ON A GENERALIZATION OF THE BERNSTEIN-MARKOV INEQUALITY

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Abstract. We show that

$$\|P'Q\|_{L_p(I)} \leq c^{1+1/p}(N+M)\log(\min(N,M+1)+1)\|PQ\|_{L_p(I)}$$

for all real trigonometric polynomials P and Q of degree N and M, respectively, where $0 < p \leq \infty,\ I := [-\pi,\pi]$, and $c > 0$ is a suitable absolute constant. We also show that

$$\|f'g\|_{L_p(J)} \leq c^{1+1/p}(N+M)^2\|fg\|_{L_p(J)}$$

for all algebraic polynomials f and g of degree N and M, respectively, where $0 < p \leq \infty,\ J := [-1,1]$, and $c > 0$ is a suitable absolute constant. Both of our trigonometric and algebraic results are sharp up to the factor $c^{1+1/p}$. In fact, we prove our results for the much wider classes of generalized trigonometric and algebraic polynomials.

1. Introduction

The function

$$P(x) := a_0 + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\,, \qquad a_k, b_k \in \mathbf{R}\,, \quad a_nb_n \neq 0\,,$$

is called a real trigonometric polynomial of degree n. It is well-known that every real trigonometric polynomial P of degree n can be written as

$$P(x) = \omega\prod_{j=1}^{2n}\sin((x-z_j)/2)\,,$$

where $\omega \in \mathbf{R}$, $z_j \in \mathbf{C}$, and the non-real zeros z_j of P form conjugate pairs. The function

$$P(x) := \omega\prod_{j=1}^{s}|\sin((x-z_j)/2)|^{r_j}\,, \qquad x \in \mathbf{R}\,, \tag{1.1}$$

where $0 < r_j \in \mathbf{R}$, $z_j \in \mathbf{C}$ are distinct (mod 2π), and $0 < \omega \in \mathbf{R}$, is called a generalized trigonometric polynomial of degree $N := \frac{1}{2}\sum_{j=1}^{s} r_j$. If P is a constant identically, then its degree is defined to be 0. Note that the absolute value of a real trigonometric polynomial of degree n may be viewed as a generalized trigonometric polynomial of degree n. Let

GTP_N denote the set of all generalized trigonometric polynomials of degree at most N. Observe that if $P \in \mathrm{GTP}_N$ is of the form (1.1), then

$$P(x) := \omega \prod_{j=1}^{s} (\sin((x-z_j)/2)\sin((x-\overline{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}\,, \qquad x \in \mathbf{R}\,,$$

where each T_j is a real trigonometric polynomial of degree 1 being nonnegative on the real line. For a $P \in \mathrm{GTP}_N$ of the form (1.1) the numbers z_j are called the zeros of P, while the exponent r_j is called the multiplicity of the zero z_j in P.

The problem arises how to define P' for a $P \in \mathrm{GTP}_N$. Observe that if $r_j \geq 1$ for each $j = 1, 2, \dots, s$ in (1.1), then, although P' may not exist at the zeros of P, the one-sided derivatives P'_- and P'_+ exist, and their absolute values are equal. This means $|P'|$ is well-defined on the real line by either $|P'_-|$ or $|P'_+|$. It is a simple exercise to check that if $f \in \mathrm{GTP}_N$ has only real zeros with multiplicities at least 1, then $|f'| \in \mathrm{GTP}_N$ has only real zeros as well, and at least one of any two adjacent zeros of $|f'|$ has multiplicity exactly 1.

It is well-known that these generalized trigonometric polynomials satisfy the following Bernstein-type inequality on $I := [-\pi, \pi]$:*

Theorem A ([1], Theorem A.4.12 and Corollary A.4.13). *Let χ be a nonnegative, nondecreasing, convex function defined on* $[0, \infty)$. *Then*

$$\int_I \chi(N^{-q}|P'(t)|^q)\,dt \leq \int_I \chi(cP(t)^q)\,dt \tag{1.2}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) *with each $r_j \geq 1$, and for every $0 < q \leq 1$. In particular,*

$$||P'||_{L_p(I)} \leq c^{1+1/p} N ||P||_{L_p(I)}\,, \qquad 0 < p \leq \infty\,, \tag{1.3}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) *with each $r_j \geq 1$.*

(1.3) can be generalized to the following (cf. Theorem 10.4 in [6]).

Theorem B. *We have*

$$||P'Q||_{L_p(I)} \leq A_{N,M} ||PQ||_{L_p(I)}, \tag{1.4}$$

where

$$A_{N,M} := c^{1+1/p}(N + M\min\{p,1\})(M+1)\,, \qquad 0 < p \leq \infty\,, \tag{1.5}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) *with each $r_j \geq 1$, and for every $Q \in \mathrm{GTP}_M$.*

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* Here and in what follows, $c, c_1, c_2, \dots$ will always denote suitable positive absolute constants, not necessarily the same at each occurrence.

