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Weighted Markov and Bernstein Type Inequalities for Generalized Non-negative Polynomials

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Weighted Markov and Bernstein type inequalities are established for generalized non-negative polynomials and generalized polynomial weight functions. The novelty of the results lies in the fact that in these estimates only the generalized degrees of the generalized polynomial and the generalized polynomial weight function, respectively, and a multiplicative absolute constant show up. To prove such inequalities was motivated by studying systems of orthogonal polynomials simultaneously, associated with generalized Jacobi weight functions.

1. INTRODUCTION

The well-known Markov–Bernstein inequality [10, pp. 39–41] asserts that

$$|p'(y)| \leqslant \min\left\{\frac{n}{\sqrt{1-y^2}}, n^2\right\} \max_{-1 \leqslant x \leqslant 1} |p(x)| \qquad (-1 < y < 1) \tag{1.1}$$

for every polynomial $p \in \Pi_n$, where Π_n denotes the set of all real algebraic polynomials of degree at most n. Markov and Bernstein type inequalities in weighted spaces play a significant role in proving inverse theorems of approximation and have their own intrinsic interest. The magnitudes of

$$\frac{\max_{-1 \leqslant x \leqslant 1} |f'(x)\, w(x)|}{\max_{-1 \leqslant x \leqslant 1} |f(x)\, w(x)|}, \tag{1.2}$$

$$\frac{|f'(y)\, w(y)|}{\max_{-1 \leqslant x \leqslant 1} |f(x)\, w(x)|} \qquad (-1 < y < 1) \tag{1.3}$$

and their corresponding L_p analogues, respectively, for polynomials $f \in \Pi_n$ and generalized Jacobi weight functions w were examined by a number of authors [1, pp. 90–111], [8, 9, 11, 13, 14], [12, pp. 161–164], but a multiplicative constant depending on the weight function appears in these

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estimates. In the next section we introduce generalized polynomials and examine the magnitudes of (1.2) and (1.3), when both f and w are the absolute values of generalized polynomials (in other words both f and w are generalized non-negative polynomials; see the remark at the end of Section 2). In our inequalities only the generalized degrees of f and w, respectively, and a multiplicative absolute constant show up. The results are new and (in a sense) sharp even when f is an ordinary polynomial. In [4] the case $w \equiv 1$ is studied and some ideas from that case play a key role in this paper as well.

Our motivation was to find tools with which to examine systems of orthogonal polynomials simultaneously, associated with generalized Jacobi, or at least generalized non-negative polynomial weight functions of degree at most $\Gamma > 0$. In a recent paper [7] we gave sharp estimates in this spirit for the Christoffel function on $[-1, 1]$, and for the distances of the consecutive zeros of orthogonal polynomials, associated with generalized non-negative polynomial weight functions of degree at most Γ.

2. Generalized Polynomials, Definitions, and Notations

Generalized algebraic and trigonometric polynomials were introduced, studied thoroughly, and applied in a sequence of papers [2–4, 6, 7]. Denote by Π_n the set of all real algebraic polynomials of degree at most n. The function

$$f = \prod_{j=1}^{k} P_{n_j}^{r_j} \qquad (P_{n_j} \in \Pi_{n_j} \setminus \Pi_{n_j - 1}, r_j > 0, j = 1, 2, ..., k) \tag{2.1}$$

will be called a generalized real algebraic polynomial of (generalized) degree

$$N = \sum_{j=1}^{k} r_j n_j. \tag{2.2}$$

To be precise, in this paper we will use the definition

$$z^r = \exp(r \log|z| + ir \arg z) \qquad (z \in \mathbb{C}, r > 0, -\pi \leqslant \arg z < \pi). \tag{2.3}$$

Obviously

$$|f| = \prod_{j=1}^{k} |P_{n_j}|^{r_j}. \tag{2.4}$$

We will denote by GRAP_N the set of all generalized real algebraic polynomials of degree at most N. We introduce the class $|\mathrm{GRAP}|_N = \{|f| : f \in \mathrm{GRAP}_N\}$. The function

$$f(z) = c \prod_{j=1}^{k} (z - z_j)^{r_j} \qquad (0 \neq c,\ z_j \in \mathbb{C},\ r_j > 0,\ j = 1, 2, ..., k) \tag{2.5}$$

will be called a generalized complex algebraic polynomial of (generalized) degree

$$N = \sum_{j=1}^{k} r_j. \tag{2.6}$$

We have

$$|f(z)| = |c| \prod_{j=1}^{k} |z - z_j|^{r_j}. \tag{2.7}$$

Denote by GCAP_N the set of all generalized complex algebraic polynomials of degree at most N. The set $\{|f| : f \in \mathrm{GCAP}_N\}$ will be denoted by $|\mathrm{GCAP}|_N$.

In the trigonometric case we denote the set of all real trigonometric polynomials by T_n. The function

$$f = \prod_{j=1}^{k} P_{n_j}^{r_j} \qquad (P_{n_j} \in T_{n_j} \backslash T_{n_j - 1},\ r_j > 0,\ j = 1, 2, ..., k) \tag{2.8}$$

will be called a generalized real trigonometric polynomial of (generalized) degree N defined by (2.2). Obviously (2.4) holds again. We will denote by GRTP_N the set of all generalized real trigonometric polynomials of degree at most N. Let $|\mathrm{GRTP}|_N = \{|f| : f \in \mathrm{GRTP}_N\}$. We say that the function

$$f(z) = c \prod_{j=1}^{k} (\sin((z - z_j)/2))^{r_j} \qquad (0 \neq c \in \mathbb{C},\ z_j \in \mathbb{C},\ r_j > 0,\ j = 1, 2, ..., k) \tag{2.9}$$

is a generalized complex trigonometric polynomial of (generalized) degree

$$N = \tfrac{1}{2} \sum_{j=1}^{k} r_j. \tag{2.10}$$

We have

$$|f(z)| = |c| \prod_{j=1}^{k} |\sin((z - z_j)/2)|^{r_j}. \tag{2.11}$$

Denote the set of all generalized complex trigonometric polynomials of degree at most N by GCTP_N. The set $\{|f|: f \in \mathrm{GCTP}_N\}$ will be denoted by $|\mathrm{GCTP}|_N$.

We remark that if $f \in |\mathrm{GCAP}|_N$, then restricted to the real line we have $f \in |\mathrm{GRAP}|_N$. Similarly if $f \in |\mathrm{GCTP}|_N$, then restricted to the real line we have $f \in |\mathrm{GRTP}|_N$. These follow from the observations

$$|z - z_j| = ((z - z_j)(z - \bar{z}_j))^{1/2} \qquad (z \in \mathbb{R}) \tag{2.12}$$

and

$$\begin{aligned} |\sin((z - z_j)/2)| &= (\sin((z - z_j)/2) \sin((z - \bar{z}_j)/2))^{1/2} \\ &= ((\cosh(\operatorname{Im} z_j) - \cos(z - \operatorname{Re} z_j))/2)^{1/2} \qquad (z \in \mathbb{R}). \end{aligned} \tag{2.13}$$

Using (2.12) and (2.13) one can easily check that restricted to the real line

$$|\mathrm{GCAP}|_N = \left\{ f = \prod_{j=1}^{k} P_j^{r_j/2} : 0 \leqslant P_j \in \Pi_2,\, r_j > 0,\, j = 1, 2, ..., k;\ \sum_{j=1}^{k} r_j \leqslant N \right\}$$

and

$$|\mathrm{GCTP}|_N = \left\{ f = \prod_{j=1}^{k} P_j^{r_j/2} : 0 \leqslant P_j \in T_1,\, r_j > 0,\, j = 1, 2, ..., k;\ \sum_{j=1}^{k} r_j \leqslant 2N \right\}.$$

The subject of this paper is the classes $|\mathrm{GCAP}|_N$ and $|\mathrm{GCTP}|_N$ restricted to the real line, and the elements of these classes can be considered as generalized non-negative polynomials in the above sense. This explains the title. Throughout this paper c_i, $i = 1, 2, ...,$ denotes a suitable positive absolute constant, and f' means df/dt, where t is real.

