DO FLAT SKEW-RECIPROCAL LITTLEWOOD POLYNOMIALS EXIST?

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Dedicated to the memory of Peter Borwein

Abstract. Polynomials with coefficients in $\{-1,1\}$ are called Littlewood polynomials. Using special properties of the Rudin-Shapiro polynomials and classical results in approximation theory such as Jackson's Theorem, de la Vallée Poussin sums, Bernstein's inequality, Riesz's Lemma, divided differences, etc., we give a significantly simplified proof of a recent breakthrough result by Balister, Bollobás, Morris, Sahasrabudhe, and Tiba stating that there exist absolute constants $\eta_2 > \eta_1 > 0$ and a sequence (P_n) of Littlewood polynomials P_n of degree n such that

$$\eta_1 \sqrt{n} \leq |P_n(z)| \leq \eta_2 \sqrt{n}\,, \qquad z \in \mathbb{C}, \ \ |z| = 1\,,$$

confirming a conjecture of Littlewood from 1966. Moreover, the existence of a sequence (P_n) of Littlewood polynomials P_n is shown in a way that in addition to the above flatness properties a certain symmetry is satisfied by the coefficients of P_n making the Littlewood polynomials P_n close to skew-reciprocal.

1. The Theorem

Polynomials with coefficients in $\{-1,1\}$ are called Littlewood polynomials.

Theorem 1.1. *There exist absolute constants $\eta_2 > \eta_1 > 0$ and a sequence (P_n) of Littlewood polynomials P_n of degree n such that*

$$\eta_1 \sqrt{n} \leq |P_n(z)| \leq \eta_2 \sqrt{n}\,, \qquad z \in \mathbb{C}, \ \ |z| = 1\,. \tag{1.1}$$

Note that Beck [B-91] showed the existence of flat unimodular polynomials P_n of degree n satisfying (1.1) with coefficients in the set of kth roots of unity and gave the value $k = 400$, but correcting a minor error in Beck's paper Belshaw [B-13] showed that the value of k in [B-91] should have been 851. Repeating Spencer's calculation Belshaw improved the value 851 to 492 in Beck's result, and an improvement of Spencer's method, due to Kai-Uwe Schmidt, allowed him to lower the value of k to 345. The recent breakthrough

2010 Mathematics Subject Classifications. 11C08, 41A17, 26C10, 30C15 result by Balister, Bollobás, Morris, Sahasrabudhe, and Tiba [B-20] formulated in Theorem 1.1 confirms a conjecture of Littlewood from 1966. Using special properties of the Rudin-Shapiro polynomials and classical results in approximation theory such as Jackson's Theorem, de la Vallée Poussin sums, Bernstein's inequality, Riesz's Lemma, divided differences, etc., in this paper we give a significantly simplified proof of this beautiful and deep theorem. Moreover, the existence of a sequence (P_n) of Littlewood polynomials P_n is shown so that in addition to (1.1) a certain symmetry is satisfied by the coefficients of P_n.

Theorem 1.2. *There exist absolute constants $0 < \eta_1 < \eta_2$, $\eta > 0$, and a sequence (P_{2n}) of Littlewood polynomials P_{2n} of the form*

$$P_{2n}(z) = \sum_{j=0}^{2n} a_{j,n} z^j \,, \qquad a_{j,n} \in \{-1,1\}\,, \;\; j = 0,1,\ldots,2n, \;\; n = 1,2,\ldots\,,$$

such that in addition to (1.1) with n replaced by $2n$ the coefficients of P_{2n} satisfy

$$a_{j,n} = -a_{2n-j,n}\,, \qquad 0 \leq j < n - m_n\,,$$

and

$$a_{j,n} = (-1)^{n-j} a_{2n-j,n}\,, \qquad n - m_n \leq j \leq n\,,$$

with some integers $0 \leq \eta n \leq m_n \leq n$.

The theorem above may be viewed as a result in an effort to answer the following question.

Problem 1.3. *Are there absolute constants $0 < \eta_1 < \eta_2$ and a sequence (P_{4n}) of skew-reciprocal Littlewood polynomials P_{4n} of the form*

$$P_{4n}(z) = \sum_{j=0}^{4n} a_{j,n} z^j \,, \qquad a_{j,n} \in \{-1,1\}\,, \;\; j = 0,1,\ldots,4n, \;\; n = 1,2,\ldots\,,$$

such that in addition to (1.1) with n replaced by $4n$ the coefficients of P_{4n} satisfy

$$a_{j,n} = (-1)^{-j} a_{4n-j,n}\,, \qquad j = 0,1,\ldots,4n\,?$$

This problem remains open. We remark that it is easy to see that every self-reciprocal Littlewood polynomial of the form

$$P_n(z) = \sum_{j=0}^{n} a_{j,n} z^j \,, \qquad a_{j,n} \in \{-1,1\}\,, \;\; j = 0,1,\ldots,n\,,$$

satisfying

$$a_{j,n} = a_{n-j,n}\,, \qquad j = 0,1,\ldots,n\,,$$

has at least one zero on the unit circle, see Theorem 2.8 in [E-01], or Corollary 6 in [M-06], for example. Hence there are no absolute constant $\eta_1 > 0$ and a sequence (P_n) of self-reciprocal Littlewood polynomials P_n of degree n such that

$$\eta_1 \sqrt{n} \leq |P_n(z)|\,, \qquad z \in \mathbb{C}, \;\; |z| = 1\,, \;\; n = 1,2,\ldots\,.$$