The function

$$f(x) := \sum_{k=0}^{n} a_k x^k \,, \qquad a_k \in \mathbf{R} \,, \quad a_n \neq 0 \,,$$

is called a real algebraic polynomial of degree n. It is well-known that every real algebraic polynomial f can be written as

$$f(x) = \omega \prod_{j=1}^{n} (x - z_j) \,,$$

where $\omega \in \mathbf{R}$, $z_j \in \mathbf{C}$, and the non-real zeros z_j of f form conjugate pairs. The function

$$(1.6) \qquad f(x) := \omega \prod_{j=1}^{s} |x - z_j|^{r_j} \,, \qquad x \in \mathbf{R} \,,$$

where $0 < r_j \in \mathbf{R}$, $z_j \in \mathbf{C}$ are distinct, and $0 < \omega \in \mathbf{R}$, is called a generalized algebraic polynomial of degree $N := \sum_{j=1}^{s} r_j$. If f is a constant identically, then its degree is defined to be 0. Note that the absolute value of a real algebraic polynomial of degree n may be viewed as a generalized algebraic polynomial of degree n. Let GAP_N be the set of all generalized algebraic polynomials of degree at most N. Observe that if $f \in \mathrm{GAP}_N$ is of the form (1.6), then

$$f(x) := \omega \prod_{j=1}^{s} \left((x - z_j)(x - \overline{z}_j)\right)^{r_j/2} = \prod_{j=1}^{s} g_j(x)^{r_j/2} \,, \qquad x \in \mathbf{R} \,,$$

where each g_j is a real algebraic polynomial of degree 2 being nonnegative on the real line. For a $P \in \mathrm{GAP}_N$ of the form (1.6) the numbers z_j are called the zeros of P, while the exponent r_j is called the multiplicity of the zero z_j in f.

The problem arises how to define f' for a $f \in \mathrm{GAP}_N$. Observe that if $r_j \geq 1$ for each $j = 1, 2, \ldots, s$ in (1.6), then, although f' may not exist at the zeros of f, the one-sided derivatives f'_- and f'_+ exist, and their absolute values are equal. This means $|f'|$ is well-defined on the real line by either $|f'_-|$ or $|f'_+|$. It is a simple exercise to check that if $f \in \mathrm{GAP}_N$ has only real zeros with multiplicities at least 1, then $|f'| \in \mathrm{GAP}_{N-1}$ has only real zeros as well, and at least one of any two adjacent zeros of $|f'| \in \mathrm{GAP}_{N-1}$ has multiplicity exactly 1.

These generalized algebraic polynomials satisfy the following Bernstein and Markov type inequalities, respectively, on $J := [-1, 1]$:

Theorem C. *We have*

$$(1.7) \qquad ||\sqrt{1-x^2} f'(x) g(x)||_{L_p(J)} \leq A_{N+1, M+1/p} ||fg||_{L_p(J)} \,, \qquad 0 < p \leq \infty \,,$$

for every $f \in \mathrm{GAP}_N$ of the form (1.6) *with each $r_j \geq 1$, and for every $g \in \mathrm{GAP}_M$, where $A_{N,M}$ is defined in* (1.5). *In particular,*

$$(1.8) \qquad ||\sqrt{1-x^2} f'(x)||_{L_p(J)} \leq c^{1+1/p} N ||f||_{L_p(J)} \,, \qquad 0 < p \leq \infty \,,$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with each* $r_j \geq 1$*.*

This can be obtained from Theorem B with a substitution

$$(1.9) \qquad P(t) = f(\cos t) \in \mathrm{GTP}_N \quad \text{and} \quad Q(t) = g(\cos t)|\sin t|^{1/p} \in \mathrm{GTP}_{M+1/p}$$

(cf. E.10 a] on p. 409 of [1]).

Theorem D (cf. [6], Theorems 10.3 and 10.6, as well as [4], Theorem 1). *We have*

$$(1.10) \qquad ||f'g||_{L_p(J)} \leq \begin{cases} A_{N,M}^2 ||fg||_{L_p(J)} & \text{if } 0 < p < \infty, \\ c(N+M)^2 ||fg||_{L_\infty(J)} & \text{if } p = \infty \end{cases}$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with each* $r_j \geq 1$*, and for every* $g \in \mathrm{GAP}_M$*. In particular,*

$$(1.11) \qquad ||f'||_{L_p(J)} \leq c^{1+1/p} N^2 ||f||_{L_p(J)}\,, \qquad 0 < p \leq \infty\,,$$

for every $f \in \mathrm{GAP}_N$ *of the form* (1.6) *with each* $r_j \geq 1$*.*

While the ordinary Bernstein and Markov type inequalities (1.3), (1.8) and (1.11) are known to be sharp as for the order of magnitude, the same is not known for (1.4), (1.7) and (1.10). In fact, it is the purpose of the present paper to improve these inequalities under certain conditions.

Generalized trigonometric and algebraic polynomials are studied in a number of papers [2–8] and most of these results may be found in the book [1] with complete proofs. We formulate four more results about generalized polynomials which are needed in the proof of the main results of this paper. In each of these, as before, $I := [-\pi, \pi]$ and $J := [-1, 1]$. The following Nikolskii-type inequalities for the classes GTP_N and GAP_N are proved in [7] (cf. Theorems 5 and 6) as well as in [1] (cf. Theorems A.4.3 and A.4.4).

Theorem E. *Let* χ *be a nonnegative, nondecreasing function defined on* $[0, \infty)$ *such that* $\chi(x)/x$ *in non-increasing on* $[0, \infty)$*. Then for* $0 < q < p \leq \infty$ *we have*

$$\|\chi(P)\|_{L_p(I)} \leq (c(1+qN))^{1/q-1/p} \|\chi(P)\|_{L_q(I)}$$

for every $P \in \mathrm{GTP}_N$*. If* $\chi(x) = x$*, then* $c = e(4\pi)^{-1}$ *is a suitable choice.*

Theorem F. *Let* χ *be a nonnegative, nondecreasing function defined on* $[0, \infty)$ *such that* $\chi(x)/x$ *is non-increasing on* $[0, \infty)$*. Then for* $0 < q < p \leq \infty$ *we have*

$$\|\chi(f)\|_{L_p(J)} \leq (c(2+qN))^{2/q-2/p} \|\chi(f)\|_{L_q(J)}$$

for every $f \in \mathrm{GAP}_N$*. If* $\chi(x) = x$*, then* $c = e^2(2\pi)^{-1}$ *is a suitable choice.*

The following Remez-type inequalities for the classes GTP_N and GAP_N are proved in [5]. The Lebesgue measure of a set $A \subset \mathbf{R}$ is denoted by $m(A)$.