3. New Results

We prove the following weighted Markov-type inequality.

Theorem 1. *Let $f \in |\mathrm{GCAP}|_N$ be of the form (2.7) with $r_j \geqslant 1$ $(1 \leqslant j \leqslant k)$ and let $w \in |\mathrm{GCAP}|_\Gamma$ be arbitrary. Then*

$$\max_{-1 \leqslant x \leqslant 1} |f'(x) \cdot w(x)| \leqslant c_1 (N + \Gamma)^2 \max_{-1 \leqslant x \leqslant 1} f(x)\, w(x),$$

where c_1 is an absolute constant.

In the trigonometric case we show

THEOREM 2. *Let $f \in |\mathrm{GCTP}|_N$ be of the form* (2.11) *with $r_j \geqslant 1$ $(1 \leqslant j \leqslant k)$ and let $w \in |\mathrm{GCTP}|_\Gamma$ be arbitrary. Then*

$$\max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, w(x)| \leqslant c_2(\Gamma+1)(N+\Gamma) \max_{-\pi \leqslant x \leqslant \pi} f(x)\, w(x),$$

where c_2 is an absolute constant.

By the substitution $x = \cos t$, from Theorem 2 we will easily obtain

THEOREM 3. *Let $f \in |\mathrm{GCAP}|_N$ be of the form* (2.7) *with $r_j \geqslant 1$ $(1 \leqslant j \leqslant k)$ and let $w \in |\mathrm{GCAP}|_\Gamma$ be arbitrary. Then*

$$|f'(y)\, w(y)| \leqslant \frac{c_2(\Gamma+1)(N+\Gamma)}{\sqrt{1-y^2}} \max_{-1 \leqslant x \leqslant 1} f(x)\, w(x) \qquad (-1 < y < 1),$$

where c_2 is the same as in Theorem 2.

Remark 3.1. The problem arises, how to define f' for an $f \in |\mathrm{GCAP}|_N$ or $f \in |\mathrm{GCTP}|_N$. Observe that though f' may not exist (even as an extended real) at the zeros of f, the one-sided derivatives exist and are equal in absolute value.

Remark 3.2. We conjecture that in the inequalities of Theorems 2 and 3 the multiplicative factor $(\Gamma+1)$ can be dropped. If this conjecture were true, we would obtain Theorem 1 as a simple consequence of Theorem 3, using a Remez-type inequality [2, Theorem 1] for generalized complex algebraic polynomials.

Remark 3.3. If $f \in |\mathrm{GCAP}|_N$ is of the form (2.7) with $r_j \geqslant 1$ $(1 \leqslant j \leqslant k)$, then $f \in |\mathrm{GRAP}|_N$ is of the form (2.4) with $r_j \geqslant \frac{1}{2}$, $P_{n_j}(z) \geqslant 0$ $(z \in \mathbb{R}$, $0 \leqslant j \leqslant h)$, and $r_j \geqslant 1$ $(h < j \leqslant k)$, where $(0 \leqslant) h\, (\leqslant k)$ is a suitable integer. Furthermore, $w \in |\mathrm{GCAP}|_\Gamma$ implies $w \in |\mathrm{GRAP}|_\Gamma$. Similar relations hold for generalized complex and real trigonometric polynomials. These follow easily from (2.12), (2.13) and the fact that we study each function restricted to the real line.

Remark 3.4. If $f \in |\mathrm{GCAP}|_N$ is of the form

$$f(z) = |c| \prod_{j=1}^{k} |z - z_j|^{r_j}$$

$$(0 \neq c \in \mathbb{C},\ z_j \in \mathbb{R} \text{ are different, } r_j \geqslant 1,\ j = 1, 2, ..., k), \tag{3.1}$$

then $|f'| \in |\mathrm{GCAP}|_N$ has only real zeros, and at least one of any two adjacent zeros of $|f'|$ has multiplicity at least 1 (for generalized polynomials the multiplicities of the zeros can be arbitrary positive real numbers). A similar statement holds for every $f \in |\mathrm{GCTP}|_N$ of the form

$$f(z) = |c| \prod_{j=1}^{k} |\sin((z - z_j)/2)|^{r_j}$$

$$(0 \neq c \in \mathbb{C},\ z_j \in \mathbb{R} \text{ are different},\ r_j \geqslant 1,\ j = 1, 2, ..., k). \tag{3.2}$$

We discuss only the trigonometric case; the same argument works in the algebraic case as well. By (3.2) we have

$$f = |c| \prod_{j=1}^{k} P_j^{r_j/2} \qquad \text{with} \quad P_j(z) = \sin^2((z - z_j)/2), \tag{3.3}$$

and using the product rule, we obtain

$$|f'| = |c/2| \prod_{j=1}^{k} P_j^{r_j/2 - 1} \left| \sum_{i=1}^{k} r_i P_i' \prod_{\substack{j=1 \\ j \neq i}}^{k} P_j \right|. \tag{3.4}$$

Observe that in (3.4)

$$Q = \sum_{i=1}^{k} r_i P_i' \prod_{\substack{j=1 \\ j \neq i}}^{k} P_j \in T_k \tag{3.5}$$

is an ordinary trigonometric polynomial. Without loss of generality we may assume that $-\pi \leqslant z_1 < z_2 < \cdots < z_k < \pi$. By Rolle's theorem $|f'|$ has a zero y_j in each of the intervals (z_j, z_{j+1}) if $1 \leqslant j \leqslant k-1$, and a zero y_k in $(z_k, z_1 + 2\pi)$. Hence from (3.3), (3.4), and (3.5) we easily deduce that

$$Q(z) = c' \prod_{j=1}^{k} \sin((z - z_j)/2) \sin((z - y_j)/2), \qquad c' \in \mathbb{R}. \tag{3.6}$$

This, together with (3.3) and (3.4), yields that

$$|f'(z)| = |c''| \prod_{j=1}^{k} |\sin(z - z_j)/2|^{r_j - 1} \prod_{j=1}^{k} |\sin((z - y_j)/2)|, \qquad c'' \in \mathbb{C}, \tag{3.7}$$

which gives the desired result.

4. Lemmas for Theorem 1

Our first two lemmas guarantee the existence of certain extremal generalized polynomials.