2. Rudin-Shapiro polynomials

Section 4 of [B-02] is devoted to the study of Rudin-Shapiro polynomials. A sequence of Littlewood polynomials that satisfy just the upper bound of Theorem 1.1 is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [G-51]. The Rudin-Shapiro polynomials are remarkably simple to construct. They are defined recursively as follows:

$$P_0(z) := 1\,, \qquad Q_0(z) := 1\,,$$
$$P_{m+1}(z) := P_m(z) + z^{2^m} Q_m(z)\,,$$
$$Q_{m+1}(z) := P_m(z) - z^{2^m} Q_m(z)\,,$$

for $m = 0, 1, 2, \ldots$. Note that both P_m and Q_m are polynomials of degree $M-1$ with $M := 2^m$ having each of their coefficients in $\{-1, 1\}$. It is well known and easy to check by using the parallelogram law that

$$|P_{m+1}(e^{it})|^2 + |Q_{m+1}(e^{it})|^2 = 2(|P_m(e^{it})|^2 + |Q_m(e^{it})|^2)\,, \qquad t \in \mathbb{R}\,.$$

Hence

$$|P_m(e^{it})|^2 + |Q_m(e^{it})|^2 = 2^{m+1} = 2M\,, \qquad t \in \mathbb{R}\,. \tag{2.1}$$

Observing that the first 2^m terms of P_{m+1} are the same as the 2^m terms of P_m, we can define the polynomial $P_{<n}$ of degree $n-1$ so that its terms are the first n terms of all P_m for all m for which $2^m \geq n$. The following bound, which is a straightforward consequence of (2.1) was proved by Shapiro [S-51].

Lemma 2.1. *We have*

$$|P_{<n}(e^{it})| \leq 5\sqrt{n}\,, \qquad t \in \mathbb{R}\,.$$

It is also well-known that

$$P_m(1) = \|P_m(e^{it})\| := \max_{t\in\mathbb{R}} |P_m(e^{it})| = 2^{(m+1)/2}$$

for every odd m and $P_m(1) = 2^{m/2}$ for every even m.

Our next lemma is stated as Lemma 3.5 in [E-16], where its proof may also be found. It plays a key role in [E-19a] [E-19b], and [E-21] as well.

Lemma 2.2. *If P_m and Q_m are the m-th Rudin-Shapiro polynomials of degree $M-1$ with $M := 2^m$, $\delta := \sin^2(\pi/8)$, and*

$$z_j := e^{it_j}\,, \quad t_j := \frac{2\pi j}{M}\,, \qquad j \in \mathbb{Z}\,,$$

then

$$\max\{|P_m(z_j)|^2, |P_m(z_{j+1})|^2\} \geq \delta 2^{m+1} = 2\delta M\,.$$

Lemma 2.3. *Using the notation of Lemma 2.2 we have*

$$|P_m(e^{it})|^2 \geq \delta M\,, \qquad t \in \left[t_j - \frac{\delta}{2M}, t_j + \frac{\delta}{2M}\right]\,,$$

for every $j \in \mathbb{Z}$ such that

$$|P_m(z_j)|^2 \geq \delta 2^{m+1} = 2\delta M\,.$$

Proof. The proof is a simple combination of the Mean Value Theorem and Bernstein's inequality (Lemma 3.4) applied to the (real) trigonometric polynomial of degree $M-1$ defined by $S(t) := P_m(e^{it})P_m(e^{-it})$. Recall that (2.1) implies $0 \leq S(t) = |P_m(e^{it})|^2 \leq 2M$ for every $t \in \mathbb{R}$. □

Let, as before, $M := 2^m$ with an odd m. We define

$$T(t) := \mathrm{Re}((1 + e^{iMt} + e^{2iMt} + \cdots + e^{8iMt})P_m(e^{it})) = \mathrm{Re}\left(\frac{e^{9iMt}-1}{e^{iMt}-1}\,P_m(e^{it})\right)\,. \tag{2.2}$$

Observe that T is a real trigonometric polynomial of degree $\mu - 1 := 9M - 1$. For every sufficiently large natural number n there is an odd integer m such that

$$2^{-75} \leq \gamma := \frac{\mu}{n} = \frac{9 \cdot 2^m}{n} < 2^{-73}\,. \tag{2.3}$$

Observe that

$$\|T\| := \max_{t\in\mathbb{R}} |T(t)| = |T(0)| = 9|P_m(1)| = 9 \cdot 2^{(m+1)/2} = 9(2M)^{1/2} = 3\sqrt{2\gamma n}\,. \tag{2.4}$$

Lemma 2.4. *In the notation of Lemmas 2.2 and 2.3, for every $j \in \mathbb{Z}$ satisfying*

$$|P_m(z_j)|^2 \geq \delta 2^{m+1} = 2\delta M$$

there are

$$a_j \in \left[t_j - \frac{3\pi}{32M}, t_j - \frac{\pi}{32M}\right] \qquad \text{and} \qquad b_j \in \left[t_j + \frac{\pi}{32M}, t_j + \frac{3\pi}{32M}\right]$$

such that

$$|T(a_j)| \geq (0.005)\|T\| = (0.015)\sqrt{2\gamma n} \qquad \text{and} \qquad |T(b_j)| \geq (0.005)\|T\| = (0.01)\sqrt{2\gamma n}\,.$$