Theorem G. *We have*

$$||P||_{L_\infty(I)} \le \exp(cNs)\,, \qquad 0 < s < \pi/2\,,$$

for every $P \in \mathrm{GTP}_N$ *satisfying* $m\{t \in I : P(t) \le 1\} \ge 2\pi - s$.

Theorem H. *We have*

$$||f||_{L_\infty(J)} \le \exp(cNs^{1/2})\,, \qquad 0 < s < 1\,,$$

for every $f \in \mathrm{GAP}_N$ *satisfying* $m\{t \in J : f(t) \le 1\} \ge 2 - s$.

2. The case of generalized trigonometric polynomials

Theorem 1. *We have*

$$(2.1)\quad ||P'Q||_{L_p(I)} \le c^{1+1/p}(N+M)\log(\min(N, M+1)+1)||PQ||_{L_p(I)}\,, \qquad 0 < p \le \infty\,,$$

for any two $P \in \mathrm{GTP}_N$ *and* $Q \in \mathrm{GTP}_M$ *such that the roots of* P *and* Q *have multiplicities at least* 1*. Moreover,* (2.1) *is sharp apart from the constant* $c^{1+1/p}$.

This improves considerably Theorem B (under the stronger condition that the multiplicities in Q are at least 1). In fact, we will prove slightly more; this is formulated as a lemma, and it is a generalization of Theorem A.

Lemma 1. *Let* χ *be a nonnegative, nondecreasing, convex function defined on* $[0,\infty)$*. Then*

$$(2.2)\qquad \int_{-\pi}^{\pi} \chi\left(\left(\frac{|P'(t)|Q(t)}{(N+M)\log(N+1)}\right)^q\right) dt \le \int_{-\pi}^{\pi} \chi(c(P(t)Q(t))^q)\, dt$$

for every $P \in \mathrm{GTP}_N$ *of the form* (1.1) *with each* $r_j \ge 1$*, for every* $Q \in \mathrm{GTP}_M$*, and for every* $0 < q \le 1$*. In particular,*

$$(2.3)\qquad ||P'Q||_{L_p(I)} \le c^{1+1/p}(N+M)\log(N+1)||PQ||_{L_p(I)}\,, \qquad 0 < p \le \infty\,,$$

for every $P \in \mathrm{GTP}_N$ *of the form* (1.1) *with each* $r_j \ge 1$*, and for every* $Q \in \mathrm{GTP}_M$.

(2.3) follows from (2.2) with $q = \min(1,p)$ and $\chi(x) = x^{\max(1,p)}$. On the other hand, (2.3) implies (2.1). Indeed, we can apply (2.3) with the roles of P and Q interchanged to get

$$||Q'P||_{L_p(I)} \le c^{1+1/p}(N+M)\log(M+1)||PQ||_{L_p(I)}\,, \qquad 0 < p \le \infty\,.$$

This coupled with the Bernstein-type inequality (cf. Theorem A)

$$||(PQ)'||_{L_p(I)} \le c^{1+1/p}(N+M)||PQ||_{L_p(I)}\,, \qquad 0 < p \le \infty\,,$$

(cf. (1.3)) yields the statement of Theorem 1.

In order to prove Lemma 1, we need several auxiliary statements.

Lemma 2. *Let $P \in \mathrm{GTP}_N$, and let Δ be an arbitrary interval with midpoint t, and*

$$\mathcal{M}(P,\Delta) := \sum_{t_j \in \Delta} r_j ,$$

where r_j is the multiplicity of the root t_j of P. Then

$$\mathcal{M}(P,\Delta) \le \left(\frac{e}{2}N|\Delta| + 1\right) \frac{||P||_{L_\infty(I)}}{P(t)} \qquad \textit{for all} \quad \Delta.$$

Proof. In the proof of E.11 on pp. 236-237 of [1] it is shown that if S is an ordinary trigonometric polynomial of degree at most n then

$$\left(\frac{2\mathcal{M}(S,\Delta)}{e|\Delta|n}\right)^{\mathcal{M}(S,\Delta)} \le \frac{||S||_{L_\infty(I)}}{|S(t)|}. \tag{2.4}$$

To prove the lemma, first we assume that in the representation of $P \in \mathrm{GTP}_N$, each r_j is rational with a common denominator $q \in \mathbf{N}$, and apply (2.4) to the ordinary trigonometric polynomial $S = P^{2q}$ of degree at most $n = 2qN$. Since evidently $\mathcal{M}(S,\Delta) = 2q\mathcal{M}(P,\Delta)$, we get

$$\left(\frac{4q\mathcal{M}(P,\Delta)}{e|\Delta|2qN}\right)^{2q\mathcal{M}(P,\Delta)} \le \left(\frac{||P||_{L_\infty(I)}}{|P(t)|}\right)^{2q},$$

i.e.

$$\left(\frac{2\mathcal{M}(P,\Delta)}{e|\Delta|N}\right)^{\mathcal{M}(P,\Delta)} \le \frac{||P||_{L_\infty(I)}}{|P(t)|}. \tag{2.5}$$

We may assume that $\mathcal{M}(P,\Delta) \ge (e/2)|\Delta|N + 1$ (otherwise there is nothing to prove), and hence we may replace the left hand side exponent in (2.5) by 1. This proves the lemma when each r_j is rational. Since (2.5) is independent of the common denominator q of the rational exponents, an obvious limit procedure yields the result for arbitrary exponents.