LEMMA 4.1. *If* $m_j \geqslant 0$ $(1 \leqslant j \leqslant k)$ *are fixed integers,* $s_j > 0$ $(1 \leqslant j \leqslant k)$ *are fixed reals such that* $\sum_{j=1}^{s} s_j m_j \leqslant \Gamma$, *then for every* $0 \not\equiv f \in |\mathrm{GCAP}|_N$ *of the form* (2.7) *with* $r_j \geqslant 1$ $(1 \leqslant j \leqslant k)$ *and for every* $-1 < \delta < 1$ *there exists a* $\tilde{w} \in |\mathrm{GRAP}|_\Gamma$ *of the form*

$$\tilde{w} = \prod_{j=1}^{k} |\tilde{Q}_{m_j}|^{s_j} \qquad (\tilde{Q}_{m_j} \in \Pi_{m_j}, 1 \leqslant j \leqslant k)$$

such that

$$\frac{|f'(1)\,\tilde{w}(1)|}{\max_{-1 \leqslant x \leqslant \delta} f(x)\,\tilde{w}(x)} = \sup_{w} \frac{|f'(1)\,w(1)|}{\max_{-1 \leqslant x \leqslant \delta} f(x)\,w(x)} = L < \infty \tag{4.1}$$

and

$$|f'(1)\,\tilde{w}(1)| = \max_{-1 \leqslant x \leqslant 1} |f'(x)\,\tilde{w}(x)|, \tag{4.2}$$

where the supremum in (4.1) *is taken for all* $0 \not\equiv w \in |\mathrm{GRAP}|_\Gamma$ *of the form*

$$w = \prod_{j=1}^{k} |Q_{m_j}|^{s_j} \qquad (Q_{m_j} \in \Pi_{m_j}, 1 \leqslant j \leqslant k)$$

such that

$$|f'(1)\,w(1)| = \max_{-1 \leqslant x \leqslant 1} |f'(x)\,w(x)|.$$

LEMMA 4.2. *If* $n_j \geqslant 0$ $(1 \leqslant j \leqslant k)$ *and* $0 \leqslant h \leqslant k$ *are fixed integers,* $r_j \geqslant \frac{1}{2}$ $(1 \leqslant j \leqslant h)$ *and* $r_j \geqslant 1$ $(h < j \leqslant k)$ *are fixed reals such that* $\sum_{j=1}^{k} r_j n_j \leqslant N$, *then for every* $0 \not\equiv w \in |\mathrm{GRAP}|_\Gamma$ *and for every* $-1 < \delta < 1$ *there exists an* $\tilde{f} \in |\mathrm{GRAP}|_N$ *of the form*

$$\tilde{f} = \prod_{j=1}^{k} |\tilde{P}_{n_j}|^{r_j} \qquad (\tilde{P}_{n_j} \in \Pi_{n_j}\,(1 \leqslant j \leqslant k),\ \tilde{P}_{n_j}(z) \geqslant 0\ (z \in \mathbb{R}, 1 \leqslant j \leqslant h))$$

such that

$$\frac{|\tilde{f}'(1)\,w(1)|}{\max_{-1 \leqslant x \leqslant \delta} \tilde{f}(x)\,w(x)} = \sup_{f} \frac{|f'(1)\,w(1)|}{\max_{-1 \leqslant x \leqslant \delta} f(x)\,w(x)} = K < \infty, \tag{4.3}$$

where the supremum in (4.3) *is taken for all* $0 \not\equiv f \in |\mathrm{GRAP}|_N$ *of the form*

$$f = \prod_{j=1}^{k} |P_{n_j}|^{r_j} \qquad (P_{n_j} \in \Pi_{n_j}\,(1 \leqslant j \leqslant k),\ P_{n_j}(z) \geqslant 0\ (z \in \mathbb{R}, 1 \leqslant j \leqslant h)).$$

Our next two lemmas show that the extremal generalized polynomials defined by Lemmas 4.1 and 4.2, respectively, have some additional properties.

LEMMA 4.3. *Let $\tilde{w} \in |\mathrm{GRAP}|_\Gamma$ be defined by Lemma 4.1. Then $\tilde{w}$ has all its zeros in $[-1, \delta]$.*

LEMMA 4.4. *Let $\tilde{f} \in |\mathrm{GRAP}|_N$ be defined by Lemma 4.2. Then $\tilde{f}$ has only zeros.*

The following Remez-type inequality was proved in [2].

LEMMA 4.5. *For every $g \in |\mathrm{GCAP}|_N$ and $0 < s < 2$ we have*

$$m(\{y \in [-1, 1]: g(y) \geqslant \exp(-N\sqrt{s}) \max_{-1 \leqslant x \leqslant 1} g(x)\}) \geqslant c_3 s,$$

where $m(\cdot)$ denotes the Lebesgue measure and $c_3 > 0$ is an absolute constant.

From Lemma 4.5, by a Phragmen–Lindelöf type argument we will easily obtain

LEMMA 4.6. *Let $N \geqslant 1$ and $g \in |\mathrm{GCAP}|_N$ be such that*

$$g(1) = \max_{-1 \leqslant x \leqslant 1} g(x). \tag{4.4}$$

Then for every $0 < c_4 \leqslant 1$ there are $c_5 > 1$ and $c_6 > 0$ depending only on c_4 such that

$$m(\{y \in [1 - c_4 N^{-2}, 1]: g(y) \geqslant c_5^{-1} \max_{-1 \leqslant x \leqslant 1} g(x)\}) \geqslant c_6 N^{-2}$$

holds.

The following lemma can be found in [5].

LEMMA 4.7. *Let $r > 0$ and $0 \not\equiv p \in \Pi_n$ be such that*

$$|p(1)| = \max_{-1 \leqslant x \leqslant 1} |p(x)|. \tag{4.5}$$

Then p has at most $c_7 n\sqrt{r}$ zeros in $[1 - r, 1]$, where c_7 is an absolute constant.

We remark that in Lemma 4.7, $c_7 = \sqrt{2}$ can be chosen. However, in the sequel we assume only that $c_7 \geqslant \sqrt{2}$ in a suitable choice. From Lemma 4.7 we will easily conclude

LEMMA 4.8. *Let $r > 0$, $0 \not\equiv g \in |\mathrm{GCAP}|_N$ be of the form* (2.7) *such that each r_j is rational, and assume that*

$$g(1) = \max_{-1 \leqslant x \leqslant 1} g(x). \tag{4.6}$$

Then the total multiplicity of the zeros of g lying in $[1-r, 1]$ is at most $c_7 N\sqrt{r}$, where c_7 is the same absolute constant as in Lemma 4.7.

From Lemmas 4.6 and 4.8 we will obtain

LEMMA 4.9. *Assume that $f \in |\mathrm{GCAP}|_N$ and $w \in |\mathrm{GCAP}|_\Gamma$ are of the forms*

$$f(z) = \prod_{j=1}^{k_1} |z - z_j|^{r_j} \qquad (z_j \text{ are real, } r_j \geqslant 1 \text{ are rational, } 1 \leqslant j \leqslant k_1) \tag{4.7}$$

and

$$w(z) = \prod_{j=1}^{k_2} |z - u_j|^{s_j} \qquad (u_j \in [-1, 1 - 2c_4(N+\Gamma)^{-2}],$$

$$c_4 = c_7^{-2},\ s_j > 0 \text{ are rational, } 1 \leqslant j \leqslant k_2), \tag{4.8}$$

respectively, and

$$|f'(1)\, w(1)| = \max_{-1 \leqslant x \leqslant 1} |f'(x)\, w(x)|. \tag{4.9}$$

Then

$$|f'(1)\, w(1)| \leqslant c_8 (N+\Gamma)^2 \max_{-1 \leqslant x \leqslant \delta} f(x)\, w(x),$$

where $\delta = 1 - 2c_4(N+\Gamma)^{-2}$ and c_8 is an absolute constant.

Our last lemma drops some assumptions from Lemma 4.9 and gives the same conclusion.

LEMMA 4.10. *Assume that $f \in |\mathrm{GCAP}|_N$ and $w \in |\mathrm{GCAP}|_\Gamma$ are of the forms*

$$f(z) = \prod_{j=1}^{k_1} |z - z_j|^{r_j} \qquad (z_j \text{ are real, } r_j \geqslant 1 \text{ are rational, } 1 \leqslant j \leqslant k_1) \tag{4.10}$$

and

$$w(z) = \prod_{j=1}^{k_2} |z - u_j|^{s_j} \qquad (u_j \in \mathbb{C},\ s_j > 0 \text{ are rational, } 1 \leqslant j \leqslant k_2), \tag{4.11}$$

respectively. Then

$$|f'(1)\, w(1)| \leqslant c_9(N+\Gamma)^2 \max_{-1 \leqslant x \leqslant \delta} f(x)\, w(x), \tag{4.12}$$

where δ is the same as in Lemma 4.9 and c_9 is an absolute constant.

