J. London Math. Soc. **57** (1998), 135–147.

10. A. Bloch and G. Pólya, *On the roots of certain algebraic equations*, Proc. London Math. Soc. **33** (1932), 102–114.

11. E. Bombieri and J. Vaaler, *Polynomials with low height and prescribed vanishing*, in Analytic Number Theory and Diophantine Problems, Birkhauser (1987), 53–73.

12. P. Borwein, *Markov's inequality for polynomials with restricted zeros*, Proc. Amer. Math. Soc. **93** (1985), 43–47.

13. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.

14. P. Borwein and T. Erdélyi, *Sharp Markov-Bernstein type inequalities for classes of polynomials with restricted zeros*, Constr. Approx. **10** (1994), 411–425.

15. P. Borwein and T. Erdélyi, *Markov and Bernstein type inequalities in L_p for classes of polynomials with constraints*, J. London Math. Soc. **51** (1995), 573–588.

16. P. Borwein and T. Erdélyi, *Upper bounds for the derivative of exponential sums*, Proc. Amer. Math. Soc. **123** (1995), 1481–1486.

17. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, N.Y., 1995.

18. P. Borwein and T. Erdélyi, *Müntz spaces and Remez inequalities*, Bull. Amer. Math. Soc. **32** (1995), 38–42.

19. P. Borwein & T. Erdélyi, *Sharp extensions of Bernstein's inequality to rational spaces*, Mathematika **43** (1996), 413–423.

20. P. Borwein and T. Erdélyi, *The full Müntz Theorem in $C[0,1]$ and $L_1[0,1]$*, J. London Math. Soc. **54** (1996), 102–110.

21. P. Borwein and T. Erdélyi, *Newman's inequality for Müntz polynomials on positive intervals*, J. Approx. Theory 85 (1996), 132-139.

22. P. Borwein and T. Erdélyi, *The L_p version of Newman's inequality for lacunary polynomials*, Proc. Amer. Math. Soc. **124** (1996), 101–109.

23. P. Borwein and T. Erdélyi, *A sharp Bernstein-type inequality for exponential sums*, J. Reine Angew. Math. **476** (1996), 127–141.

24. P. Borwein and T. Erdélyi, *The integer Chebyshev problem*, Math. Comp. **65** (1996), 661–681.

25. P. Borwein and T. Erdélyi, *Generalizations of Müntz's Theorem via a Remez-type inequality for Müntz spaces*, J. Amer. Math. Soc. **10** (1997), 327–349.

26. P. Borwein and T. Erdélyi, *On the zeros of polynomials with restricted coefficients*, Illinois J. Math. **41** (1997), 667–675.

27. P. Borwein and T. Erdélyi, *Pointwise Remez- and Nikolskii-type inequalities for exponential sums*, Math. Ann. **316** (2000), 39–60.

28. P. Borwein and T. Erdélyi, *Markov- and Bernstein-type inequalities for polynomials with restricted coefficients*, Ramanujan J. **1** (1997), 309–323.

29. P. Borwein & T. Erdélyi, *Littlewood-type problems on subarcs of the unit circle*, Indiana Univ. Math. J. **46** (1997), 1323–1346.

30. P. Borwein and T. Erdélyi, *Markov-Bernstein type inequalities under Littlewood-type coefficient constraints*, Indag. Math. (N.S.) **11** (2000), 159–172.

31. P. Borwein and T. Erdélyi, *Lower bounds for the merit factors of trigonometric polynomials from Littlewood classes*, J. Approx. Theory **125** (2003), 190–197.

32. P. Borwein and T. Erdélyi, *Nikolskii-type inequalities for shift invariant function spaces*, Proc. Amer. Math. Soc. **134** (2006), 3243–3246.

33. P. Borwein and T. Erdélyi, *Lower bounds for the number of zeros of cosine polynomials: a problem of Littlewood*, Acta Arith. **128** (2007), 377–384.