Lemma 3 ([1], E.7 b] on p. 409; cf. also [5]). *The inequality*

$$m(\{t \in [-\pi,\pi) : P(t) \ge \lambda ||P||_{L_\infty(I)}\}) \ge \frac{\mu(\lambda)}{N+1}$$

holds for every $P \in \mathrm{GTP}_N$, where $0 < \lambda < 1$ is arbitrary, and $\mu(\lambda) > 0$ depends only on λ.

We use the notation $\mathbf{T} := \mathbf{R} \pmod{2\pi}$.

Lemma 4. *Suppose $P \in \mathrm{GTP}_N$. Let $a \in [-\pi,\pi)$ be such that*

$$P(a) = ||P||_{L_\infty(I)}\,. \tag{2.6}$$

Then

$$m\{t \in [a-1/N, a+1/N] : P(t) \geq c_2^{-1}||P||_{L_\infty(I)}\} \geq \frac{c_1}{N}\,.$$

Proof of Lemma 4. With $\omega := \pi - 1/N$ and $n := \lfloor N \rfloor$, let

$$R_N(t) := \left| T_n \left(\frac{\sin((t-\pi-a)/2)}{\sin(\omega/2)} \right) \right| \in \mathrm{GTP}_N\,,$$

where $T_n(x) = \cos(nt)$, $x = \cos t$, is the Chebyshev polynomial of degree n. Then there is an absolute constant $c_2 > 1$ such that

$$c_2 \leq |R_N(a)| = ||R_N||_{L_\infty(I)} \tag{2.7}$$

and

$$|R_N(t)| \leq 1\,, \qquad t \in \mathbf{T} \setminus [a-1/N, a+1/N]\,. \tag{2.8}$$

Assume (2.6) holds. Let

$$A := \{t \in [a-1/N, a+1/N] : P(t) \leq c_2^{-1}||P||_{L_\infty(I)}\}\,, \tag{2.9}$$

and $Q := PR_N \in \mathrm{GTP}_{2N}$. Then, using (2.6)–(2.9), we obtain

$$Q(t) \leq c_2^{-1}||Q||_{L_\infty(I)}\,, \qquad t \in A \cup (\mathbf{T} \setminus [a-1/N, a+1/N])\,. \tag{2.10}$$

Let

$$B := [a-1/N, a+1/N] \setminus A\,. \tag{2.11}$$

Observe that (2.10), (2.11), and Lemma 3 applied to $Q \in \mathrm{GTP}_{2N}$ imply that

$$m(B) \geq m\{t \in \mathbf{T} : Q(t) \geq c_2^{-1}||Q||_{L_\infty(I)}\} \geq \frac{c_1}{N}.$$

Hence

$$m(B) = m\{t \in [a-1/N, a+1/N] : P(t) \geq c_2^{-1}||P||_{L_\infty(I)}\} \geq \frac{c_1}{N}$$

and the lemma is proved.

Proof of Lemma 1. First we prove (2.3) for $p = \infty$. By considering a shift, if it is necessary, we need to prove only that

$$|(P'Q)(\pi)| \leq c(N+M)\log(N+1)||PQ||_{L_\infty(I)} \tag{2.12}$$

for every $P \in \mathrm{GTP}_N$ of the form (1.1) with each $r_j \geq 1$, and for every $Q \in \mathrm{GTP}_M$. To prove (2.12) it is sufficient to handle the case when P has only real zeros. This can be seen by Lemmas 5.1–5.3 in [4]. However, for the sake of completeness, we present the arguments. Let $Q \in \mathrm{GTP}_M$ be fixed, and let each $r_j \geq 1$ in the representation (1.1) of P be fixed. Let

$$P(x) := \omega \prod_{j=1}^{s} (\sin((x - z_j)/2)\sin((x - \overline{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}\,, \qquad x \in \mathbf{R}\,,$$

where each T_j is a real trigonometric polynomial of degree 1 being nonnegative on the real line. Let $\delta \in (0, \pi)$ be fixed, and let $I_\delta := [-\pi + \delta, \pi - \delta]$. A simple compactness argument shows that for the fixed exponents $r_1, r_2, \ldots, r_s$, and for the fixed parameter $\delta \in (0, \pi)$ there are real trigonometric polynomials $\widetilde{T}_j$ of degree 1 being nonnegative on the real line so that with

$$\widetilde{S}(x) := \prod_{j=1}^{s} \widetilde{T}_j(x)^{r_j/2}\,, \qquad x \in \mathbf{R}\,,$$

we have

$$\max_S \left\{ \frac{|(S'Q)(\pi)|}{||SQ||_{L_\infty(I_\delta)}} \right\} = \frac{|(\widetilde{S}'Q)(\pi)|}{||\widetilde{S}Q||_{L_\infty(I_\delta)}}\,,$$

where the maximum is taken for all $S \in \mathrm{GTP}_N$ of the form

$$S(x) := \omega \prod_{j=1}^{s} (\sin((x - z_j)/2)\sin((x - \overline{z}_j)/2))^{r_j/2} = \prod_{j=1}^{s} T_j(x)^{r_j/2}\,, \qquad x \in \mathbf{R}\,, \tag{2.13}$$

where each T_j is a real trigonometric polynomial of degree 1 being nonnegative on the real line. To see this we can normalize S so that in (2.13)

$$||T_j||_{L_\infty(I_\delta)} = 1\,, \qquad j = 1, 2, \ldots, s\,,$$

hence the existence of the above real trigonometric polynomials $\widetilde{T}_j$ of degree 1 is really just a simple compactness argument. Now we can easily show that each $\widetilde{T}_j$ has only real zeros. Suppose to the contrary that $\widetilde{T}_k(x) \geq \eta > 0$ on the real line for some k. Consider

$$S_\varepsilon(x) := \prod_{j=1}^{s} T_{\varepsilon,j}(x)^{r_j/2}\,,$$

where $T_{\varepsilon,j} := \widetilde{T}_j$ if $j \neq k$, and

$$T_{\varepsilon,k}(x) := \widetilde{T}_k(x) - \varepsilon \sin^2 \frac{x - \pi}{2}\,.$$