5. Lemmas for Theorem 2

Our lemmas for Theorem 2 are very similar to the corresponding ones from Section 4. The unwanted factor $\Gamma+1$ appears in Lemma 5.9 first, since Lemma 5.8, though it is sharp, cannot be exploited to such an extent in Lemma 5.9 (see (7.14)) as Lemma 4.8 in Lemma 4.9 in the corresponding algebraic case.

Lemma 5.1. *If $m_j \geqslant 0$ $(1 \leqslant j \leqslant k)$ are fixed integers, $s_j > 0$ $(1 \leqslant j \leqslant k)$ are fixed reals such that $\sum_{j=1}^k s_j m_j \leqslant N$, then for every $0 \not\equiv f \in |\mathrm{GCTP}|_N$ of the form (2.11) with $r_j \geqslant 1$ $(1 \leqslant j \leqslant k)$ and for every $0 < \delta < \pi$ there exists a $\tilde{w} \in |\mathrm{GRTP}|_\Gamma$ of the form*

$$\tilde{w} = \prod_{j=1}^{k} |\tilde{Q}_{m_j}|^{s_j} \qquad (\tilde{Q}_{m_j} \in T_{m_j},\ 1 \leqslant j \leqslant k)$$

such that

$$\frac{|f'(\pi)\, \tilde{w}(\pi)|}{\max_{-\delta \leqslant x \leqslant \delta} f(x)\, \tilde{w}(x)} = \sup_w \frac{|f'(\pi)\, w(\pi)|}{\max_{-\delta \leqslant x \leqslant \delta} f(x)\, w(x)} = L < \infty \tag{5.1}$$

and

$$|f'(\pi)\, \tilde{w}(\pi)| = \max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, \tilde{w}(x)|, \tag{5.2}$$

where the supremum in (5.1) is taken for all $0 \not\equiv w \in |\mathrm{GRTP}|_\Gamma$ of the form

$$w = \prod_{j=1}^{k} |Q_{m_j}|^{s_j} \qquad (Q_{m_j} \in T_{m_j},\ 1 \leqslant j \leqslant k)$$

such that

$$|f'(\pi)\, w(\pi)| = \max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, w(x)|.$$

Lemma 5.2. *If $n_j \geqslant 0$ $(1 \leqslant j \leqslant k)$ and $0 \leqslant h \leqslant k$ are fixed integers, $r_j \geqslant \frac{1}{2}$ $(1 \leqslant j \leqslant h)$ and $r_j \geqslant 1$ $(h < j \leqslant k)$ are fixed reals such that $\sum_{j=1}^k r_j n_j \leqslant N$,* *then for every* $0 \not\equiv w \in |\mathrm{GRTP}|_r$ *and for every* $0 < \delta < \pi$ *there exists an* $\tilde{f} \in |\mathrm{GRTP}|_N$ *of the form*

$$\tilde{f} = \prod_{j=1}^{k} |\tilde{P}_{n_j}|^{r_j} \qquad (\tilde{P}_{n_j} \in T_{n_j}\ (1 \leqslant j \leqslant k),\ \tilde{P}_{n_j}(z) \geqslant 0\ (z \in \mathbb{R},\ 1 \leqslant j \leqslant h))$$

such that

$$\frac{|\tilde{f}'(\pi)\, w(\pi)|}{\max_{-\delta \leqslant x \leqslant \delta} \tilde{f}(x)\, w(x)} = \sup_f \frac{|f'(\pi)\, w(\pi)|}{\max_{-\delta \leqslant x \leqslant \delta} f(x)\, w(x)} = K < \infty, \qquad (5.3)$$

where the supremum is taken for all $0 \not\equiv f \in |\mathrm{GRTP}|_N$ *of the form*

$$f = \prod_{j=1}^{k} |P_{n_j}|^{r_j} \qquad (P_{n_j} \in T_{n_j}\ (1 \leqslant j \leqslant k),\ P_{n_j}(z) \geqslant 0\ (z \in \mathbb{R},\ 1 \leqslant j \leqslant h)).$$

LEMMA 5.3. *Let* $\tilde{w} \in |\mathrm{GRTP}|_r$ *be defined by Lemma* 5.1. *Then* $\tilde{w}$ *has all its zeros in* $[-\delta, \delta]$.

LEMMA 5.4. *Let* $\tilde{f} \in |\mathrm{GRTP}|_N$ *be defined by Lemma* 5.2. *Then* $\tilde{f}$ *has only real zeros.*

The following Remez-type inequality was proved in [2].

LEMMA 5.5. *For every* $g \in |\mathrm{GCTP}|_N$ *and* $0 < s < 2\pi$ *we have*

$$m(\{y \in [-\pi, \pi) \colon g(y) \geqslant \exp(-Ns) \max_{-\pi \leqslant x \leqslant \pi} g(x)\}) \geqslant c_{10} s,$$

where $m(\cdot)$ *denotes the Lebesgue measure and* c_{10} *is a positive absolute constant.*

LEMMA 5.6. *Let* $N \geqslant 1$ *and* $g \in |\mathrm{GCTP}|_N$ *be such that*

$$g(\pi) = \max_{-\pi \leqslant x \leqslant \pi} g(x). \qquad (5.4)$$

Then for every $0 < c_4 \leqslant 1$ *there are* $c_5 > 1$ *and* $c_6 > 0$ *depending only on* c_4 *such that*

$$m(\{y \in [\pi - c_4 N^{-1}, \pi + c_4 N^{-1}] \colon g(y) \geqslant c_5^{-1} \max_{-\pi \leqslant x \leqslant \pi} g(x)\}) \geqslant c_6 N^{-1}$$

holds.

The following lemma can be found in [5].

LEMMA 5.7. *Let $r>0$ and $0 \not\equiv p \in T_n$ be such that*

$$|p(\pi)| = \max_{-\pi \leqslant x \leqslant \pi} |p(x)|. \tag{5.5}$$

Then p has at most $3nr$ zeros in $[\pi - r, \pi + r]$.

LEMMA 5.8. *Let $r>0$ and $0 \not\equiv g \in |\mathrm{GCTP}|_N$ be of the form (2.11) such that each r_j is rational, and assume that*

$$g(\pi) = \max_{-\pi \leqslant x \leqslant \pi} g(x). \tag{5.6}$$

Then the total multiplicity of the zeros of g lying in $[\pi - r, \pi + r]$ is at most $3Nr$.

LEMMA 5.9. *Assume that $f \in |\mathrm{GCTP}|_N$ and $w \in |\mathrm{GCTP}|_\Gamma$ are of the forms*

$$f(z) = \prod_{j=1}^{k_1} \left| \sin \frac{z - z_j}{2} \right|^{r_j} \qquad (z_j \text{ are real, } r_j \geqslant 1 \text{ are rational, } 1 \leqslant j \leqslant k_1) \tag{5.7}$$

and

$$w(z) = \prod_{j=1}^{k_2} \left| \sin \frac{z - u_j}{2} \right|^{s_j}$$

$$(u_j \in [-\pi + \tfrac{2}{3}(N+\Gamma)^{-1}, \pi - \tfrac{2}{3}(N+\Gamma)^{-1}],$$

$$s_j > 0 \text{ are rational, } 1 \leqslant j \leqslant k_2), \tag{5.8}$$

respectively, and

$$|f'(\pi)\, w(\pi)| = \max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, w(x)|. \tag{5.9}$$

Then

$$|f'(\pi)\, w(\pi)| \leqslant c_{11}(\Gamma + 1)(N + \Gamma) \max_{-\delta \leqslant x \leqslant \delta} f(x)\, w(x),$$

where $\delta = \pi - \frac{2}{3}(N+\Gamma)^{-1}$ and c_{11} is an absolute constant.