34. P. Borwein, T. Erdélyi, and G. Kós, *Littlewood-type problems on* $[0, 1]$, Proc. London Math. Soc. **(3) 79** (1999), 22–46.

35. P. Borwein, T. Erdélyi, and J. Zhang, *Müntz systems and orthogonal Müntz-Legendre polynomials*, Trans. Amer. Math. Soc. **342** (1994), 523–542.

36. P. Borwein, T. Erdélyi, and J. Zhang, *Chebyshev polynomials and Bernstein-Markov type inequalities for rational spaces*, J. London Math. Soc. **50** (1994), 501–519.

37. P. Borwein, T. Erdélyi, and F. Littmann, *Zeros of polynomials with finitely many different coefficients*, Trans. Amer. Math. Soc. **360** (2008), 5145–5154.

38. P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart, *On the zeros of cosine polynomials : an old problem of Littlewood*, Ann. of Math. (to appear).

39. P. Borwein, C. Pinner, and I. Pritsker, *Monic integer Chebyshev problem*, Math. Comp. **72**, 1901 - 1916.

40. D. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients*, Math. Comput. **66** (1997), 1697–1703.

41. Braess, D., *Nonlinear Approximation Theory*, Springer-Verlag, Berlin, 1986.

42. J.A. Clarkson and P. Erdős, *Approximation by polynomials*, Duke Math. J. **10** (1943), 5–11.

43. B. Conrey, A. Granville, B. Poonen, and K. Soundararajan, *Zeros of Fekete polynomials*, Ann. Inst. Fourier (Grenoble) **50** (2000), 865–889.

44. R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

45. Z. Ditzian and D. Lubinsky, *Jackson and smoothness theorems for Freud weights*, Constr. Approx. **13** (1997), 99–152.

46. T. Erdélyi, *The Remez inequality on the size of polynomials*, in: Approximation Theory VI, C. K. Chui, L. L. Schumaker, & J. D. Wards, Eds., Academic Press, Boston, 1989, pp. 243–246.

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47. T. Erdélyi, *Markov-type estimates for certain classes of constrained polynomials*, Constr. Approx. **5** (1989), 347–356.

48. T. Erdélyi, *Estimates for the Lorentz degree of polynomials*, J. Approx. Theory **67** (1991), 187–198.

49. T. Erdélyi, *Bernstein-type inequality for the derivative of constrained polynomials*, Proc. Amer. Math. Soc. **112** (1991), 829–838.

50. T. Erdélyi, *Bernstein and Markov type theorems for generalized nonnegative polynomials*, Canad. J. Math. **43** (1991), 1–11.

51. T. Erdélyi, *Remez-type inequalities on the size of generalized polynomials*, J. London Math. Soc. **45** (1992), 255–264.

52. T. Erdélyi, *Remez-type inequalities and their applications*, J. Comp. and Applied Math. **47** (1993), 167–210.

53. T. Erdélyi, *Markov-Bernstein type inequalities for constrained polynomials with real versus complex coefficients*, Jour d'Analyse Math. **74** (1998), 165–181.

54. T. Erdélyi, *Markov-type inequalities for constrained polynomials with complex coefficients*, Illinois J. Math. **42** (1998), 544–563.

55. T. Erdélyi, *Markov- and Bernstein-type inequalities for Müntz polynomials and exponential sums in L_p*, J. Approx. Theory **104** (2000), 142–152.

56. T. Erdélyi, *The resolution of Saffari's Phase Problem*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000, 10), 803–808.

57. T. Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints*, Michigan Math. J. **49** (2001), 97–111.

58. T. Erdélyi, *How far is an ultraflat sequence of unimodular polynomials from being conjugate reciprocal?*, Michigan Math. J. **49** (2001), 259–264.

59. T. Erdélyi, *The phase problem of ultraflat unimodular polynomials: The resolution of the conjecture of Saffari*, Math. Ann. **321** (2001), 905–924.