Proof. We prove the statement about the existence of b_j as the proof of the statement about the existence of a_j is essentially the same. Let

$$P_m(e^{it}) = R(t)e^{i\alpha(t)}\,, \qquad R(t) = |P_m(e^{it})|\,,$$

where the function α could be chosen so that it is differentiable on any interval where $P_m(e^{it})$ does not vanish. Then

$$ie^{it}P_m'(e^{it}) = R'(t)e^{i\alpha(t)} + R(t)e^{i\alpha(t)}(i\alpha'(t)),$$

hence

$$\alpha'(t) = \mathrm{Re}\left(\frac{e^{it}P_m'(e^{it})}{P_m(e^{it})}\right)$$

on any interval where $P_m(e^{it})$ does not vanish. Combining Bernstein's inequality (Lemma 3.4), Lemma 2.3, and $\|P_m\| \le (2M)^{1/2}$, we obtain

$$|\alpha'(t)| \le \frac{M(2M)^{1/2}}{(\delta M)^{1/2}} = \left(\frac{2}{\delta}\right)^{1/2} M \le (3.7)M, \qquad t \in \left[t_j, t_j + \frac{\delta}{2M}\right]. \tag{2.5}$$

Now let

$$\frac{e^{9iMt}-1}{e^{iMt}-1} = \left|\frac{e^{9iMt}-1}{e^{iMt}-1}\right| e^{4Mt}, \qquad t \in \left(t_j - \frac{2\pi}{9M}, t_j + \frac{2\pi}{9M}\right). \tag{2.6}$$

By writing

$$(1 + e^{iMt} + e^{2iMt} + \cdots + e^{8iMt})P_m(e^{it}) = \left|\frac{e^{9iMt}-1}{e^{iMt}-1}P_m(e^{it})\right| e^{i(\alpha(t)+4Mt)},$$

we see by (2.5) and (2.6) that $\beta(t) := \alpha(t) + 4Mt$ satisfies

$$(0.3)M = 4M - (3.7)M \le 4M - |\alpha'(t)| \le |\beta'(t)|, \qquad t \in \left[t_j, t_j + \frac{\delta}{M}\right]. \tag{2.7}$$

It is also simple to see that

$$\left|\frac{e^{9iMt}-1}{e^{iMt}-1}\right| \ge \left|\frac{e^{iM\pi}-1}{e^{iM\pi/9}-1}\right| = \frac{2}{2\sin(\pi/18)} \ge \frac{18}{\pi}, \qquad t \in \left[t_j - \frac{\pi}{9M}, t_j + \frac{\pi}{9M}\right]. \tag{2.8}$$

Observe that (2.7) and (2.8) imply that there are

$$b_j \in \left[t_j + \frac{\pi}{32M}, t_j + \frac{3\pi}{32M}\right]$$

for which

$$\left|\frac{e^{9iMb_j}-1}{e^{iMb_j}-1}\right| \ge \frac{18}{\pi} \tag{2.9}$$

and

$$\cos(\beta(b_j)) \ge \cos\left(\frac{\pi}{2} - \frac{(0.15)\pi}{16}\right) \ge 0.0294. \tag{2.10}$$

Combining (2.9), (2.10), Lemma 2.3, and (2.4) we obtain

$$\begin{aligned}|T(b_j)| &= \left|\mathrm{Re}\left(\frac{e^{9iMb_j}-1}{e^{iMb_j}-1}P_m(e^{ib_j})\right)\right| = \left|\frac{e^{9iMb_j}-1}{e^{iMb_j}-1}\right| \left|P_m(e^{ib_j})\right| |\cos(\beta(b_j)| \\ &\ge \frac{18}{\pi}(\delta M)^{1/2}(0.0294) \ge \frac{(0.5292)\sin(\pi/8)}{9\sqrt{2}\pi}9(2M)^{1/2} \ge (0.005)9(2M)^{1/2} \\ &\ge (0.005)\|T\|.\end{aligned}$$

□

3. Tools from Approximation Theory

Let $\mathcal{T}_\nu$ denote the set of all real trigonometric polynomials of degree at most ν. Let $\|T\|$ denote the maximum modulus of a trigonometric polynomial T on $\mathbb{R}$.

Definition 3.1 Let $n > 0$ be an integer divisible by 10. Let $\mathcal{I}$ be a collection of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/2\pi\mathbb{Z}$. We call the collection $\mathcal{I}$ *suitable* if

(a) the endpoints of each interval in $\mathcal{I}$ are in $(10\pi/n)\mathbb{Z}$;
(b) $\mathcal{I}$ is invariant under the maps $\theta \to \pi \pm \theta$;
(c) $|\mathcal{I}| = 4N$ for some $N \leq \gamma n$.

We call a suitable collection $\mathcal{I}$ of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/(2\pi\mathbb{R})$ *well-separated* if

(d) $|I| \leq 3990\pi/n$ for each $I \in \mathcal{I}$;
(e) $d(I, J) \geq 10\pi/n$ for each $I, J \in \mathcal{I}$ with $I \neq J$;
(f) the sets $\bigcup_{I\in\mathcal{I}} I$ and $(\pi/2)\mathbb{Z} + (-5\pi/n, 5\pi/n)$ are disjoint;

where in (e) $d(I, J)$ denotes the distance between the intervals I and J.