If $\varepsilon > 0$ is sufficiently small, then S_ε contradicts the maximality of $\widetilde{S}$. This contradiction implies that each $\widetilde{T}_j$, and hence $\widetilde{S}$, has only real zeros. Also, by definition,

$$\frac{|(P'Q)(\pi)|}{||PQ||_{L_\infty(I)}} \leq \frac{|(P'Q)(\pi)|}{||PQ||_{L_\infty(I_\delta)}} \leq \frac{|(\widetilde{S}'Q)(\pi)|}{||\widetilde{S}Q||_{L_\infty(I_\delta)}} \leq (1 + \eta) \frac{|(\widetilde{S}'Q)(\pi)|}{||\widetilde{S}Q||_{L_\infty(I)}}\,,$$

where, as it can be seen by the Remez-type inequality of Theorem G, the numbers $\eta > 0$ tend to 0 as the numbers $\delta > 0$ tend to 0. This finishes the proof of our claim that it is sufficient to prove (2.12) only for $P \in \mathrm{GTP}_N$ of the form (1.1) with each $r_j \geq 1$ having only real zeros.

To prove (2.12) for $P \in \mathrm{GTP}_N$ of the form (1.1) with each $r_j \geq 1$ having only real zeros, without loss of generality we may assume that $|P'(\pi)|Q(\pi) = ||P'Q||_{L_\infty(I)}$. We now apply Lemma 4 with $|P'|Q \in \mathrm{GTP}_{N+M}$ and $a = \pi$ to obtain

$$|P'(t)|Q(t) \geq \frac{1}{c_2}||P'Q||_{L_\infty(I)}\,, \qquad t \in B\,, \tag{2.14}$$

where

$$B \subset K := \left[\pi - \frac{1}{N+M}, \pi + \frac{1}{N+M}\right], \quad |B| \geq \frac{c_1}{N+M}. \tag{2.15}$$

By Lemma 2 with $t = \pi$ we get $\mathcal{M}(|P'|Q, B) \leq \mathcal{M}(|P'|Q, K) \leq e + 1 < 4$. Thus $\mathcal{M}(|P'|, B) < 4$ which implies $\mathcal{M}(P, B) < 5$. Denote the different zeros of P in $(-\pi, \pi]$ by α_j with respective multiplicities $r_j \geq 1$, $j = 1, 2, \dots, m$ (then, of course, $2N = \sum_{j=1}^m r_j$). Thus (2.15), $\mathcal{M}(P, B) < 5$ and $r_j \geq 1$, $j = 1, 2, \dots, m$ yield that there exists a $t \in B$ such that

$$|t - \alpha_j| > \frac{c}{N+M}\,, \qquad j = 1, 2, \dots, m\,.$$

Fixing this t, we introduce the following intervals:

$$I_k := \left[t - \frac{2^k c}{N+M}\,, t + \frac{2^k c}{N+M}\right)\,, \qquad k = 0, 1, \dots, [\log_2 N]\,,$$

and let

$$I_{[\log_2 N]+1} := [t - \pi, t + \pi)\,.$$

Using (2.14) for our t, we can easily deduce that

$$\mathcal{M}(P, I_k) \leq \mathcal{M}(|P'|, I_k) + 1 \leq \mathcal{M}(|P'|Q, I_k) + 1 \leq$$

$$\leq \left(\frac{e}{2}(N+M)|I_k| + 1\right)\frac{||P'Q||_{L_\infty(I)}}{|P'(t)|Q(t)} + 1 = ecc_2 2^k + c_2 + 1 \leq c_3 2^k\,, \qquad k = 0, 1, \dots, [\log_2 N]\,.$$

Also, because of the choice of t, P does not have a zero in I_0. Therefore we can estimate as follows:

$$\frac{|P'(\pi)|Q(\pi)}{||PQ||_{L_\infty(I)}} \leq c_2 \frac{|P'(t)Q(t)|}{|P(t)Q(t)|} = \frac{c_2}{2}\left|\sum_{j=1}^m r_j \cot\frac{t - \alpha_j}{2}\right| \leq$$

$$\leq \sum_{k=1}^{[\log_2 N]+1} \frac{c_2}{2} \sum_{j \in I_k \setminus I_{k-1}} r_j \left|\cot\frac{t - \alpha_j}{2}\right| \leq c_2 \sum_{k=1}^{[\log_2 N]+1} \frac{\mathcal{M}(P, I_k)}{c_4|I_{k-1}|} \leq$$

$$\leq c_2 \sum_{k=1}^{[\log_2 N]+1} \frac{c_3(N+M)}{cc_4} \leq c_5(N+M)\log(N+1).$$

This proves (2.3) for $p = \infty$.