60. T. Erdélyi, *A proof of Saffari's "near-orthogonality" conjecture for ultraflat sequences of unimodular polynomials*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001, 7), 623–628.

61. T. Erdélyi, *The full Clarkson-Erdős-Schwartz Theorem on the uniform closure of non-dense Müntz spaces*, Studia Math. **155 (2)** (2002), 145–152.

62. T. Erdélyi, *Extremal properties of the derivatives of the Newman polynomials*, Proc. Amer. Math. Soc. **131** (2003), 3129–3134.

63. T. Erdélyi, *On the real part of ultraflat sequences of unimodular polynomials: consequences implied by the resolution of the Phase Problem*, Math. Ann. **326** (2003), 489–498.

64. T. Erdélyi, *The full Müntz Theorem revisited*, Constr. Approx. **21 (3)** (2005), 319-335.

65. T. Erdélyi, *Bernstein-type inequalities for linear combinations of shifted Gaussians*, Bull. London Math. Soc. **38** (2006), 124–138.

66. T. Erdélyi, *Markov-Nikolskii-type inequalities for exponential sums on a finite interval*, Adv. Math. **208** (2007), 135–146.

67. T. Erdélyi, *The Remez inequality for linear combinations of shifted Gaussians*, Math. Proc. Cambridge Phil. Soc. (to appear).

68. T. Erdélyi, *Newman's inequality for increasing exponential sums*, Number Theory and Polynomials, Series: London Mathematical Society Lecture Note Series (No. 352), J. McKee and Ch. Smyth (Eds.) (2008).

69. T. Erdélyi, *Inequalities for exponential sums via interpolation and Turán-type reverse Markov inequalities*, Frontiers in Interpolation and Approximation, dedicated to the memory of Ambikeshvar Sharma, Chapman & Hall/CRC, Taylor & Francis, New York, N.K. Govil, H.N. Mhaskar, Ram Mohapatra, Zuhair Nashed, J. Szabados (Eds.) (2006), 119–144.

70. T. Erdélyi, *An improvement of the Erdős-Turán theorem on the distribution of zeros of polynomials.*, C. R. Math. Acad. Sci. Paris 346 (2008, 5-6), 267–270..

71. T. Erdélyi, *Extensions of the Bloch-Pólya theorem on the number of distinct real zeros of polynomials*, manuscript (2008).

72. T. Erdélyi and W. Johnson, *The "Full Müntz Theorem" in $L_p[0,1]$ for $0 < p < \infty$*, Journal d'Analyse Math. **84** (2001), 145–172.

73. T. Erdélyi and J. Szabados, *On polynomials with positive coefficients*, J. Approx. Theory **54** (1988), 107–122.

74. T. Erdélyi, A. Magnus, and P. Nevai, *Generalized Jacobi weights, Christoffel functions and Jacobi polynomials*, SIAM J. Math. Anal. **25** (1994), 602–614.

75. T. Erdélyi and P. Nevai, *Lower bounds for the derivatives of polynomials and Remez-type inequalities*, Trans. Amer. Math. Soc. **349** (1997), 4953 – 4972.

76. T. Erdélyi, A. Máté, and P. Nevai, *Inequalities for generalized nonnegative polynomials*, Constr. Approx. **8** (1992), 241–255.

77. T. Erdélyi and J. Szabados, *On a generalization of the Bernstein-Markov inequality*, St. Petersburg Math. J. **4** (2003), 563–765.

78. P. Erdős, *On extremal properties of the derivatives of polynomials*, Ann. of Math. **2** (1940), 310–313.

79. P. Erdős, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

80. P. Erdős and P. Turán, *On the distribution of roots of polynomials*, Ann. of Math. **57** (1950), 105–119.

81. J. Erőd, *Bizonyos polinomok maximumának alsó korlátjáról*, Matematikai és Fizikai Lapok **XLVI** (1939).

82. C. Frappier, *Quelques problèmes extrémaux pour les polynomes et les functions entières de type exponentiel*, Ph. D. Dissertation, Université de Montréal.