We will denote the intervals in a suitable and well-separated collection $\mathcal{I}$ by

$$I_j, \qquad j = 1, 2, \dots, 4N,$$

where $I_1, I_2, \dots, I_N \subset (0, \pi/2)$. Associated with an interval $[a, b] \subset [-\pi + 5\pi/n, \pi - 5\pi/n]$ we define

$$\Phi_{[a,b]}(t) := \begin{cases} 1, & \text{if } t \in [a, b]\,, \\ 0, & \text{if } t \in [-\pi, a - 5\pi/n] \cup [b + 5\pi/n, \pi]\,, \\ (n/(5\pi))(t - (a - 5\pi/n)), & \text{if } t \in [a - 5\pi/n, a]\,, \\ (n/(5\pi))((b + 5\pi/n) - t), & \text{if } t \in [b, b + 5\pi/n]\,. \end{cases}$$

We call the coloring $\alpha : \mathcal{I} \to \{-1, 1\}$ *symmetric* if $\alpha(I) = \alpha(\pi - I)$ and $\alpha(I) = -\alpha(\pi + I)$. Associated with a symmetric $\mathcal{I} :\to \{-1, 1\}$ let

$$g_\alpha := \sum_{j=1}^{4N} \alpha(I_j)\Phi_{I_j} \qquad \text{and} \qquad G_\alpha := K\sqrt{n}\, g_\alpha\,.$$

Let $S_o := \{1, 3, \dots, 2n-1\}$ be the set of odd numbers between 1 and $2n-1$. Let $C_{2\pi}$ denote the set of all continuous 2π periodic functions defined on $\mathbb{R}$. Associated with $f \in C_{2\pi}$ we define the nth partial sums

$$S_n(f, t) := a_0 + \sum_{k=1}^{n} (a_k \cos(kt) + b_k \sin(kt))$$

of the Fourier series expansion of f, where

$$a_0 = a_0(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt\,,$$

$$a_k = a_k(f) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\, dt\,, \qquad k = 1, 2, \dots\,,$$

and

$$b_k = b_k(f) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\, dt\,, \qquad k = 1, 2, \dots\,.$$

Observe that if $\alpha : \mathcal{I} \to \{-1, 1\}$ is symmetric, then

$$S_{2n}(G_\alpha, t) = S_{2n-1}(G_\alpha, t) = \sum_{k=1}^{n} b_{2k-1}(G_\alpha) \sin((2k-1)t)\,.$$

Associated with $f \in C_{2\pi}$ we also define

$$E_n(f) := \min_{Q \in \mathcal{T}_n} \|f - Q\|$$

and

$$\omega(f, \delta) := \max_{t \in \mathbb{R}} |f(t + \delta) - f(t)|\,.$$

In the proof of Theorem 6.1 we will use D. Jackson's theorem on best uniform approximation of continuous periodic functions with exact constant. The result below is due to Korneichuk [K-62].

Lemma 3.2. *If $f \in C_{2\pi}$ then*

$$E_n(f) \le \omega\left(f, \frac{\pi}{n+1}\right)\,.$$

In the proof of Theorem 6.1 we will also use the following result of De La Vallée Poussin, the proof of which may be found on pages 273–274 in [D-93].

Lemma 3.3. *Associated with $f \in C_{2\pi}$ let*

$$V_n(f, t) := \frac{1}{n} \sum_{j=n}^{2n-1} S_j(f, t)\,.$$

We have

$$\max_{t \in \mathbb{R}} |V_n(f, t) - f(t)| \le 4E_n(f)\,.$$

The following inequality is known as Bernstein's inequality and plays an important role in the proof of Lemma 3.5.

Lemma 3.4. *We have*

$$\|U^{(k)}\| \le \nu^k \|U\|\,, \qquad U \in \mathcal{T}_\nu\,, \qquad \nu = 1, 2, \dots\,, \quad k = 1, 2, \dots\,.$$

Lemma 3.5. *Suppose $U \in \mathcal{T}_\nu$, $\tau \in [0, 2\pi/\nu]$, $A \geq 0.005$, and $|U(\tau)| \geq A\|U\|$. Let*

$$I_{j,\nu} := \left[\frac{j\eta}{\nu}, \frac{(j+1)\eta}{\nu}\right] \subset \left[\tau, \tau + \frac{18\pi}{\nu}\right], \qquad j = u, u+1, \dots, k\,. \tag{3.1}$$

We have

$$\min_{t \in I_{j,\nu}} |U(t)| \geq \frac{A}{400}\left(\frac{\eta}{18\pi}\right)^{200} \|U\|$$

for at least one $j \in \{v, v+1, \dots, v+399\}$ for every $v \in \{u, u+1, \dots, k-399\}$.

Proof. Suppose the statement of the lemma is false, and there are $v \in \{u, u+1, \dots, k-399\}$ and

$$x_j \in I_{j,\nu} := \left[\frac{j\eta}{\nu}, \frac{(j+1)\eta}{\nu}\right] \subset \left[\tau, \tau + \frac{18\pi}{\nu}\right] \tag{3.2}$$

such that

$$|U(x_j)| < \frac{A}{400}\left(\frac{\eta}{2\pi}\right)^{200} \|U\|\,, \qquad j \in \{v, v+1, \dots, v+399\}\,.$$

Let $y_j := x_{v+2j-1}$ for $j \in \{1, 2, \dots, 200\}$. Then the points y_j satisfy

$$y_1 - \tau \geq \frac{\eta}{\nu} \quad \text{and} \quad y_{j+1} - y_j \geq \frac{\eta}{\nu}\,, \qquad j \in \{1, 2, \dots, 200\}\,.$$