We now turn to the proof of (2.2). Applying (2.3) with $p = \infty$ to the generalized trigonometric polynomials P and QR, with

$$R(t) := \left|\frac{\sin(N+M)t}{\sin t}\right|^{2/q},$$

instead of P and Q, respectively, then using the Nikolskii type inequality

$$||\chi(P)||_{L_p(I)} \leq (c(1+qN))^{1/q-1/p}||\chi(P)||_{L_p(I)}\,, \qquad P \in \mathrm{GTP}_N\,, 0 < q < p \leq \infty\,,$$

of Theorem E with $\chi(x) = x$, $p = \infty$ and with PQR instead of P, we obtain

$$||P'QR||^q_{L_\infty(I)} \leq c_1^q(N+M)^q \log^q(N+1)||PQR||^q_{L_\infty(I)} \leq$$

$$\leq c(N+M)^{q+1}\log^q(N+1)||PQR||^q_{L_q(I)}.$$

Since $R(0)^q = (N+M)^2$, the latter inequality implies

$$|P'(0)|^q Q(0)^q \leq c(N+M)^{q-1}\log^q(N+1)||PQR||^q_{L_q(I)}.$$

Using this with $P(\cdot+\tau)$ and $Q(\cdot+\tau)$ instead of $P(\cdot)$ and $Q(\cdot)$, respectively (τ is a fixed parameter), we obtain

$$\left(\frac{|P'(\tau)|Q(\tau)}{(N+M)\log(N+1)}\right)^q \leq \int_{-\pi}^{\pi} c(P(t)Q(t))^q \frac{R(t-\tau)^q}{||R^q||_{L_1}}\,dt,$$

since evidently $||R^q||_{L_1} \sim N+M$. Hence by Jensen's inequality

$$\chi\left(\int_a^b S(t)w(t)\,dt\right) \leq \int_a^b \chi(S(t))w(t)\,dt$$

(cf. [1], E.20 on p. 414) applied with $[a,b] = [-\pi,\pi]$, $S = c(PQ)^q$, and

$$w(t) = \frac{R(t)^q}{||R^q||_{L_1(I)}}$$

we obtain

$$\chi\left(\left(\frac{|P'(\tau)|Q(\tau)}{(N+M)\log(N+1)}\right)^q\right) \leq \int_{-\pi}^{\pi} \chi(c(P(t)Q(t))^q)\frac{R(t-\tau)^q}{||R^q||_{L_1(I)}}\,dt.$$

Integrating with respect to τ and using Fubini's theorem yields the desired inequality.

It remains to show that (2.1) is sharp. To see this, we may assume that N and M are positive integers. Let $q = \min(1, p)$ and

$$S_{N+M}(t) := \left| \frac{\sin((N+M+1)t)}{\sin t} \right|^{2/q} = C_{N+M} \left| \prod_{k=1}^{N+M} \sin \frac{t-\alpha_k}{2} \prod_{k=1}^{N+M} \sin \frac{t+\alpha_k}{2} \right|^{2/q},$$

where

$$\alpha_k := \frac{k\pi}{N+M+1}, \qquad k = 1, 2, \dots, N+M.$$

Let P_N and Q_M be generalized trigonometric polynomials of degree $2N/q$ and $2M/q$, respectively, defined by

$$P_N(t) := \begin{cases} \left| \prod_{k=1}^{2N} \sin \frac{t-\alpha_k}{2} \right|^{2/q} & \text{if } \ N \le M, \\ \left| \prod_{k=1}^{N+M} \sin \frac{t-\alpha_k}{2} \prod_{k=2M+1}^{N+M} \sin \frac{t+\alpha_k}{2} \right|^{2/q} & \text{if } \ N > M, \end{cases}$$

and

$$Q_M(t) := \frac{S_{N+M}(t)}{P_N(t)},$$

respectively. It is easy to see that

$$S_{N+M}(0) = \max_{t \in (-\pi, \pi]} S_{N+M}(t).$$

Hence

$$\frac{\max_{t \in (-\pi,\pi]} |P_N'(t)| Q_M(t)}{\max_{t \in (-\pi,\pi]} P_N(t) Q_M(t)} \ge \frac{|P_N'(0)| Q_M(0)}{P_N(0) Q_M(0)} =$$

$$= \begin{cases} \frac{1}{q} \sum_{k=1}^{2N} \cot \frac{\alpha_k}{2} \ge c(N+M) \log(N+1) & \text{if } \ N \le M, \\ \frac{1}{q} \sum_{k=1}^{2M} \cot \frac{\alpha_k}{2} \ge c(N+M) \log(M+1) & \text{if } \ N > M. \end{cases}$$

This proves the sharpness for $p = \infty$.

Now let $0 < p < \infty$. We have

$$P_N'(t) Q_M(t) = \begin{cases} \frac{1}{q} S_{N+M}(t) \sum_{k=1}^{2N} \cot \frac{t-\alpha_k}{2} & \text{if } \ N \le M, \\ \frac{1}{q} S_{N+M}(t) \left(\sum_{k=1}^{N+M} \cot \frac{t-\alpha_k}{2} + \sum_{k=2M+1}^{N+M} \cot \frac{t+\alpha_k}{2} \right) & \text{if } \ N > M. \end{cases}$$

We shall give a lower estimate of the L_p norm of this polynomial over the interval $[0, \alpha_1/2]$. Evidently

$$S_{N+M}(t) \ge c(N+M)^{2/q}, \qquad 0 \le t \le \alpha_1/2.$$

On the other hand, if $t \in [0, \alpha_1/2]$ and $N \le M$, then

$$\left| \sum_{k=1}^{2N} \cot \frac{t-\alpha_k}{2} \right| \ge \sum_{k=1}^{N} \cot \frac{\alpha_k - \alpha_1/2}{2} \ge c(N+M) \log(N+1),$$

while if $t \in [0, \alpha_1/2]$ and $N > M$, then

$$\left| \sum_{k=1}^{N+M} \cot \frac{t-\alpha_k}{2} + \sum_{k=2M+1}^{N+M} \cot \frac{t+\alpha_k}{2} \right| \geq$$

$$\geq \sum_{k=2j}^{M} \cot \frac{\alpha_k - \alpha_j}{2} - \sum_{k=2M+1}^{N+M} \left| \cot \frac{t-\alpha_k}{2} + \cot \frac{t+\alpha_k}{2} \right| \geq$$