83. G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford, 1971.

84. S. Güntürk, *Approximation by power series with ± 1 coefficients.*, Int. Math. Res. Not. **26** (2005), 1601–1610.

85. G. Halász, *Markov-type inequalities for polynomials with restricted zeros*, J. Approx. Theory **101** (1999), 148–155.

86. K. Hare and C. Smyth, *The monic integer transfinite diameter*, Math. Comp. **75** (2006), 1997-2019.

87. F. Hausdorff, *Summationsmethoden und Momentfolgen I*, Math. Zeit. **9** (1921), 74-109.

88. L.K. Hua, *Introduction to Number Theory*, Springer-Verlag, Berlin Heidelberg, New York, 1982.

89. M. Kac, *On the average number of real roots of a random algebraic equation, II*, Proc. London Math. Soc. **50** (1948), 390–408.

90. J.P. Kahane, *Sur les polynomes à coefficient unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342.

91. T. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224.

92. S. Konyagin, *On a question of Pichorides*, C.R. Acad. Sci. Paris Sér I Math. **324** (1997), 385–388.

93. S.V. Konyagin, *On a problem of Littlewood*, Mathematics of the USSR, Izvestia **18** (1981), 205–225.

94. S.V. Konyagin and V.F. Lev, *Character sums in complex half planes*, J. Théor. Nombres Bordeaux **16** (2004), no. 3, 587–606.

95. N. Levenberg and J. Poletcky, *A reverse Markov inequality*, Ann. Acad. Sci. Fenn. Math. **37** (2002), 173–182.

96. J.E. Littlewood, *On the mean values of certain trigonometrical polynomials*, J. London Math. Soc. **36** (1961), 307–334.

97. J.E. Littlewood, *On the real roots of real trigonometrical polynomials (II)*, J. London Math. Soc. **39** (1964), 511–552.

98. J.E. Littlewood, *On polynomials* $\sum \pm z^m$ *and* $\sum e^{\alpha_m i} z^m$, $z = e^{\theta i}$, J. London Math. Soc. **41** (1966), 367–376.

99. J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

100. J.E. Littlewood and A.C. Offord, *On the number of real roots of a random algebraic equation, II*, Proc. Cam. Phil. Soc. **35** (1939), 133-148.

101. J. Szabados and A. Kroó, *Constructive properties of self-reciprocal polynomials*, Analysis **14** (1994), 319–339.

102. G.G. Lorentz, *The degree of approximation by polynomials with positive coefficients*, Math. Ann. **151** (1963), 239–251.

103. G.G. Lorentz, *Notes on Approximation*, J. Approx. Theory **56** (1989), 360–365.

104. G.G. Lorentz, *Approximation of Functions, 2nd ed.*, Chelsea, New York, N.Y., 1986.

105. G.G. Lorentz, M. von Golitschek, and Y. Makovoz, *Constructive Approximation: Advanced Problems*, Springer-Verlag, Berlin, 1996.

106. Milovanović, G.V., D.S. Mitrinović, and Th.M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

107. O.C. McGehee, L. Pigno, and B. Smith, *Hardy's inequality and the L_1 norm of exponential sums*, Ann. of Math. **113** (1981), 613–618.

108. I.D. Mercer, *Unimodular roots of special Littlewood polynomials*, Canad. Math. Bull. **49** (2006), no. 3, 438–447.

109. H.N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation*, World Scientific, Singapore, 1996.

110. H.N. Mhaskar, *When is approximation by Gaussian networks necessarily a linear process?*, Neural Networks **17 (7)** (2004), 989 – 1001.