LEMMA 5.10. *Assume that $f \in |\mathrm{GCTP}|_N$ and $w \in |\mathrm{GCTP}|_\Gamma$ are of the forms*

$$f(z) = \prod_{j=1}^{k_1} \left| \sin \frac{z - z_j}{2} \right|^{r_j} \qquad (z_j \text{ are real, } r_j \geqslant 1 \text{ are rational, } 1 \leqslant j \leqslant k_1) \tag{5.10}$$

and

$$w(z)=\prod_{j=1}^{k_2}\left|\sin\frac{z-u_j}{2}\right|^{s_j}\qquad (u_j\in\mathbb{C},\ s_j>0 \text{ are rational},\ 1\leqslant j\leqslant k_2),\tag{5.11}$$

respectively. Then

$$|f'(\pi)\,w(\pi)|\leqslant c_{12}(\Gamma+1)(N+\Gamma)\max_{-\delta\leqslant x\leqslant\delta} f(x)\,w(x),\tag{5.12}$$

where $\delta=\pi-\frac{2}{3}(N+\Gamma)^{-1}$ *and* c_{12} *is an absolute constant.*

6. Proofs of the Lemmas from Section 4

Proof of Lemma 4.1. Choose $w_i\in|\mathrm{GRAP}|_\Gamma$ $(i=1,2,\ldots)$ of the form

$$w_i=\prod_{j=1}^{k}|Q_{m_j,i}|^{s_j}\qquad (Q_{m_j,i}\in\Pi_{m_j},\ 1\leqslant j\leqslant k)\tag{6.1}$$

such that

$$\frac{|f'(1)\,w_i(1)|}{\max_{-1\leqslant x\leqslant\delta} f(x)\,w_i(x)}\geqslant\min\{L-i^{-1},\ i\}\qquad (i=1,2,\ldots)$$

and

$$|f'(1)\,w_i(1)|=\max_{-1\leqslant x\leqslant 1}|f'(x)\,w_i(x)|.$$

We may assume that

$$\max_{-1\leqslant x\leqslant 1}|Q_{m_j,i}(x)|=1\quad (1\leqslant j\leqslant k,\ i=1,2,\ldots).$$

For every $1\leqslant j\leqslant k$ we can select a subsequence of $\{Q_{m_j,i}\}_{i=1}^{\infty}$ (without loss of generality we may assume that this is $\{Q_{m_j,i}\}_{i=1}^{\infty}$ itself) such that

$$\lim_{i\to\infty}\max_{-1\leqslant x\leqslant 1}|Q_{m_j,i}(x)-\tilde{Q}_{m_j}(x)|=0$$

holds for every $1\leqslant j\leqslant k$ with some limit polynomials $\tilde{Q}_{m_j}\in\Pi_{m_j}$. Then it is easy to see that

$$\tilde{w}=\prod_{j=1}^{k}|\tilde{Q}_{m_j}|^{s_j}$$

has the desired properties. ∎

The proof of Lemma 4.2 is quite similar to that of Lemma 4.1, so we omit the details.

Proof of Lemma 4.3. If $\tilde{w}(z_0)=0$ for a $z_0 \in \mathbb{C}\backslash\mathbb{R}$ (we may assume that $\tilde{Q}_{m_1}(z_0)=0$), then the function

$$\tilde{w}_\varepsilon(z)=\left(\prod_{j=2}^{k} |\tilde{Q}_{m_j}(z)|^{s_j}\right)\left|\tilde{Q}_{m_1}(z)\left(1-\frac{\varepsilon(z-1)^2}{(z-z_0)(z-\bar{z}_0)}\right)\right|^{s_1} \in |\text{GRAP}|_\Gamma$$

with a sufficiently small $\varepsilon>0$ contradicts the maximality of $\tilde{w}$. If $\tilde{w}(z_0)=0$ for a $z_0 \in \mathbb{R}\backslash[-1, \delta]$ (we may assume that $\tilde{Q}_{m_1}(z_0)=0$), then the function

$$\tilde{w}_\varepsilon(z)=\left(\prod_{j=2}^{k} |\tilde{Q}_{m_j}(z)|^{s_j}\right)\left|\tilde{Q}_{m_1}(z)\left(1-\varepsilon\,\text{sgn}(z_0)\frac{1-z}{z_0-z}\right)\right|^{s_1} \in |\text{GRAP}|_\Gamma$$

with a sufficiently small $\varepsilon>0$ contradicts the maximality of $\tilde{w}$. Thus the lemma is proved. ∎

Proof of Lemma 4.4. If $\tilde{f}(z_0)=0$ for a $z_0 \in \mathbb{C}\backslash\mathbb{R}$, then there is an index $1 \leqslant i \leqslant k$ such that $\tilde{P}_{n_i}(z_0)=0$. Then the function

$$\tilde{f}_\varepsilon(z)=\left(\prod_{\substack{j=1\\ j\neq i}}^{k} |\tilde{P}_{n_j}|^{r_j}\right)\left|\tilde{P}_{n_j}(z)\left(1-\frac{\varepsilon(z-1)^2}{(z-z_0)(z-\bar{z}_0)}\right)\right|^{r_i} \in |\text{GRAP}|_N$$

with a sufficiently small $\varepsilon>0$ contradicts the maximality of $\tilde{f}$. Thus the lemma is proved. ∎

Lemma 4.5 was proved in [2, Theorem 1].

Proof of Lemma 4.6. Let $T_n(x)=\cos(n \arccos x)$ $(-1 \leqslant x \leqslant 1)$ be the Chebyshev polynomial of degree n, and with $n=[N] \geqslant 1$ we define

$$Q_n(x) := T_n(x+c_4 N^{-2}). \tag{6.2}$$

It is easy to check that the well-known explicit formula for T_n outside $(-1, 1)$ implies

$$Q_n(1)>c_5>1, \tag{6.3}$$

where c_5 is a constant depending only on c_4. We study the product

$$G=Q_n g, \tag{6.4}$$

where $g \in |\mathrm{GCAP}|_N$ satisfies (4.4). From (6.2), (6.3), (6.4), and (4.4) we easily deduce that

$$|G(y)| \leqslant \max_{-1 \leqslant x \leqslant 1} g(x) < c_5^{-1} Q_n(1)\, g(1) \leqslant c_5^{-1} \max_{-1 \leqslant x \leqslant 1} |G(x)|$$
$$(-1 \leqslant y \leqslant 1 - c_4 N^{-2}). \tag{6.5}$$

Applying Lemma 4.5 to $|G| \in |\mathrm{GCAP}|_{2N}$ with $s = (\log c_5)^2 (2N)^{-2}$, we obtain

$$\begin{aligned} &m(\{y \in [-1, 1]: |G(y)| \geqslant \exp(-\log c_5) \max_{-1 \leqslant x \leqslant 1} |G(x)|\}) \\ &\quad = m(\{y \in [-1, 1]: |G(y)| \geqslant \exp(-2N\sqrt{s}) \max_{-1 \leqslant x \leqslant 1} |G(x)|\}) \\ &\quad \geqslant c_3 (\log c_5)^2 (2N)^{-2} = c_6 N^{-2}. \end{aligned} \tag{6.6}$$

This, together with (6.5), yields

$$m(\{y \in [1 - c_4 N^{-2}, 1]: |G(y)| \geqslant c_5^{-1} \max_{-1 \leqslant x \leqslant 1} |G(x)|\}) \geqslant c_6 N^{-2},$$

hence by (6.2), (6.3), and (6.4) we get

$$m(\{y \in [1 - c_4 N^{-2}, 1]: g(y) \geqslant c_5^{-1} \max_{-1 \leqslant x \leqslant 1} g(x)\}) \geqslant c_6 N^{-2};$$

thus the lemma is proved. ∎

The proof of Lemma 4.7 can be found in [5, Lemma 1 and Corollary 1].