$$\geq c_1(N+M)\log(M+1) - c_2 \sum_{k=2M+1}^{N+M} \frac{\sin t}{\sin^2 \frac{\alpha_k}{2}} \geq$$

$$\geq c_1(N+M)\log(M+1) - c_3 \frac{N+M}{M} \geq c_4(N+M)\log(M+1).$$

Thus we obtain

$$\int_0^{\alpha_1/2} |P_N'(t)Q_M(t)|^p \, dt \geq \frac{c}{N+M}((N+M)^{1+2/q}\log(\min(N,M)+1))^p \geq$$

$$\geq c(N+M)^{p-1+2p/q}\log^p(\min(N,M)+1).$$

This compared with

$$||P_N Q_M||_{L_p}^p = ||S_{N+M}||_{L_p}^p \sim (N+M)^{2p/q-1}$$

proves that

$$||P_N' Q_M||_{L_p} \geq c(N+M)\log(\min(N,M)+1)||P_N Q_M||_{L_p}.$$

3. The case of generalized algebraic polynomials

In this section we improve the estimates of the inequalities (1.7) and (1.10).

Theorem 2. *Let $0 < p \leq \infty$. We have*

$$(3.1) \qquad ||\sqrt{1-x^2} f'(x)g(x)||_{L_p(J)} \leq c^{1+1/p}(N+M)\log(\min(N, M+1)+1)||fg||_{L_p(J)}$$

for any two $f \in \mathrm{GAP}_N$ and $g \in \mathrm{GAP}_M$ such that the roots of f and g have multiplicities at least 1. Moreover, (3.1) *is sharp apart from the constant $c^{1+1/p}$.*

Proof. Using the substitution (1.9) in (2.3), we obtain

$$||\sqrt{1-x^2} f'(x)g(x)||_{L_p(J)} \leq c^{1+1/p}(N+M)\log(N+1)||fg||_{L_p(J)}, \qquad 0 < p \leq \infty.$$

Exchanging the roles of f and g, and using the Bernstein type inequality (1.8) with fg instead of f, we obtain (3.1). The sharpness follows from the sharpness of the trigonometric analogue (2.1).

Theorem 3. *Let χ be a nonnegative, nondecreasing, convex function defined on $[0,\infty)$. Then*

$$\int_{-1}^{1} \chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) dx \leq 2\int_{-1}^{1} \chi(c(f(x)g(x))^q)\, dx \tag{3.2}$$

for every $f \in \mathrm{GAP}_N$ of the form (1.6) with each $r_j \geq 1$, for every $g \in \mathrm{GAP}_M$, and for every $0 < q \leq 1$. In particular,

$$||f'g||_{L_p(J)} \leq c^{1+1/p}(N+M)^2||fg||_{L_p(J)}\,, \qquad 0 < p \leq \infty\,, \tag{3.3}$$

for every $f \in \mathrm{GAP}_N$ of the form (1.6) with each $r_j \geq 1$, and for every $g \in \mathrm{GAP}_M$. The latter inequality is sharp up to the factor $c^{1+1/p}$ for all $N, M \geq 1$.

Proof. (3.3) readily follows from (3.2) by putting $q = \min(1,p)$ and $\chi(x) = x^{\max(1,p)}$. In order to prove (3.2) we mention that (3.3) for $p = \infty$, i.e.

$$||f'g||_{L_\infty(J)} \leq c(N+M)^2||fg||_{L_\infty(J)} \tag{3.4}$$

is nothing else but the corresponding inequality in (1.10) proved in [4].

Now consider an ordinary algebraic polynomial $h \geq 0$ on $[-1,1]$ of degree at most $N+M$ such that

$$\int_{-1}^{1} h(t)\, dt \leq \frac{c}{(N+M)^2} \quad \text{and} \quad h(1) = 1. \tag{3.5}$$

The existence of such a polynomial is guaranteed by known estimates for the Christoffel functions of the orthogonal Legendre polynomials (cf. e.g. Freud [9], Problem 10 on p. 132). Let $0 \leq x \leq 1$, let y be a fixed parameter to be specified later, and apply (3.4) on the interval $[x-1,x]$ instead of $[-1,1]$ for the generalized polynomials $f(x)$ and $g(x)h(x-y)^{1/q}$ (instead of $g(x)$):

$$|f'(x)|g(x)h(x-y)^{1/q} \leq c_2 q^{-2}(N+M)^2 \max_{x-1\leq t\leq x} f(t)g(t)h(t-y)^{1/q}\,, \qquad 0 \leq x, y \leq 1\,.$$

Using the Nikolskii type inequality of Theorem F with $\chi(x) = x$, $p = \infty$, and with $f(t)g(t)h(t-y)$ instead of f in the interval $[x-1,x]$ instead of J, we get

$$(|f'(x)|g(x))^q h(x-y) \leq c_1(N+M)^{2q+2}\int_{x-1}^{x} (f(t)g(t))^q h(t-y)\, dt\,, \qquad 0 \leq x, y \leq 1\,.$$

Putting $y = x-1$ and recalling (3.5), we deduce that

$$(|f'(x)|g(x))^q \leq c(N+M)^{2q+2}\int_{x-1}^{x} (f(t)g(t))^q h(t-x+1)\, dt$$

$$\leq c(N+M)^{2q} \int_{x-1}^{x} \frac{(f(t)g(t))^q h(t-x+1)}{\int_{x-1}^{x} h(u-x+1)\,du}\,dt\,, \qquad 0 \leq x \leq 1\,.$$

Rearranging this and using (2.13) with $[a,b] = [x-1,x]$, $S = c(fg)^q$, and

$$w(t) := \frac{h(t-x+1)}{\int_{x-1}^{x} h(u-x+1)\,du} = \frac{h(t-x+1)}{\int_0^1 h(v)\,dv}$$