111. H.L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS, Vol. 84, Amer. Math. Soc., R. I., 1980.

112. C. Müntz, *Über den Approximationsatz von Weierstrass*, H.A. Schwartz Festschrift, Berlin (1914).

113. F. Nazarov, *Local estimates for exponential polynomials and their applications to inequalities of the uncertainty type*, Algebra i Analiz (4) **5** (1993), 3–66.

114. D.J. Newman, *Derivative bounds for Müntz polynomials*, J. Approx. Theory **18** (1976), 360–362.

115. D.J. Newman, *Approximation with rational functions*, vol. 41, Regional Conference Series in Mathematics, Providence, Rhode Island, 1978.

116. S.M. Nikolskii, *Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables*, Trudy Mat. Inst. Steklov **38** (1951), 244–278.

117. A. Odlyzko and B. Poonen, *Zeros of polynomials with* $0,1$ *coefficients*, Ens. Math. **39** (1993), 317–348.

118. P.P. Petrushev and V.A. Popov, *Rational Approximations of Real Functions*, Cambridge University Press, 1987.

119. S.K. Pichorides, *Notes on trigonometric polynomials.*, in: Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I (Chicago, Illinois, 1981), 84–94, Wadsworth Math. Ser., Wadsworth, Belmont, Calif., 1983..

120. Q.I. Rahman and G. Schmieisser, *Les Inegalités de Markoff et de Bernstein*, Les Presses de L'Université de Montréal, 1983.

121. Q.I. Rahman and G. Schmieisser, *Analytic Theory of Polynomials*, Oxford University Press, Oxford, 2002.

122. E.J. Remez, *Sur une propriété des polynômes de Tchebyscheff*, Comm. Inst. Sci. Kharkow **13** (1936), 93–95.

123. Sz. Révész, Turn-type reverse Markov inequalities for compact convex sets, J. Approx. Theory 141 (2006), 162–173.

124. H. Queffelec and B. Saffari, *Unimodular polynomials and Bernstein's inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995, 3), 513–318.

125. H. Queffelec and B. Saffari, *On Bernstein's inequality and Kahane's ultraflat polynomials*, J. Fourier Anal. Appl. **2** (1996, 6), 519–582.

126. B. Saffari, *The phase behavior of ultraflat unimodular polynomials*, in Probabilistic and Stochastic Methods in Analysis, with Applications (1992), Kluwer Academic Publishers, Printed in the Netherlands.

127. J.T. Scheick, *Inequalities for derivatives of polynomials of special type*, J. Approx. Theory **6** (1972), 354–358.

128. E. Schmidt, *Zur Kompaktheit der Exponentialsummen*, J. Approx. Theory **3** (1970), 445–459.

129. I. Schur, *Untersuchungen über algebraische Gleichungen.*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1933), 403–428.

130. T. Sheil-Small, *Complex polynomials*, Cambridge University Press, Cambridge, 2002.

131. G. Somorjai, *A Müntz-type problem for rational approximation*, Acta. Math. Hung. **27** (1976), 197–199.

132. B. Solomyak, *On the random series* $\sum \pm\lambda^n$ *(an Erdős problem)*, Ann. of Math. **142** (1995), 611–625.

133. G. Szegő, *Bemerkungen zu einem Satz von E. Schmidt uber algebraische Gleichungen.*, Sitz. Preuss. Akad. Wiss., Phys.-Math. Kl. (1934), 86–98.

Ref

134. G. Szegő and A. Zygmund, *On certain meanvalues of polynomials*, J. Anal. Math. **3** (1954), 225–244.

135. V. Totik and P. Varjú, *Polynomials with prescribed zeros and small norm*, Acta Sci. Math. (Szeged) (to appear).

136. P. Turán, *Über die Ableitung von Polynomen*, Comp. Math. (1939), 89-95.

137. P. Turán, *On a New Method of Analysis and its Applications*, Wiley, New York, NY, 1984.

138. Q. Wu, *A new exceptional polynomial for the integer transfinite diameter of* $[0, 1]$, J. Thorie des Nombres de Bordeaux **15** (2003), 847-861.

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