By the well-known formula for divided differences we have

$$U(\tau)\prod_{h=1}^{200}(\tau - y_h)^{-1} + \sum_{j=1}^{200} U(y_j)(\tau - y_j)^{-1} \prod_{\substack{h=1 \\ h \neq j}}^{200} (y_h - y_j)^{-1} = \frac{1}{200!} U^{(200)}(\xi)\,,$$

and combining this with $|U(\tau)| \geq A\|U\|$, (3.1), and (3.2), we get

$$A\|U\|\left(\frac{18\pi}{\nu}\right)^{-200} \leq 200\,\frac{A}{400}\left(\frac{\eta}{18\pi}\right)^{200}\|U\|\left(\frac{\eta}{\nu}\right)^{-200} + \frac{1}{200!}|U^{(200)}(\xi)|\,,$$

with some $\xi \in [\tau, \tau + 18\pi/\nu]$. Therefore Bernstein's inequality (Lemma 3.4) yields that

$$A\|U\|\left(\frac{18\pi}{\nu}\right)^{-200} \leq 200\,\frac{A}{400}\left(\frac{\eta}{18\pi}\right)^{200}\|U\|\left(\frac{\eta}{\nu}\right)^{-200} + \frac{1}{200!}\nu^{200}\|U\|\,,$$

that is,

$$A \leq \frac{2(18\pi)^{200}}{200!} \leq 2\left(\frac{18\pi e}{200}\right)^{200} < 0.005\,,$$

which contradicts our assumption $A \geq 0.005$. □

The following lemma ascribed to M. Riesz is well-known and can easily be proved by a simple zero counting argument (see [B-95], for instance).

Lemma 3.6. *If $T \in \mathcal{T}_\nu$, $t_0 \in \mathbb{R}$, and $|T(t_0)| = \|T\|$, then*

$$|T(t)| \geq |T(t_0)| \cos(\nu(t - t_0)), \qquad t \in \mathbb{R}, \ |t - t_0| \leq \frac{\pi}{2\nu}.$$

We will also need the following simple corollary of the above lemma.

Lemma 3.7. *If $L = 32n$,*

$$t_r := \frac{(2r-1)\pi}{4L}, \qquad r = 1, 2, \ldots, 4L,$$

and $T \in \mathcal{T}_{2n}$, then

$$\max_{t \in \mathbb{R}} |T(t)| \leq (\cos(\pi/64))^{-1} \max_{1 \leq r \leq 4L} |T(t_r)| \leq (1.0013) \max_{1 \leq r \leq 4L} |T(t_r)|.$$

4. Minimizing Discrepancy

Associated with a vector $\mathbf{x} = \langle x_1, x_2, \ldots, x_v \rangle \in \mathbb{R}^v$ let

$$\|\mathbf{x}\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_v|\}.$$

A crucial ingredient in [B-20] is the main "partial coloring" lemma of Spencer [S-85] based on a technique of Beck [B-81]. In Section 4 of [B-20] a simple consequence of a variant of this due to Lovett and Meka [L-15, Theorem 4] is observed, and it plays an important part in the proof of Theorem 6.1. This can be stated as follows.

Lemma 4.1. *Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_u \in \mathbb{R}^v$ and $\mathbf{x}_0 \in [-1,1]^v$. If $c_1, c_2, \ldots, c_u \geq 0$ are such that*

$$\sum_{r=1}^{u} \exp(-(c_r/14)^2) \leq \frac{v}{16}, \tag{4.1}$$

then there exists an $\mathbf{x} \in \{-1,1\}^v$ such that

$$|\langle \mathbf{x} - \mathbf{x}_0, \mathbf{y}_r \rangle| \leq (c_r + 30)\sqrt{u}\, \|\mathbf{y}_r\|_\infty, \qquad r = 1, 2, \ldots, u.$$

5. The Cosine Polynomial

Theorem 5.1. *Let $n > 0$ be a sufficiently large integer divisible by 10. Let $\mu = \gamma n$ defined by (2.3). There exist a cosine polynomial*

$$c(t) = \sum_{k=0}^{\mu} \varepsilon_k \cos(2kt), \qquad \varepsilon_0 = 1, \quad \varepsilon_k \in \{-1,1\}, \quad k = 1, 2, \ldots, \mu, \tag{5.1}$$

and a suitable and well-separated collection $\mathcal{I}$ of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/(2\pi\mathbb{Z})$ such that

$$c(t) \geq \eta_1 \sqrt{n}\,, \qquad t \notin \bigcup_{I \in \mathcal{I}} I\,,$$

and

$$c(t) \leq \sqrt{n}\,, \qquad t \in \mathbb{R}\,,$$

where $\eta_1 > 0$ is an absolute constant.

Proof. Let $c(t) := U(t) := T(2t)$, where $T \in \mathcal{T}_{\mu-1}$ with $\mu := 9M$ is defined by (2.2) and $U \in \mathcal{T}_{\nu-2}$ with $\nu := 2\mu$. Observe that c is of the form (5.1). It follows from (2.1), (2.3), and (2.4) that

$$|c(t)| \leq 9\sqrt{2M} \leq 3\sqrt{2\mu} \leq \sqrt{n}\,.$$

Set

$$\eta := 20\pi\gamma = 20\pi\mu/n = 10\pi\nu/n \qquad \text{and} \qquad \eta_1 := \frac{0.005}{400}\left(\frac{\eta}{18\pi}\right)^{200}.$$

We partition $\mathbb{R}/(2\pi\mathbb{Z})$ into $n/5$ intervals

$$I_j := \left[\frac{10\pi j}{n}, \frac{10\pi(j+1)}{n}\right], \qquad j = 0, 1, \ldots, n/5 - 1\,,$$

and say that an interval I_j is good if

$$\min_{t \in I_j} |U(t)| \geq \frac{0.005}{400}\left(\frac{\eta}{18\pi}\right)^{200} \|U\|\,.$$

Let $\mathcal{J}$ be the collection of maximal unions of consecutive good intervals I_j, and let $\mathcal{I}$ be the collection of the remaining intervals (that is, the maximal unions of consecutive bad intervals). We claim that $\mathcal{I}$ is the required suitable and well-separated collection.