(note that $\int_{x-1}^{x} w(t)\,dt = 1$), we obtain

$$\chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right) \leq \frac{\int_{x-1}^{x} \chi(c(f(t)g(t))^q)h(t-x+1)\,dt}{\int_0^1 h(v)\,dv}$$

$$= \frac{\int_{-1}^{1} \chi(c(f(t)g(t))^q)\varphi_{[x-1,x]}(t)h(t-x+1)\,dt}{\int_0^1 h(v)\,dv},$$

where $\varphi_{[a,b]}(t)$ is the characteristic function of the interval $[a,b]$. Integrating both sides with respect to x on $[0,1]$ and using Fubini's theorem, we get

$$\int_0^1 \chi\left(\frac{|f'(x)|g(x)}{(N+M)^{2q}}\right)\,dx \leq \frac{\int_{-1}^{1} \chi(c(f(t)g(t))^q) \int_0^1 \varphi_{[x-1,x]}(t)h(t-x+1)\,dx\,dt}{\int_0^1 h(v)\,dv}.$$

Here an easy calculation shows that

$$\int_0^1 \varphi_{[x-1,x]}(t)h(t-x+1)\,dx = \begin{cases} \int_0^{t+1} h(v)\,dv & \text{if } -1 \leq t \leq 0, \\ \int_t^1 h(v)\,dv & \text{if } 0 \leq t \leq 1, \end{cases}$$

which can be estimated by $\int_0^1 h(v)\,dv$ in both cases. Hence (3.2) is proved without the factor 2 when the integral is taken over $[0,1]$ rather than $[-1,1]$ on the left hand side. Similar arguments yield (3.2) without the factor 2 when the integral is taken over $[-1,0]$ rather than $[-1,1]$ on the left hand side. In conclusion (3.2) holds with the factor 2.

To prove the sharpness of (3.3), let $u_n^{(\lambda)}(x)$ be the ultraspherical Jacobi polynomial of degree n with parameter $\lambda \geq 0$ normalized such that $u_n^{(\lambda)}(1) = 1$. Then the absolute maximum of $u_n^{(\lambda)}(x)$ is attained at ± 1 (cf. [10], p. 168). Without loss of generality we may assume that $N \geq 1$ and $M \geq 0$ are integers. Let f_N be the monic polynomial of degree N which has N roots of the polynomial $u_{N+M}^{(\lambda)}(x)$ closest to 1, and let g_M be defined by $f_N g_M = u_{N+M}^{(\lambda)}$. Then for $p = \infty$ we get

$$f_N'(1)g_M(1) = \frac{f_N'(1)}{f_N(1)} u_{N+M}^{(\lambda)}(1) \geq \frac{||f_N g_M||_{L_p(J)}}{1-x_1} \geq c(N+M)^2 ||f_N g_M||_{L_p(J)},$$

where x_1 is the largest root of $u_{N+M}^{(\lambda)}$ (cf. [10], (6.6.6), where $1 - x_1 \le c(N+M)^{-2}$ is shown). We remark that in the rest of this proof $c, c_1, c_2, \ldots$ may depend on λ. Now let $0 < p < \infty$ and $\lambda > 2/p$. Using the estimates

$$|u_n^{(\lambda)}(\cos t)| \le \begin{cases} c_2 & \text{if } \ 0 \le t \le c_1/n, \\ c_3(nt)^{-\lambda} & \text{if } \ c_1/n \le t \le \pi/2 \end{cases}$$

(cf. [10], (7.33.6)), we get

$$||f_N g_M||_{L_p(J)} = ||u_{N+M}^{(\lambda)}||_{L_p(J)} = 2^{1/p}||u_{N+M}^{(\lambda)}||_{L_p([0,1])}$$

$$= 2^{1/p}\left(\int_0^{\pi/2} |u_{N+M}^{(\lambda)}(\cos t)|^p \sin t\, dt\right)^{1/p}$$

$$\le c_4^{1/p}\left(\int_0^{c_1/(N+M)} t\, dt + (N+M)^{-p\lambda}\int_{c_1/(N+M)}^{\pi/2} t^{1-p\lambda}\, dt\right)^{1/p} \le c_5^{1/p}(N+M)^{-2/p}.$$

On the other hand, Markov's inequality (cf. Theorem D with $p := \infty$ and $g := 1$) guarantees a constant c_6 such that

$$x_1 \le 1 - \frac{2c_6}{(N+M)^2}$$

and

$$u_{N+M}^{(\lambda)}(x) \ge \frac{1}{2} \quad \text{if} \quad 1 - \frac{c_6}{(N+M)^2} \le x \le 1\,.$$

Thus

$$||f_N' g_M||_{L_p(J)} \ge \left(\int_{1-c_6/(N+M)^2}^{1} \left|\frac{f_N'(x)}{f_N(x)} u_{N+M}^{(\lambda)}(x)\right|^p dx\right)^{1/p}$$

$$\ge \left(\int_{1-c_6/(N+M)^2}^{1} \left|\frac{u_{N+M}^{(\lambda)}(x)}{x - x_1}\right|^p dx\right)^{1/p} \ge \left(\frac{c_6}{(N+M)^2}\left(\frac{(N+M)^2}{c_6}\right)^p \left(\frac{1}{2}\right)^p\right)^{1/p} =$$

$$= \frac{1}{2}{c_6}^{1/p-1}(N+M)^{2-2/p} = \frac{1}{2}{c_6}^{1/p-1}c_5^{-1/p}(N+M)^2||f_N g_M||_{L_p(J)},$$

which proves the sharpness of (3.3).

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