Proof of Lemma 4.8. Let $r_j = q_j/q$ $(1 \leqslant j \leqslant k)$ with some positive integers. Applying Lemma 4.7 to the polynomial

$$|c|^{2q} \prod_{j=1}^{k} ((z - z_j)(z - \bar{z}_j))^{q_j} \in \Pi_{2qN},$$

and taking the $(2q)$th root of its modulus we get the lemma immediately. ∎

Proof of Lemma 4.9. By Remark 3.4., (4.7) implies that $|f'| \in |\mathrm{GCAP}|_N$ has only real zeros, and at least one of any two adjacent zeros of $|f'|$ has multiplicity at least 1. Hence, applying Lemma 4.8 to $g = |f'w| \in |\mathrm{GCAP}|_{N+\Gamma}$ with $r = c_4(N+\Gamma)^{-2} = (c_7(N+\Gamma))^{-2}$, we can deduce that $|f'|$ does not have two different zeros in $[1 - c_4(N+\Gamma)^{-2}, 1]$. Since w does not have any zero in $[1 - c_4(N+\Gamma)^{-2}, 1]$ by assumption, the set

$$\{y \in [1 - c_4(N+\Gamma)^{-2}, 1]: g(y) \geqslant c_5^{-1} \max_{-1 \leqslant x \leqslant 1} g(x)\} \tag{6.7}$$

is the union of at most two intervals. Since $|f'|$ does not have two different zeros in $[1-c_4(N+\Gamma)^{-2}, 1]$, by Rolle's theorem we deduce that f has at most two different zeros in $[1-c_4(N+\Gamma)^{-2}, 1]$. Hence, applying Lemma 4.6 to $g=|f'w| \in |\mathrm{GCAP}|_{N+\Gamma}$, we can find an interval

$$[a, b]=I\subset[1-c_4(N+\Gamma)^{-2}, 1] \tag{6.8}$$

such that

$$m(I)=b-a\geqslant\frac{c_6}{6}(N+\Gamma)^{-2}, \tag{6.9}$$

$$g(t)\geqslant c_5^{-1}\max_{-1\leqslant x\leqslant 1} g(x) \qquad (t\in I) \tag{6.10}$$

and

$$f \text{ is positive (hence differentiable) on } I. \tag{6.11}$$

By (6.8) and (4.8)

$$w \text{ is positive (hence differentiable) on } I. \tag{6.12}$$

Because of (6.11) and (6.12) we can use the partial integration formula for $g=|f'w|$ on I, and we obtain

$$\begin{aligned}\frac{c_6}{6c_5}&(N+\Gamma)^{-2}\max_{-1\leqslant x\leqslant 1}|f'(x)\,w(x)|\\&\leqslant\int_a^b|f'(t)\,w(t)|\,dt=\left|\int_a^b f'(t)\,w(t)\,dt\right|\\&\leqslant|f(b)\,w(b)-f(a)\,w(a)|+\int_a^b|f(t)\,w'(t)|\,dt.\end{aligned} \tag{6.13}$$

Here we used the fact that $|f'(t)|$ does not vanish on $[a, b]=I$ because of (6.10). To handle the term $\int_a^b |f(t)\,w'(t)|\,dt$ we use Lemma 4.8. We introduce the intervals

$$I_\alpha=[1-2c_4(\alpha+1)^4(N+\Gamma)^{-2}, 1-2c_4\alpha^4(N+\Gamma)^{-2}] \qquad (\alpha=0, 1, 2, ...). \tag{6.14}$$

By assumption (4.8), w does not vanish in I_0, and applying Lemma 4.8 to $g=|f'w|$ (we can do so by assumption (4.9)) we deduce that the total multiplicity of the zeros of g lying in I_α is at most $c_7\sqrt{2c_4}\,(N+\Gamma)(\alpha+1)^2(N+\Gamma)^{-1}=\sqrt{2}(\alpha+1)^2$. Now let

$$t\in I\subset[1-c_4(N+\Gamma)^{-2}, 1]. \tag{6.15}$$

Then $u_j < t$ by (4.8), therefore

$$\begin{aligned} \frac{|w'(t)|}{w(t)} &= \sum_{j=1}^{k_2} \frac{s_j}{t-u_j} = \sum_{\alpha=1}^{\infty} \sum_{u_j \in I_\alpha} \frac{s_j}{t-u_j} \\ &\leqslant \sum_{\alpha=1}^{\infty} \sqrt{2}(\alpha+1)^2 \frac{1}{c_4 \alpha^4 (N+\Gamma)^{-2}} \\ &\leqslant \frac{\sqrt{2}}{c_4} \left(\sum_{\alpha=1}^{\infty} \alpha^{-2} \right) (N+\Gamma)^2 \leqslant \frac{2 \cdot \sqrt{2}}{c_4} (N+\Gamma)^2. \end{aligned} \tag{6.16}$$

Therefore, recalling (6.8), we obtain

$$\begin{aligned} \int_a^b |f(t)\, w'(t)|\, dt &\leqslant \frac{\sqrt{8}}{c_4} (N+\Gamma)^2 \int_a^b f(t)\, w(t)\, dt \\ &\leqslant \frac{\sqrt{8}}{c_4} (N+\Gamma)^2 (b-a) \max_{a \leqslant x \leqslant b} f(x)\, w(x) \\ &\leqslant \sqrt{8} \max_{-1 \leqslant x \leqslant 1} f(x)\, w(x). \end{aligned} \tag{6.17}$$

This, together with (6.13), yields

$$\max_{-1 \leqslant x \leqslant 1} |f'(x)\, w(x)| \leqslant c_{13} (N+\Gamma)^2 \max_{-1 \leqslant x \leqslant 1} f(x)\, w(x). \tag{6.18}$$

Observe that the Remez-type inequality of Lemma 4.5 implies

$$\max_{-1 \leqslant x \leqslant 1} f(x)\, w(x) \leqslant c_{14} \max_{-1 \leqslant x \leqslant \delta} f(x)\, w(x) \tag{6.19}$$

with $\delta = 1 - 2c_4(N+\Gamma)^{-2}$, where c_{14} is an absolute constant. Now (6.18) and (6.19) gives the lemma. ∎

Proof of Lemma 4.10. First assume that (4.9) holds. Then using Lemmas 4.1, 4.3, 4.9 and Remark 3.3, we obtain (4.12). Now we can easily drop assumption (4.9). Choose an $x_0 \in [-1, 1]$ such that

$$|f'(x_0)\, w(x_0)| = \max_{-1 \leqslant x \leqslant 1} |f'(x)\, w(x)|. \tag{6.20}$$

Without loss of generality we may assume that $0 \leqslant x_0 \leqslant 1$. Applying the already proved part of the lemma transformed linearly to the interval $[-1, x_0]$, we get

$$|f'(1)\,w(1)| \leqslant |f'(x_0)\,w(x_0)| \leqslant \frac{2c_8}{1+x_0}(N+\Gamma)^2 \max_{-1 \leqslant x \leqslant x_0} f(x)\,w(x)$$
$$\leqslant 2c_8(N+\Gamma)^2 \max_{-1 \leqslant x \leqslant 1} f(x)\,w(x)$$
$$\leqslant c_9(N+\Gamma)^2 \max_{-1 \leqslant x \leqslant \delta} f(x)\,w(x), \tag{6.21}$$

where the last inequality follows from (6.19) with $c_9 = 2c_8c_{14}$. Thus the lemma is proved. ∎

7. Proofs of the Lemmas from Section 5

The proofs of Lemmas 5.1 and 5.2 are the same as those of the corresponding lemmas from Section 4.