First, to see that $\mathcal{I}$ is suitable, note that the endpoints of each of the intervals I_j are in $10\pi\mathbb{Z}$. The set $\mathcal{I}$ is invariant under the maps $\theta \to \pi \pm \theta$ by the symmetries of the functions $\cos(2kt)$, $k = 0, 1, \ldots, \mu$. To see that $4N = |\mathcal{I}| \leq 4\gamma n$, note that a real trigonometric polynomial of degree at most ν has at most 2ν real zeros in a period, and hence there are at most 4ν values of t in a period for which

$$U(t) = \frac{\pm 0.005}{400}\left(\frac{\eta}{18\pi}\right)^{200} \|U\|\,.$$

Since each $I \in \mathcal{I}$ must contain at least two such points (counted with multiplicities), we have $4N := |\mathcal{I}| \leq 2\nu = 4\mu = 4\gamma n$. Thus $\mathcal{I}$ has each of the properties (a), (b), and (c) in the definition of a suitable collection.

We now show that $\mathcal{I}$ is well-separated. By Lemmas 2.2 3.4, and 3.5 any 400 consecutive intervals I_j must contain a good interval, and hence $|I| \leq 3990\pi/n$ for each $I \in \mathcal{I}$. Thus $\mathcal{I}$ has property (d) in the definition of a well-separated collection. The fact that $\mathcal{I}$ has property (e) in the definition of a suitable collection is obvious by the construction. Finally observe that for an odd m we have $|P_m(1)| = 2^{(m+1)/2} = \|P_m(e^{it})\|$, from which

$$|T(0)| = |T(\pi)| = \|T\|$$

follows. Hence, property (f) in the definition of a well-separated collection follows from the Riesz's Lemma stated as Lemma 3.6 (recall that $\nu = 2\mu = 2\gamma n < 2^{-72} n$). $\square$

6. The Sine Polynomials

Theorem 6.1. *Let $n > 0$ be an integer divisible by 10. Let $\mathcal{I}$ be a suitable and well-separated collection of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/(2\pi\mathbb{Z})$. There exists a sine polynomial*

$$s_o(t) = \sum_{k=1}^{n} \varepsilon(2k-1)\sin((2k-1)t)\,, \qquad \varepsilon(2k-1) \in \{-1,1\}\,,$$

such that

$$|s_o(t)| \geq 36\sqrt{n}\,, \qquad t \in \bigcup_{I\in\mathcal{I}} I\,, \qquad \text{and} \qquad |s_o(t)| \leq 1090\sqrt{n}\,, \qquad t \in \mathbb{R}\,.$$

To prove Theorem 6.1 we need some lemmas.

Lemma 6.2. *Let $\mathcal{I}$ be a suitable and well-separated collection of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/(2\pi\mathbb{R})$. There exists a symmetric coloring $\alpha : \mathcal{I} \to \{-1,1\}$ such that*

$$a_k(G_\alpha) = 0\,, \qquad k = 0,1,\ldots,2n\,,$$

$$b_{2k}(G_\alpha) = 0 \quad \text{and} \quad |b_{2k-1}(G_\alpha)| \leq 1\,, \qquad k = 1,2,\ldots,n\,.$$

Proof. As before, we denote the intervals in a suitable and well-separated collection $\mathcal{I}$ by I_j, $j = 1,2,\ldots,4N$, where $I_1, I_2, \ldots, I_N \subset (0,\pi/2)$. As we have already observed before, we have $a_k(G_\alpha) = 0$, $k = 0,1,\ldots,2n$, and $b_{2k}(G_\alpha) = 0$, $k = 1,2,\ldots,n$, for every symmetric coloring $\alpha : \mathcal{I} \to \{-1,1\}$, so we have to show only that there exists a symmetric coloring $\alpha : \mathcal{I} \to \{-1,1\}$ such that $|b_{2k-1}(G_\alpha)| \leq 1$, $k = 1,2,\ldots,n$. To this end let

$$\mathbf{y}_k := \langle y_{k,1}, y_{k,2}, \ldots, y_{k,N}\rangle\,, \qquad k = 1,2,\ldots,n\,,$$

with

$$y_{k,j} := \frac{4K\sqrt{n}}{\pi}\int_{-\pi}^{\pi} \Phi_{I_j}(t)\sin((2k-1)t)\,dt\,, \qquad k = 1,2,\ldots,n,\ \ j = 1,2,\ldots,N\,.$$

If $\alpha : \mathcal{I} \to \{-1,1\}$ is a symmetric coloring, then by the symmetry conditions on $\mathcal{I}$ we have

$$b_{2k-1}(G_\alpha) := \frac{1}{\pi}\int_{-\pi}^{\pi} G_\alpha(t)\sin((2k-1)t))\,dt = \sum_{j=1}^{N} \alpha(I_j) y_{k,j}\,, \qquad k = 1,2,\ldots,n\,.$$

We apply Lemma 4.1 with $u := n$, $v := N$, $\mathbf{x}_0 := \mathbf{0} \in [-1,1]^N$, and

$$c_1 = c_2 = \cdots = c_n := 14\sqrt{\log(16n/N)}\,.$$

Observe that

$$\sum_{r=1}^{u} \exp(-c_r^2/14^2) = n\frac{N}{16n} = \frac{N}{16}\,,$$

so (4.1) is satisfied. It follows from Lemma 4.1 that there exists an

$$\langle \alpha(I_1), \alpha(I_2), \dots, \alpha(I_N)\rangle = \mathbf{x} \in \{-1,1\}^N$$

such that

$$|\langle \mathbf{x}, \mathbf{y}_k\rangle| \le (c_k + 30)\sqrt{N}\,\|\mathbf{y}_k\|_\infty\,, \qquad k = 1,2,\dots,n\,.$$

As $\mathcal{I}$ is well-separated, by part (d) of the definition we have

$$|y_{k,j}| \le \frac{4K\sqrt{n}}{\pi}(|I_j| + 10\pi/n) \le \frac{4K\sqrt{n}}{\pi}\frac{4000\pi}{n} = \frac{16000K}{\sqrt{n}}$$

for every $k = 1,2,\dots,n$ and $j = 1,2,\dots,N$. It follows that

$$|b_{2k-1}(G_\alpha)| = |\langle \mathbf{x}, \mathbf{y}_k\rangle| \le (14\sqrt{\log(16n/N)} + 30)\sqrt{N/n}\cdot 16000K\,, \qquad k = 1,2,\dots,n\,.$$

As the right-hand side above is an increasing function of N for $N/n \le \gamma < 1$, we have

$$|b_{2k-1}(G_\alpha)| = |\langle \mathbf{x}, \mathbf{y}_k\rangle| \le (14\sqrt{\log(16/\gamma)} + 30)\sqrt{\gamma}\cdot 16000K \le 1\,, \qquad k = 1,2,\dots,n\,,$$

where the last inequality follows from $K := 2^9$ and the inequality $2^{-75} \le \gamma < 2^{-73}$. Hence the desired symmetric coloring is given by setting

$$\langle \alpha(I_1), \alpha(I_2), \dots, \alpha(I_N)\rangle := \mathbf{x}\,.$$

□

From now on let $\alpha : \mathcal{I} \to \{-1,1\}$ denote the symmetric coloring guaranteed by Lemma 6.2. Then we have

$$V_n(G_\alpha, t) = \sum_{k=0}^{n} \widetilde{\varepsilon}(2k-1)\sin((2k-1)t)\,, \qquad |\widetilde{\varepsilon}(2k-1)| \le 1\,.$$

Lemma 6.3. *There is a coloring $\varepsilon : S_o \to \{-1,1\}$ such that with the notation*

$$s_o(t) = \sum_{k=1}^{n} \varepsilon(2k-1)\sin((2k-1)t)$$

we have

$$|s_o(t) - V_n(G_\alpha, t)| \le 66\sqrt{n}\,, \qquad t \in \mathbb{R}\,.$$

Proof. Let $L := 32n$,

$$t_r := \frac{(2r-1)\pi}{4L}\,, \qquad r = 1,2,\dots,4L\,,$$

$$y_{r,k} := \sin((2k-1)t_r)\,, \qquad r = 1,2,\dots,L\,, \;\; k = 1,2,\dots,n\,,$$
$$\mathbf{y}_r := \langle y_{r,1}, y_{r,2}, \dots, y_{r,n}\rangle\,, \qquad r = 1,2,\dots,L\,.$$

Observe that

$$s_o(t_r) - V_n(G_\alpha, t_r) = \sum_{k=1}^{n} (\varepsilon(2k-1) - \widetilde{\varepsilon}(2k-1))y_{r,k} = \langle \mathbf{e} - \widetilde{\mathbf{e}}, \mathbf{y}_r\rangle\,, \tag{6.1}$$

where

$$\mathbf{e} := \langle \varepsilon(1), \varepsilon(3), \dots, \varepsilon(2n-1)\rangle \qquad \text{and} \qquad \widetilde{\mathbf{e}} := \langle \widetilde{\varepsilon}(1), \widetilde{\varepsilon}(3), \dots, \widetilde{\varepsilon}(2n-1)\rangle\,.$$

We apply Lemma 4.1 with $u := L$, $v := n$, $\mathbf{x}_0 := \widetilde{\mathbf{e}}$, and

$$c_1 = c_2 = \cdots = c_n := 42\sqrt{\log 2}\,.$$

Observe that

$$\sum_{r=1}^{u} \exp(-c_r^2/14^2) = L2^{-9} = \frac{n}{16}\,,$$

so (4.1) is satisfied. It follows from Lemma 4.1 that there exists an $\mathbf{e} \in \{-1,1\}^n$ such that

$$|\langle \mathbf{e} - \widetilde{\mathbf{e}}, \mathbf{y}_r\rangle| \le (c_r + 30)\sqrt{n}\|\mathbf{y}_r\|_\infty \le (c_r + 30)\sqrt{n} \le 65\sqrt{n}\,, \qquad r = 1,2,\dots,L\,. \tag{6.2}$$