Proof of Lemma 5.3. If $\tilde{w}(z_0) = 0$ for a $z_0 \in \mathbb{C}\setminus\mathbb{R}$ (we may assume that $\tilde{Q}_{m_1}(z_0) = 0$), then the function

$$\tilde{w}_\varepsilon(z) = \left(\prod_{j=2}^{k} |\tilde{Q}_{m_j}(z)|^{s_j}\right)$$
$$\times \left|\tilde{Q}_{m_1}(z)\left(1 - \varepsilon\frac{\sin^2((z-\pi)/2)}{\sin((z-z_0)/2)\sin((z-\bar{z}_0)/2)}\right)\right|^{s_1} \in |\mathrm{GRTP}|_\Gamma$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $\tilde{w}$. If $\tilde{w}(z_0) = 0$ for a $z_0 \in [-\pi, \pi)\setminus[-\delta, \delta]$ (we may assume that $\tilde{Q}_{m_1}(z_0)$), then the function

$$\tilde{w}_\varepsilon(z) = \left(\prod_{j=2}^{k} |\tilde{Q}_{m_j}(z)|^{s_j}\right)$$
$$\times \left|\tilde{Q}_{m_1}(z)\left(1 - \varepsilon\,\mathrm{sgn}(z_0)\frac{\sin((z-\pi)/2)}{\sin((z-z_0)/2)}\right)\right|^{s_1} \in |\mathrm{GRTP}|_\Gamma$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $\tilde{w}$. Thus the lemma is proved. ∎

Proof of Lemma 5.4. If $\tilde{f}(z_0) = 0$ for a $z_0 \in \mathbb{C}\setminus\mathbb{R}$, then there is an index $1 \leqslant i \leqslant k$ such that $\tilde{P}_{n_i}(z_0) = 0$. But then the function

$$\tilde{f}_\varepsilon(z) = \left(\prod_{\substack{j=1\\ j\neq i}}^{k} |\tilde{P}_{n_j}(z)|^{r_j}\right)$$
$$\times \left|\tilde{P}_{n_i}(z)\left(1 - \varepsilon\frac{\sin^2((z-\pi)/2)}{\sin((z-z_0)/2)\sin((z-\bar{z}_0)/2)}\right)\right|^{r_i} \in |\mathrm{GRTP}|_N$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $\tilde{f}$. Thus the lemma is proved. ▮

Lemma 5.5 was proved in [2, Theorem 2].

Proof of Lemma 5.6. Let $T_n(x) = \cos(n \arccos x)$ $(-1 \leqslant x \leqslant 1)$ be the Chebyshev polynomial of degree n. Then

$$Q_{n,\omega}(x) = T_{2n}\left(\frac{\sin(x/2)}{\sin(\omega/2)}\right) \qquad (0 < \omega \leqslant \pi)$$

is a trigonometric polynomial of degree n. We define

$$Q_n(x) = Q_{n,\omega}(x) \qquad \text{with} \quad n = [N] \qquad \text{and} \qquad \omega = \pi - c_4 N^{-1}. \tag{7.1}$$

It is easy to verify that the well-known explicit formula for T_n outside $(-1, 1)$ implies

$$Q_n(\pi) > c_5 > 1, \tag{7.2}$$

where c_5 is a constant depending only on c_4. We examine the product

$$G = Q_n g. \tag{7.3}$$

From (7.1), (7.2), and (7.3) we conclude

$$|G(y)| \leqslant \max_{-\pi \leqslant x \leqslant \pi} g(x) < c_5^{-1} Q_n(\pi)\, g(\pi) \leqslant c_5^{-1} \max_{-\pi \leqslant x \leqslant \pi} |G(x)|$$

$$(-\omega \leqslant y \leqslant \omega). \tag{7.4}$$

Applying Lemma 5.5 to $|G| \in |\mathrm{GCTP}|_{2N}$ with $s = (\log c_5)(2N)^{-1}$, we obtain

$$\begin{aligned} &m(\{y \in [-\pi, \pi) : |G(y)| \geqslant \exp(-\log c_5) \max_{-\pi \leqslant x \leqslant \pi} |G(x)|\}) \\ &\quad = m(\{y \in [-\pi, \pi) : |G(y)| \geqslant \exp(-2Ns) \max_{-\pi \leqslant x \leqslant \pi} |G(x)|\}) \\ &\quad \geqslant c_{10}(\log c_5)(2N)^{-1} = c_6 N^{-1}. \end{aligned} \tag{7.5}$$

This, together with (7.4), yields

$$m(\{y \in [-\pi, \pi) \backslash (-\omega, \omega] : |G(y)| \geqslant c_5^{-1} \max_{-\pi \leqslant x \leqslant \pi} |G(x)|\}) \geqslant c_6 N^{-1},$$

hence by (7.1), (7.2), and (7.3) we deduce

$$m(\{y \in [\pi - c_4 N^{-1}, \pi + c_4 N^{-1}] : g(y) \geqslant c_5^{-1} \max_{-\pi \leqslant x \leqslant \pi} g(x)\}) \geqslant c_6 N^{-1},$$

thus the lemma is proved. ▮

The proof of Lemma 5.7 can be found in [5, Lemma 2].

Proof of Lemma 5.8. Let $r_j = q_j/q$ $(1 \leqslant j \leqslant k)$ with some positive integers. Applying Lemma 5.7 to the trigonometric polynomial

$$|c|^{2q} \prod_{j=1}^{k} \left(\sin\frac{z-z_j}{2}\sin\frac{z-\bar{z}_j}{2}\right)^{q_j} \in T_{2qN}$$

and taking the $(2q)$th root of its modulus we get the desired result immediately. ∎

Proof of Lemma 5.9. From (5.7) and Remark 3.4 we can deduce that $|f'| \in |\text{GCTP}|_N$ has only real zeros, and at least one of any two adjacent zeros of $|f'|$ has multiplicity at least 1. Therefore, applying Lemma 5.8 to $g = |f'w| \in |\text{GCTP}|_{N+\Gamma}$ with $r = \frac{1}{3}(N+\Gamma)^{-1}$, we deduce that $|f'|$ does not have two different zeros in $[\pi - \frac{1}{3}(N+\Gamma)^{-1}, \pi + \frac{1}{3}(N+\Gamma)^{-1}]$. Since w does not have any zero in $[\pi - \frac{1}{3}(N+\Gamma)^{-1}, \pi + \frac{1}{3}(N+\Gamma)^{-1}]$ by assumption, the set

$$\{y \in [\pi - 3^{-1}(N+\Gamma)^{-1}, \pi + 3^{-1}(N+\Gamma)^{-1}] : g(y) \geqslant c_5^{-1} \max_{-\pi \leqslant x \leqslant \pi} g(x)\} \tag{7.6}$$

is the union of at most two intervals. Since $|f'|$ does not have two different zeros in $[\pi - \frac{1}{3}(N+\Gamma)^{-1}, \pi + \frac{1}{3}(N+\Gamma)^{-1}]$, Rolle's theorem implies that f has at most two different zeros in $[\pi - \frac{1}{3}(N+\Gamma)^{-1}, \pi + \frac{1}{3}(N+\Gamma)^{-1}]$. Hence, applying Lemma 5.6 to $g = |f'w| \in |\text{GCTP}|_{N+\Gamma}$, we can find and interval