Combining (6.1) and (6.2) we obtain

$$|s_o(t_r) - V_n(G_\alpha, t_r)| \le 65\sqrt{n}\,, \qquad r = 1,2,\dots,L\,.$$

Note that by the special form of the trigonometric polynomials s_o and $V_n(G_\alpha, \cdot)$ we have

$$\max_{1 \le r \le L} |s_o(t_r) - V_n(G_\alpha, t_r)| = \max_{1 \le r \le 4L} |s_o(t_r) - V_n(G_\alpha, t_r)|\,,$$

hence

$$|s_o(t_r) - V_n(G_\alpha, t_r)| \le 65\sqrt{n}\,, \qquad r = 1,2,\dots,4L\,.$$

This, together with Lemma 3.7 gives the lemma. $\square$

Lemma 6.4. *We have*

$$|V_n(G_\alpha, t)| \ge \frac{K\sqrt{n}}{5}\,, \qquad t \in \bigcup_{I \in \mathcal{I}} I\,, \qquad \textit{and} \qquad |V_n(G_\alpha, t)| \le 2K\sqrt{n}\,, \qquad t \in \mathbb{R}\,.$$

Proof. Combining Lemmas 3.3 and 3.2 we have

$$\max_{t \in \mathbb{R}} |V_n(G_\alpha, t) - G_\alpha(t)| \le 4E_n(G_\alpha) \le 4\omega(G_\alpha, \pi/n) \le \frac{4K\sqrt{n}}{5}\,,$$

and the lemma follows. $\square$

Let $\mu = 9M = 9 \cdot 2^m$ be the same as in Section 2, and let

$$s_e(t) := \mathrm{Im}(P_{<(n+1)}(e^{2it})) - \mathrm{Im}(P_{<\mu}(e^{2it}))\,.$$

Lemma 6.5. *We have*

$$\|s_e\| \leq 6\sqrt{n}\,.$$

Proof. This is an obvious consequence of Lemma 2.1. Recall that $\mu = \gamma n < 2^{-73} n$. □

Proof of Theorem 6.1. Let $\mathcal{I}$ be a suitable and well-separated collection of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/(2\pi\mathbb{Z})$. By Lemma 6.3 there is a coloring $\varepsilon : S_o \to \{-1, 1\}$ such that if $\alpha : \mathcal{I} \to \{-1, 1\}$ is the symmetric coloring given by Lemma 6.2, then

$$|s_o(t) - V_n(G_\alpha, t)| \leq 66\sqrt{n}\,, \qquad t \in \mathbb{R}\,.$$

Hence by Lemma 6.4 and $K := 2^9$ we have

$$|s_o(t)| \geq |V_n(G_\alpha, t)| - |s_o(t) - V_n(G_\alpha, t)| \geq 102\sqrt{n} - 66\sqrt{n} \geq 36\sqrt{n}\,, \qquad t \in \bigcup_{I \in \mathcal{I}} I\,,$$

and

$$|s_o(t)| \leq |V_n(G_\alpha, t)| + |s_o(t) - V_n(G_\alpha, t)| \leq 2^{10}\sqrt{n} + 66\sqrt{n} \leq 1090\sqrt{n}\,, \quad t \in \mathbb{R}\,.$$

□

7. Proof of Theorems 1.1 and 1.2

Proof of the Theorems 1.2. It is sufficient to prove the theorem with $2n$ replaced by $4n$, and without loss of generality we may assume that $n > 0$ is an integer divisible by 10. Since the Littlewood polynomial $P_{4n}(z) := 1 - z - z^2 - \cdots - z^{4n}$ does not vanish on the unit circle, we may assume also that n is sufficiently large. By Theorems 5.1 and 6.1 the Littlewood polynomial P_{4n} of degree $4n$ defined by

$$P_{4n}(e^{it})e^{-2int} = (-1 + 2c(t)) + 2i(s_o(t) + s_e(t))$$

has the properties required by the theorem. It is obvious from the construction that the coefficients of P_{4n} satisfy the requirements. To see that the required inequalities are satisfied let $\mathcal{I}$ be a suitable and well-separated collection of disjoint closed intervals (with nonempty interiors) in $\mathbb{R}/(2\pi\mathbb{Z})$ on which (5.1) holds. Then Theorem 5.1 gives that

$$|P_{4n}(e^{it})| \geq |-1 + 2c(t)| \geq \eta_1\sqrt{n}\,, \qquad t \notin \bigcup_{I \in \mathcal{I}} I\,,$$

while Theorem 6.1 gives that

$$|P_{4n}(e^{it})| \geq |2(s_o(t) + s_e(t))| \geq |2s_o(t)| - |2s_e(t)| \geq 72\sqrt{n} - 12\sqrt{n} = 60\sqrt{n}\,, \quad t \in \bigcup_{I \in \mathcal{I}} I\,.$$

Combining the two inequalities above gives the lower bound of the theorem. The upper bounds of the theorem follows from combining the upper bounds of Theorems 5.1 and 6.1 by

$$\begin{aligned} |P_{4n}(e^{it})| &\leq |-1 + 2c(t)| + |2(s_o(t) + s_e(t))| \leq 1 + 2\sqrt{n} + 2180\sqrt{n} + 12\sqrt{n} \\ &\leq 1 + 2194\sqrt{n}\,, \quad t \in \mathbb{R}\,. \end{aligned}$$

For the value m_n in the theorem we have $m_n = 2\mu = 2\gamma n$, so $\eta = 2\gamma > 0$ can be chosen. □

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