$$[a, b] = I \subset [\pi - 3^{-1}(N+\Gamma)^{-1}, \pi + 3^{-1}(N+\Gamma)^{-1}] \tag{7.7}$$

such that

$$m(I) = b - a \geqslant \frac{c_6}{6}(N+\Gamma)^{-1}, \tag{7.8}$$

$$g(y) \geqslant c_5^{-1} \max_{\pi \leqslant x \leqslant \pi} g(x) \qquad (y \in I) \tag{7.9}$$

and

$$f \text{ is positive (hence differentiable) on } I, \tag{7.10}$$

where $c_5 > 1$ and $c_6 > 0$ are chosen for $c_4 = \frac{1}{3}$ by Lemma 5.6. By (7.7) and (5.8)

$$w \text{ is positive (hence differentiable) on } I. \tag{7.11}$$

Because of (7.10) and (7.11) we can use the partial integration formula for $g = |f'w|$ on I, which yields

$$\frac{c_6}{6c_5}(N+\Gamma)^{-1} \max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, w(x)| \leqslant \int_a^b |f'(t)\, w(t)|\, dt$$
$$= \left| \int_a^b f'(t)\, w(t)\, dt \right| \leqslant |f(b)\, w(b) - f(a)\, w(a)| + \int_a^b |f(t)\, w'(t)|\, dt. \tag{7.12}$$

Here we used the fact that $|f'(t)|$ does not vanish on I because of (7.9). To handle the term $\int_a^b |f(t)\, w'(t)|\, dt$ we use Lemma 5.8. Let

$$t \in I \subset [\pi - 3^{-1}(N+\Gamma)^{-1}, \pi + 3^{-1}(N+\Gamma)^{-1}]. \tag{7.13}$$

By assumption (5.8) we easily obtain

$$\frac{|w'(t)|}{w(t)} = \left| \frac{1}{2} \sum_{j=1}^{k_2} s_j \cot \frac{t-u_j}{2} \right| \leqslant 3\Gamma(N+\Gamma) \qquad (t \in I). \tag{7.14}$$

Hence, recalling (7.7), we obtain

$$\int_a^b |f(t)\, w'(t)|\, dt \leqslant 3\Gamma(N+\Gamma) \int_a^b f(t)\, w(t)\, dt$$
$$\leqslant 3\Gamma(N+\Gamma)(b-a) \max_{a \leqslant x \leqslant b} f(x)\, w(x)$$
$$\leqslant 2\Gamma \max_{a \leqslant x \leqslant b} f(x)\, w(x). \tag{7.15}$$

This, together with (7.12), gives

$$\max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, w(x)| \leqslant c_{15}(\Gamma + 1)(N+\Gamma) \max_{-\pi \leqslant x \leqslant \pi} f(x)\, w(x). \tag{7.16}$$

Now observe that the Remez-type inequality of Lemma 5.5 yields

$$\max_{-\pi \leqslant x \leqslant \pi} f(x)\, w(x) \leqslant c_{16} \max_{-\delta \leqslant x \leqslant \delta} f(x)\, w(x) \tag{7.17}$$

with $\delta = \pi - \frac{2}{3}(N+\Gamma)^{-1}$, where c_{16} is an absolute constant, and this, together with (7.16), gives the lemma. ∎

Proof of Lemma 5.10. First we assume (5.9) holds. Then Lemmas 5.1, 5.3, 5.9 and Remark 3.3 imply (5.12). Now we drop assumption (5.9). We choose an $\alpha \in [-\pi, \pi)$ such that

$$|f'(\alpha)\, w(\alpha)| = \max_{-\pi \leqslant x \leqslant \pi} |f'(x)\, w(x)|. \tag{7.18}$$

Applying the already proved part of the lemma to $\hat{f}(z)=f(z+\alpha-\pi)\in |\mathrm{GCTP}|_N$ and $\hat{w}(z)=w(z+\alpha-\pi)\in|\mathrm{GCTP}|_\Gamma$, we obtain

$$|f'(\pi)\,w(\pi)|\leqslant|f'(\alpha)\,w(\alpha)|\leqslant c_{11}(\Gamma+1)(N+\Gamma)\max_{-\pi\leqslant x\leqslant\pi} f(x)\,w(x)$$

$$\leqslant c_{12}(\Gamma+1)(N+\Gamma)\max_{-\delta\leqslant x\leqslant\delta} f(x)\,w(x), \tag{7.19}$$

where the last inequality follows from (7.17) with $c_{12}=c_{11}c_{16}$. Thus the lemma is proved. ∎

8. Proofs of Theorems 1, 2, and 3

Proof of Theorem 1. It is sufficient to prove that

$$|f'(1)\,w(1)|\leqslant c_{17}(N+\Gamma)^2\max_{-1\leqslant x\leqslant\delta} f(x)\,w(x), \tag{8.1}$$

where δ is defined by Lemmas 4.9 and 4.10, and c_{17} is an absolute constant. To estimate $|f'(y)\,w(y)|$ $(-1\leqslant y\leqslant 1)$ we can use the above inequality transformed linearly to the interval $[-1, y]$ if $y\geqslant 0$, or to the interval $[y, 1]$ if $y<0$, and we obtain the desired inequality with $c_1=2c_{17}$. To show (8.1) we may assume that

$$f(z)=\prod_{j=1}^{k_1}|z-z_j|^{r_j}\qquad (z_j\in\mathbb{C},\ r_j\geqslant 1 \text{ are rational, } 1\leqslant j\leqslant k_1) \tag{8.2}$$

and

$$w(z)=\prod_{j=1}^{k_2}|z-u_j|^{s_j}\qquad (u_j\in\mathbb{C},\ s_j>0 \text{ are rational, } 1\leqslant j\leqslant k_2), \tag{8.3}$$

since, if the inequality of Theorem 1 holds for these functions, then we get the theorem in the general case by approximation. By Lemmas 4.2 and 4.4 we may also assume that each z_j $(1\leqslant j\leqslant k_1)$ in (8.2) is real; therefore from Lemma 4.10 we obtain (8.1). Thus the theorem is proved. ∎

Proof of Theorem 2. Because of the periodicity it is sufficient to prove that

$$|f'(\pi)\,w(\pi)|\leqslant c_{18}(\Gamma+1)(N+\Gamma)\max_{-\delta\leqslant x\leqslant\delta} f(x)\,w(x), \tag{8.4}$$

where δ is defined by Lemmas 5.9 and 5.10 and c_{18} is an absolute constant. By using a density argument, it is sufficient to prove (8.4) when

$$f(z)=\prod_{j=1}^{k_1}\left|\sin\frac{z-z_j}{2}\right|^{r_j}\qquad (z_j\in\mathbb{C},\ r_j\geqslant 1 \text{ are rational, } 1\leqslant j\leqslant k_1) \tag{8.5}$$

and

$$w(z) = \prod_{j=1}^{k_2} \left| \sin \frac{z - u_j}{2} \right|^{s_j} \qquad (u_j \in \mathbb{C},\ s_j > 0 \text{ are rational},\ 1 \leqslant j \leqslant k_2). \quad (8.6)$$

By Lemmas 5.2 and 5.4 we may also assume that each z_j $(1 \leqslant j \leqslant k_1)$ in (8.5) is real, hence Lemma 5.10 implies the theorem. ∎

Proof of Theorem 3. The inequality of Theorem 3 follows immediately from Theorem 2 and Remark 3.3, by the substitution $y = \cos x$. ∎

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